

## A NOVEL PARAMETRIC METHOD FOR PROVING SOME ANALYTIC INEQUALITIES AND DETERMINING MINIMAX APPROXIMATIONS

MILOŠ MIĆOVIĆ, BRANKO MALEŠEVIĆ\*,  
TATJANA LUTOVAC AND BOJANA MIHAILOVIĆ

**Abstract.** In this paper, we give a novel method for proving inequalities and determining minimax approximants of the stratified families of functions. In the theory of analytic inequalities, there are numerous inequalities based on which one can form a family of functions that is stratified, i.e. monotonic with respect to the introduced parameter. The introduced method is based on identifying those functions from the family that have a stationary point on the observed interval. The applications of this method are demonstrated to the Cusa-Huygens, Mitrinović-Adamović-type and Jordan-type inequalities.

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