

ON BERNSTEIN-TYPE ESTIMATES FOR LINEAR OPERATORS ACTING ON POLYNOMIALS

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Abstract. We establish new Bernstein-type inequalities for a class of linear operator T_g acting on complex polynomials, generalizing classical results in geometric function theory. Notably, we introduce a family of operators T_g induced by polynomials g with all zeros in the open unit disk, whose action is governed by the Schur-Szegő composition theorem and derive sharp bounds relating the maximum norms of $T_g[(P \circ f_R)(z) - \alpha(P \circ f_r)(z)]$, where $f_a(z) = az + S$, $a \in \{r, R\}$, and P on the unit disk. Our framework offers a unified extension of foundational inequalities due to Bernstein, Lax, Ankeny-Rivlin, and Aziz-Dawood, specifically for polynomials with restricted zeros. The results not only encapsulate known theorems as special cases but also introduce new bounds for a wider class of operators. This study enhances the connection between operator theory and polynomial inequalities, furnishing tools with potential applications in approximation theory and related domains.

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