

SHARP MULTIDIMENSIONAL MULTIPLICATIVE INEQUALITIES FOR WEIGHTED L_p SPACES WITH HOMOGENEOUS WEIGHTS

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Abstract. Let Ω be an arbitrary cone in \mathbb{R}^n with the origin as a vertex. A multidimensional multiplicative inequality for weighted $L_p(\Omega)$ -spaces with homogeneous weights is proved. The inequality is sharp and all cases of equality are pointed out. In particular, this inequality may be regarded as a weighted multidimensional extension of previous inequalities of Carlson, Beurling and Levin.

1. Introduction

Let f be a nonnegative function on $[0, \infty)$ such that $f^2(x)$ and $x^2 f^2(x)$ are integrable on $[0, \infty)$. Then the inequality

$$\int_0^\infty f(x) dx \leq \sqrt{\pi} \left(\int_0^\infty f^2(x) dx \right)^{1/4} \left(\int_0^\infty x^2 f^2(x) dx \right)^{1/4} \quad (1)$$

holds and $\sqrt{\pi}$ is the best possible constant. This is the well-known Carlson inequality [2]. It was generalized in many directions. We formulate next a general result obtained by V. I. Levin [3]. For other generalizations see the books [4, 5] and the references given there.

THEOREM 1. ([3]) *Let $p > 1$, $q > 1$, $\lambda > 0$, $\mu > 0$ and let f be a nonnegative function such that $x^{p-1-\lambda} f^p(x)$ and $x^{q-1+\mu} f^q(x)$ are integrable on $[0, \infty)$. Then*

$$\int_0^\infty f(x) dx \leq C \left(\int_0^\infty x^{p-1-\lambda} f^p(x) dx \right)^s \left(\int_0^\infty x^{q-1+\mu} f^q(x) dx \right)^t \quad (2)$$

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where¹

$$C = \left(\frac{1}{ps}\right)^s \left(\frac{1}{qt}\right)^t \left(\frac{1}{\lambda + \mu} B\left(\frac{s}{1-s-t}, \frac{t}{1-s-t}\right)\right)^{1-s-t}$$

and

$$s = \frac{\mu}{p\mu + q\lambda}, \quad t = \frac{\lambda}{p\mu + q\lambda}.$$

The constant C is best possible.

We note that the inequality (2) is equivalent to the inequality

$$\int_{-\infty}^{\infty} f(x) dx \leq 2^{1-s-t} C \left(\int_{-\infty}^{\infty} |x|^{p-1-\lambda} f^p(x) dx \right)^s \left(\int_{-\infty}^{\infty} |x|^{q-1+\mu} f^q(x) dx \right)^t. \quad (3)$$

In particular, the inequality (1) is equivalent to the inequality

$$\int_{-\infty}^{\infty} f(x) dx \leq \sqrt{2\pi} \left(\int_{-\infty}^{\infty} f^2(x) dx \right)^{1/4} \left(\int_{-\infty}^{\infty} x^2 f^2(x) dx \right)^{1/4}$$

first established by A. Beurling [1]. Indeed, in order to deduce (2) from (3) it is enough to apply (3) to the even extension of f from $[0, \infty)$ to $(-\infty, \infty)$. On the other hand from (2) and Hölder's inequality for sums it follows that

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx \\ &\leq C \left(\left(\int_{-\infty}^0 |x|^{p-1-\lambda} f^p(x) dx \right)^s \left(\int_{-\infty}^0 |x|^{q-1+\mu} f^q(x) dx \right)^t \right. \\ &\quad \left. + \left(\int_0^{\infty} |x|^{p-1-\lambda} f^p(x) dx \right)^s \left(\int_0^{\infty} |x|^{q-1+\mu} f^q(x) dx \right)^t \right) \\ &\leq 2^{1-s-t} C \left(\int_{-\infty}^{\infty} |x|^{p-1-\lambda} f^p(x) dx \right)^s \left(\int_{-\infty}^{\infty} |x|^{q-1+\mu} f^q(x) dx \right)^t \end{aligned}$$

since $a_1 b_1 + a_2 b_2 \leq 2^{1-s-t} \left(a_1^{1/s} + b_1^{1/s} \right)^s \left(a_2^{1/t} + b_2^{1/t} \right)^t$, $a_i, b_i \geq 0$.

Our aim is to give generalizations of (2) and (3) to the multidimensional case and general homogeneous weights.

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¹Here $B(\sigma, \tau)$, $\sigma, \tau > 0$, is the beta-function: $B(\sigma, \tau) = \frac{\Gamma(\sigma)\Gamma(\tau)}{\Gamma(\sigma+\tau)}$, where $\Gamma(\sigma)$, $\sigma > 0$, is the gamma-function.

2. The Main Result

Let Ω be an arbitrary infinite cone in \mathbb{R}^n with the origin as a vertex, i.e.,

$$\Omega = \{x \in \mathbb{R}^n : x = \rho\sigma, 0 < \rho < \infty, \sigma \in S\} \quad (4)$$

where $S \subseteq S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ is a measurable set. We note that for arbitrary positive ε we have $\varepsilon\Omega = \Omega$.

THEOREM 2. *Let Ω be defined by (4) and $w_i, i = 0, 1, 2$, be functions defined on Ω . Suppose that w_i are positive, measurable and homogeneous² of orders $\alpha_i \in \mathbb{R}, i = 0, 1, 2$. Moreover, suppose that*

$$0 < p_0 < p_1, p_2 < \infty, \quad 0 < \theta < 1, \quad (5)$$

and put

$$d_i = \alpha_i + \frac{n}{p_i}, \quad i = 0, 1, 2.$$

1. *In order that for some $A > 0$ and for all functions f measurable on Ω and satisfying $\|f w_i\|_{L_{p_i}(\Omega)} < \infty, i = 1, 2$,*

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq A \|f w_1\|_{L_{p_1}(\Omega)}^\theta \|f w_2\|_{L_{p_2}(\Omega)}^{1-\theta}, \quad (6)$$

it is necessary and sufficient that

$$d_0 = \theta d_1 + (1 - \theta) d_2, \quad d_1 \neq d_2, \quad (7)$$

and

$$\left\| \frac{w_0}{w_1^\theta w_2^{1-\theta}} \right\|_{L_q(S)} < \infty, \quad (8)$$

where

$$\frac{1}{q} = \frac{1}{p_0} - \frac{\theta}{p_1} - \frac{1-\theta}{p_2}. \quad (9)$$

2. *If (7) and (8) are satisfied, then the minimal possible value A_* of A in (6) has the form*

$$\begin{aligned} A_* &= \left(\frac{\theta}{p_1} \right)^{-\frac{\theta}{p_1}} \left(\frac{1-\theta}{p_2} \right)^{-\frac{1-\theta}{p_2}} (p_1 p_2)^{-\frac{1}{p_0}} \left(\frac{B(\theta \frac{q}{p_1}, (1-\theta) \frac{q}{p_2})}{|d_1 - d_2|} \right)^{1/q} \times \\ &\times \left(\frac{1}{p_0} - \frac{1}{q} \right)^{-\frac{1}{q}} \left\| \frac{w_0}{w_1^\theta w_2^{1-\theta}} \right\|_{L_q(S)}. \end{aligned} \quad (10)$$

²I.e., for each $x \in \Omega$ and $\varepsilon > 0$, $w_i(\varepsilon x) = \varepsilon^{\alpha_i} w_i(x)$.

3. Equality in (6) with $A = A_*$ holds if and only if, for some $B \geq 0, \gamma > 0$,

$$|f(x)| = Bf_*(\gamma x) \quad (11)$$

for almost all $x \in \Omega$, where

$$f_* = \left(k \frac{w_0^{p_0}}{w_1^{p_1}} \right)^{\frac{1}{p_1 - p_0}}. \quad (12)$$

and the positive function k is uniquely defined by

$$\left(k^{1/p_1} \frac{w_0}{w_1} \right)^{r_1} = \left((1 - k)^{1/p_2} \frac{w_0}{w_2} \right)^{r_2}, \quad (13)$$

where

$$\frac{1}{r_i} = \frac{1}{p_0} - \frac{1}{p_i}, \quad i = 1, 2. \quad (14)$$

REMARK 1. It is enough to prove Theorem 2 for the case in which $p_0 = 1, w_0 = 1$. The general case follows from this case if we replace w_i by $\left(\frac{w_i}{w_0} \right)^{p_0}$, p_i by $\frac{p_i}{p_0}$, $i = 1, 2$ and f by $(f w_0)^{p_0}$.

3. Proof of The Main Result

First we state the following lemma of independent interest:

LEMMA 1. Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set and let w_0, w_1, w_2 be positive and measurable functions on Ω . Moreover, suppose that $0 < p_0 < p_1, p_2 < \infty$, and $\delta > 0$. If f is measurable on Ω and $\|f w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$, then

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \left(M_1^{p_0}(\delta) \|f w_1\|_{L_{p_1}(\Omega)}^{p_0} + M_2^{p_0}(\delta) \|f w_2\|_{L_{p_2}(\Omega)}^{p_0} \right)^{1/p_0}, \quad (15)$$

where

$$M_i(\delta) = \left\| a_{i,\delta}^{1/p_0} \frac{w_0}{w_i} \right\|_{L_{r_i}(\Omega)}, \quad i = 1, 2,$$

and r_i are defined by (14).

Here $0 < a_{1,\delta}(x) < 1$, $x \in \Omega$, is uniquely defined by

$$\left(a_{1,\delta}^{1/p_1}(x) \frac{w_0(x)}{w_1(x)} \right)^{r_1} = \delta \left((1 - a_{1,\delta}(x))^{1/p_2} \frac{w_0(x)}{w_2(x)} \right)^{r_2} \quad (16)$$

and $a_{2,\delta}(x) = 1 - a_{1,\delta}(x)$.

Inequality (15) is sharp. Moreover equality holds if, and only if, for some nonnegative λ ,

$$|f| = \lambda \left(a_{1,\delta} \frac{w_0^{p_0}}{w_1^{p_1}} \right)^{\frac{1}{p_1 - p_0}} \quad (17)$$

almost everywhere on Ω .

Proof. Let the functions a_1, a_2 be positive and measurable on Ω and $a_1 + a_2 = 1$. Then

$$\int_{\Omega} |f|^{p_0} w_0^{p_0} dx = \int_{\Omega} a_1 \left(\frac{w_0}{w_1} \right)^{p_0} (|f| w_1)^{p_0} dx + \int_{\Omega} a_2 \left(\frac{w_0}{w_2} \right)^{p_0} (|f| w_2)^{p_0} dx.$$

For $p > 1$ we use as usual the notation $p' = \frac{p}{p-1}$. Since $p_i > p_0$ we have, by Hölder's inequality,

$$\begin{aligned} \int_{\Omega} |f|^{p_0} w_0^{p_0} dx &\leq \left\| a_1 \left(\frac{w_0}{w_1} \right)^{p_0} \right\|_{L_{\left(\frac{p_1}{p_0}\right)'(\Omega)}} \left\| (|f| w_1)^{p_0} \right\|_{L_{\frac{p_1}{p_0}}(\Omega)} \\ &\quad + \left\| a_2 \left(\frac{w_0}{w_2} \right)^{p_0} \right\|_{L_{\left(\frac{p_2}{p_0}\right)'(\Omega)}} \left\| (|f| w_2)^{p_0} \right\|_{L_{\frac{p_2}{p_0}}(\Omega)} \\ &= \left(\left\| a_1^{1/p_0} \frac{w_0}{w_1} \right\|_{L_{r_1}(\Omega)} \|f w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} \\ &\quad + \left(\left\| a_2^{1/p_0} \frac{w_0}{w_2} \right\|_{L_{r_2}(\Omega)} \|f w_2\|_{L_{p_2}(\Omega)} \right)^{p_0}. \end{aligned} \quad (18)$$

Next we choose a_1 and a_2 in the optimal way keeping in mind that in (18) equality holds if and only if, for some $\lambda_1, \lambda_2 \geq 0$

$$\begin{aligned} (|f| w_1)^{p_1} &= \lambda_1 \left(a_1 \left(\frac{w_0}{w_1} \right)^{p_0} \right)^{\left(\frac{p_1}{p_0}\right)'}, \\ (|f| w_2)^{p_2} &= \lambda_2 \left(a_2 \left(\frac{w_0}{w_2} \right)^{p_0} \right)^{\left(\frac{p_2}{p_0}\right)'}, \end{aligned}$$

almost everywhere on Ω , or

$$|f| = \lambda \left(a_1 \frac{w_0^{p_0}}{w_1^{p_1}} \right)^{\frac{1}{p_1 - p_0}}$$

where $\lambda = \lambda_1^{1/p_1}$ and

$$\left(a_1^{1/p_1} \frac{w_0}{w_1} \right)^{r_1} = \delta \left(a_2^{1/p_2} \frac{w_0}{w_2} \right)^{r_2}$$

where $\delta = \left(\lambda_2^{1/p_2} \lambda_1^{-1/p_1} \right)^{p_0}$.

Hence the statement of the lemma follows.

Next we state a corollary of Lemma 1, which is useful for our later purposes.

LEMMA 2. *Let Ω be defined by (4) and w_i be functions defined on Ω which are positive, measurable and homogeneous of orders α_i , $i = 1, 2$. Moreover, suppose that $0 < p_0 < p_1, p_2 < \infty$ and $\varepsilon > 0$.*

If f is measurable on Ω and $\|f w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$, then, for every $\varepsilon > 0$,

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \left(\left(\varepsilon^{(d_1-d_0)} A_1 \|f w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} + \left(\varepsilon^{(d_2-d_0)} A_2 \|f w_2\|_{L_{p_2}(\Omega)} \right)^{p_0} \right)^{1/p_0}, \quad (19)$$

where

$$A_i = \left\| k_i^{1/p_0} \frac{w_0}{w_i} \right\|_{L_{r_i}(\Omega)}, \quad i = 1, 2, \quad (20)$$

$k_1 \equiv k$, $k_2 \equiv 1 - k$ and d_i , r_i and k are defined in Theorem 2.

Inequality (19) is sharp. Moreover, equality holds if, and only if, for some $\mu \geq 0$

$$|f(x)| = \mu f_*(\varepsilon x), \quad (21)$$

where f_* is defined by (12).

Proof: Replacing x by εx in (13) we get

$$\begin{aligned} \left(k^{1/p_1}(\varepsilon x) \frac{w_0(x)}{w_1(x)} \right)^{r_1} &= \varepsilon^{(\alpha_0-\alpha_2)r_2-(\alpha_0-\alpha_1)r_1} \left((1-k(\varepsilon x))^{1/p_2} \frac{w_0(x)}{w_2(x)} \right)^{r_2} \\ &= \varepsilon^{(d_0-d_2)r_2-(d_0-d_1)r_1} \left((1-k(\varepsilon x))^{1/p_2} \frac{w_0(x)}{w_2(x)} \right)^{r_2}. \end{aligned}$$

Consequently, if $\delta = \varepsilon^{(d_0-d_2)r_2-(d_0-d_1)r_1}$ and $a_{i,\delta}(x) = k_i(\varepsilon x)$, then the equality (16) is satisfied.

Hence, by Lemma 1, the inequality (15) is valid, where

$$\begin{aligned} M_i(\delta) &= \left\| k_i^{1/p_0}(\varepsilon x) \frac{w_0(x)}{w_i(x)} \right\|_{L_{r_i}(\Omega)} \\ &= \varepsilon^{-\frac{n}{r_i}} \left\| k_i^{1/p_0}(y) \frac{w_0(\frac{y}{\varepsilon})}{w_i(\frac{y}{\varepsilon})} \right\|_{L_{r_i}(\varepsilon\Omega)} \\ &= \varepsilon^{-\frac{n}{r_i}-\alpha_0+\alpha_i} \left\| k_i^{1/p_0} \frac{w_0}{w_i} \right\|_{L_{r_i}(\Omega)} = \varepsilon^{d_i-d_0} A_i \end{aligned}$$

and (19) follows.

Furthermore, by (17), equality in (19) holds if, and only if,

$$\begin{aligned} |f(x)| &= \lambda \left(k(\varepsilon x) \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{1}{p_1 - p_0}} \\ &= \lambda \varepsilon^{\frac{\alpha_1 p_1 - \alpha_0 p_0}{p_1 - p_0}} f_*(\varepsilon x) = \mu f_*(\varepsilon x). \end{aligned}$$

The proof is complete.

Proof of Theorem 2. Our proof is organized in six steps:

1. Let (7) and (8) be satisfied. According to (7) it yields that $d_1 - d_0 = (1 - \theta)(d_1 - d_2)$ and $d_2 - d_0 = -\theta(d_1 - d_2)$. We set $\varepsilon^{(d_1 - d_2)p_0} = \delta$ in Lemma 2. Since $d_1 \neq d_2$, by (19) we have, for all $\delta > 0$,

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \left(\delta^{1-\theta} \left(A_1 \|f w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} + \delta^{-\theta} \left(A_2 \|f w_2\|_{L_{p_2}(\Omega)} \right)^{p_0} \right)^{1/p_0}.$$

Since for $a, b > 0$

$$\min_{\delta > 0} (a\delta^{1-\theta} + b\delta^{-\theta}) = (a\delta^{1-\theta} + b\delta^{-\theta}) \Big|_{\delta = \frac{\theta}{1-\theta} \frac{b}{a}} = (\theta^\theta (1 - \theta)^{1-\theta})^{-1} a^\theta b^{1-\theta}$$

we obtain, by setting in (19) $\varepsilon = \varepsilon_0$, where

$$\varepsilon_0 = \varepsilon_0(f) = \left(\left(\frac{\theta}{1-\theta} \right)^{1/p_0} \frac{A_2 \|f w_2\|_{L_{p_2}(\Omega)}}{A_1 \|f w_1\|_{L_{p_1}(\Omega)}} \right)^{\frac{1}{d_1 - d_2}}, \quad (22)$$

that inequality (6) is valid with $A = \tilde{A}$, where

$$\tilde{A} = \left(\theta^\theta (1 - \theta)^{1-\theta} \right)^{-1/p_0} A_1^\theta A_2^{1-\theta}. \quad (23)$$

2. Let there be equality in (6) with $A = \tilde{A}$. Then, for $\varepsilon = \varepsilon_0(f)$ defined by (22) there is equality in (19). Hence, by (21), for some $\mu \geq 0$,

$$|f(x)| = \mu f_*(\varepsilon_0(f)x). \quad (24)$$

Thus an extremal function in (6) is necessarily of the form (11).

Let us verify that each function f defined by (11) is extremal. By Lemma 2 with $\varepsilon = 1$ and the elementary arithmetic-geometric mean inequality

$$x^\theta y^{1-\theta} \leq \theta^\theta (1 - \theta)^{1-\theta} (x + y), \quad x, y \geq 0, \quad 0 < \theta < 1, \quad (25)$$

it follows that

$$\begin{aligned} \|f_* w_0\|_{L_{p_0}(\Omega)} &= \left(\left(A_1 \|f_* w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} + \left(A_2 \|f_* w_2\|_{L_{p_2}(\Omega)} \right)^{p_0} \right)^{1/p_0} \\ &\geq \tilde{A} \|f_* w_1\|_{L_{p_1}(\Omega)}^\theta \|f_* w_2\|_{L_{p_2}(\Omega)}^{1-\theta}. \end{aligned} \quad (26)$$

Consequently, keeping in mind inequality (6) for $f = f_*$ with $A = \tilde{A}$, we see that there is equality in (26).

Thus \tilde{A} is the minimal possible value of A in (6) and f_* is an extremal function, if $\|f_* w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$. This will be proved in step 4 of the proof. Moreover for every $B \geq 0$ and $\gamma > 0$ the function f defined by (11) is also an extremal function. This easily follows if we replace γx by y and apply the relation (7).

3. Next we note that

$$\|f_* w_i\|_{L_{p_i}(\Omega)} = \left\| k_i^{1/p_0} \frac{w_0}{w_i} \right\|_{L_{r_i}(\Omega)}^{r_i} = A_i^{r_i/p_i}, \quad i = 1, 2. \quad (27)$$

Since equality in (25) holds if, and only if, $\frac{x}{\theta} = \frac{y}{1-\theta} (= x + y)$ and there is equality in (26), we have, applying in addition (27), that

$$\frac{A_1^{r_1}}{\theta} = \frac{A_2^{r_2}}{1-\theta} = \|f_* w_0\|_{L_{p_0}(\Omega)}^{p_0}. \quad (28)$$

In particular, we note that due to (28) equation (22) is satisfied for the function f defined by (24). Moreover, we use (23) and (27) and find that

$$\begin{aligned} \tilde{A} &= \left(\theta^\theta (1-\theta)^{1-\theta} \right)^{-1/p_0} \theta^{\frac{\theta}{r_1}} (1-\theta)^{\frac{1-\theta}{r_2}} \|f_* w_0\|_{L_{p_0}(\Omega)}^{p_0 \left(\frac{\theta}{r_1} + \frac{1-\theta}{r_2} \right)} \\ &= \theta^{-\frac{\theta}{p_1}} (1-\theta)^{-\frac{1-\theta}{p_2}} \left\| k^{\frac{1}{p_1}} \frac{w_0}{w_1} \right\|_{L_{r_1}(\Omega)}^{\frac{r_1}{q}}. \end{aligned} \quad (29)$$

4. Since the functions $w_i(x)$ are homogeneous of orders α_i we have $w_i(x) = \rho^{\alpha_i} w_i(\sigma)$ for $x \in \Omega$, where $\rho = |x|$ and $\sigma = \frac{x}{|x|} \in S$. Substituting this into (13) we get

$$k^{\frac{r_1}{p_1}}(x) = \left(\frac{w_1(\sigma)}{w_0(\sigma)} \right)^{r_1} \left(\frac{w_0(\sigma)}{w_2(\sigma)} \right)^{r_2} \rho^{(\alpha_0 - \alpha_2)r_2 - (\alpha_0 - \alpha_1)r_1} (1 - k(x))^{\frac{r_2}{p_2}}$$

or

$$k^{\frac{1}{p_1 r_2}}(x) = \left(\frac{w_1(\sigma)}{w_0(\sigma)} \right)^{\frac{1}{r_2}} \left(\frac{w_0(\sigma)}{w_2(\sigma)} \right)^{\frac{1}{r_1}} \rho^{\left(\frac{\theta}{r_1} + \frac{1-\theta}{r_2} \right)(d_1 - d_2)} (1 - k(x))^{\frac{1}{p_2 r_1}}.$$

Thus

$$k^{\frac{1}{p_1 r_q}}(x) = (w(\sigma) \rho)^{\frac{d_1 - d_2}{q}} (1 - k(x))^{\frac{1}{p_2 r_1}},$$

where

$$w(\sigma) = \left(\left(\frac{w_1(\sigma)}{w_0(\sigma)} \right)^{\frac{1}{r_2}} \left(\frac{w_0(\sigma)}{w_2(\sigma)} \right)^{\frac{1}{r_1}} \right)^{\frac{q}{d_1 - d_2}}.$$

Consequently

$$k(x) = \varphi(w(\sigma)\rho)$$

where the function φ of one variable is defined by

$$\varphi^{1/p_1 r_2}(z) = z^{\frac{d_1-d_2}{q}} (1 - \varphi(z))^{1/p_2 r_1}. \quad (30)$$

Furthermore,

$$\begin{aligned} \|f_* w_0\|_{L_{p_0}(\Omega)}^{p_0} &= \int_{\Omega} \left(k^{1/p_1}(x) \frac{w_0(x)}{w_1(x)} \right)^{r_1} dx \\ &= \int_S \left(\int_0^\infty \varphi^{r_1/p_1}(w(\sigma)\rho) \left(\frac{w_0(\sigma)}{w_1(\sigma)} \right)^{r_1} \rho^{(\alpha_0-\alpha_1)r_1+n-1} d\rho \right) d\sigma \\ &= \int_S w(\sigma)^{-(d_0-d_1)r_1} \left(\frac{w_0(\sigma)}{w_1(\sigma)} \right)^{r_1} d\sigma \int_0^\infty \varphi^{r_1/p_1}(z) z^{(d_0-d_1)r_1-1} dz \\ &= \int_S \left(\frac{w_0}{w_1^\theta w_2^{1-\theta}} \right)^q d\sigma \int_0^\infty \varphi^{r_1/p_1}(z) z^{-(1-\theta)(d_1-d_2)r_1-1} dz. \end{aligned} \quad (31)$$

The latter integral can be evaluated in the following way: Substitute $\varphi(z) = y$ so that, by (30), $z = \left(y^{\frac{1}{p_1 r_2}} (1-y)^{-\frac{1}{r_1 p_2}} \right)^{\frac{q}{d_1-d_2}}$. Then

$$\begin{aligned} &\int_0^\infty \varphi^{r_1/p_1}(z) z^{-(1-\theta)(d_1-d_2)r_1-1} dz \\ &= \int_0^1 y^{\frac{r_1}{p_1}} \left(y^{\frac{1}{p_1 r_2}} (1-y)^{-\frac{1}{r_1 p_2}} \right)^{\frac{-[q(1-\theta)(d_1-d_2)r_1+1]}{|d_1-d_2|}} |z'(y)| dy \\ &= \frac{1}{p_1 r_2} \frac{q}{|d_1-d_2|} B\left(\theta \frac{q}{p_1}, (1-\theta) \frac{q}{p_2} + 1\right) \\ &\quad + \frac{1}{r_1 p_2} \frac{q}{|d_1-d_2|} B\left(\theta \frac{q}{p_1} + 1, (1-\theta) \frac{q}{p_2}\right). \end{aligned}$$

We now only use the facts that $B(a, b+1) = \frac{a}{a+b} B(a, b)$ and $B(a, b) = B(b, a)$ and find that

$$\int_0^\infty \varphi^{r_1/p_1}(z) z^{-(1-\theta)(d_1-d_2)r_1-1} dz = \frac{B\left(\theta \frac{q}{p_1}, (1-\theta) \frac{q}{p_2}\right)}{|d_1-d_2|(\theta p_2 + (1-\theta p_1))}. \quad (32)$$

Hence $\|f_* w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$, and $\tilde{A} = A_*$ where A_* is defined by (10).

5. Let us prove that the first condition (7) and condition (8) are necessary. Suppose that inequality (6) is valid for all functions measurable on Ω satisfying $\|f w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$. Consider a fixed function f_0 which is not equivalent to zero, satisfying these conditions and apply (6) to $f_0\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$. Replacing $\frac{x}{\varepsilon}$ by y , we obtain

$$\|f_0\left(\frac{x}{\varepsilon}\right) w_i(x)\|_{L_{p_i}(\Omega)} = \varepsilon^{n/p_i} \|f_0 w_i(\varepsilon y)\|_{L_{p_i}(\frac{1}{\varepsilon}\Omega)} = \varepsilon^{d_i} \|f_0 w_i\|_{L_{p_i}(\Omega)}.$$

Hence $\varepsilon^{d_0 - \theta d_1 - (1-\theta)d_2} \leq A_1$ for all $\varepsilon > 0$, where A_1 is independent of ε , and it follows that $d_0 = \theta d_1 + (1-\theta)d_2$.

Next we set $w_{i,N}(x) = w_i(x)\chi_N\left(\frac{x}{|x|}\right)$, $i = 0, 1, 2$, and $f_{*,N}(x) = f_*(x)\chi_N\left(\frac{x}{|x|}\right)$, where $N \in \mathbb{N}$ and χ_N is the characteristic function of the set $S_N \subset S$ of those $\sigma \in S$ for which $w_i(\sigma) \leq N$, $i = 0, 1, 2$. Then, according to step 4,

$$\begin{aligned} A &\geq \frac{\|f_{*,N} w_0\|_{L_{p_0}(\Omega)}}{\|f_{*,N} w_1\|_{L_{p_1}(\Omega)}^\theta \|f_{*,N} w_2\|_{L_{p_2}(\Omega)}^{1-\theta}} \\ &= \frac{\|f_{*,N} w_{0,N}\|_{L_{p_0}(\Omega)}}{\|f_{*,N} w_{1,N}\|_{L_{p_1}(\Omega)}^\theta \|f_{*,N} w_{2,N}\|_{L_{p_2}(\Omega)}^{1-\theta}} \\ &= C \left\| \frac{w_0}{w_1^\theta w_2^{1-\theta}} \right\|_{L_q(S_N)}, \end{aligned}$$

where C is independent of N . If $\left\| \frac{w_0}{w_1^\theta w_2^{1-\theta}} \right\|_{L_q(S_N)} = \infty$, then, by passing to the limit as $N \rightarrow \infty$, we see that for any $A > 0$ inequality (6) cannot be valid. We conclude that also (8) holds.

6. Finally, we prove that also the second condition $d_1 \neq d_2$ in (7) is necessary. Suppose that the first condition (7) and condition (8) are satisfied and $d_1 = d_2$. Hence by the first condition (7) it yields that $d_0 = d_1 = d_2$. Consider the family of functions

$$f_\varepsilon(x) = f_*(x)|x|^\varepsilon \chi(x),$$

where $\varepsilon > 0$, χ is the characteristic function of the unit ball in \mathbb{R}^n and f_* is defined by (12). By step 4 it follows that the function k depends on $\sigma = \frac{x}{|x|}$ only, i.e., k is homogeneous of order 0. Consequently,

$$\|f_\varepsilon w_i\|_{L_{p_i}(\Omega)} = \left(\int_{\Omega} \left(k(x) \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{p_i}{p_1 - p_0}} w_i^{p_i}(x) |x|^{\varepsilon p_i} \chi(x) dx \right)^{\frac{1}{p_i}}$$

$$\begin{aligned}
&= \left(\int_S \left(k(\sigma) \frac{w_0^{p_0}(\sigma)}{w_1^{p_1}(\sigma)} \right)^{\frac{p_i}{p_1-p_0}} w_i^{p_i}(\sigma) d\sigma \right)^{\frac{1}{p_i}} \times \\
&\quad \times \left(\int_0^1 \rho^{\left(\frac{\alpha_0 p_0 - \alpha_1 p_1}{p_1-p_0} + \alpha_i + \varepsilon \right) p_i + n-1} d\rho \right)^{\frac{1}{p_i}} \\
&= \|f_* w_i\|_{L_{p_i}(S)} \left(\int_0^1 \rho^{(-d_i + \alpha_i + \varepsilon) p_i + n-1} d\rho \right)^{\frac{1}{p_i}} \\
&= p_i^{-\frac{1}{p_i}} \|f_* w_i\|_{L_{p_i}(S)} \varepsilon^{-\frac{1}{p_i}} = C_i \varepsilon^{-\frac{1}{p_i}}.
\end{aligned}$$

We note that $\|f_* w_i\|_{L_{p_i}(S)} < \infty$, $i = 0, 1, 2$. This follows from (8) and the inequality

$$(f_* w_i)^{p_i} \leq \left(\frac{w_0}{w_1^\theta w_2^{1-\theta}} \right)^q, \quad i = 0, 1, 2.$$

In fact, if $\xi = \frac{\theta q}{r_1}$, then $1 - \xi = \frac{(1-\theta)q}{r_2}$ and (13) implies that

$$\begin{aligned}
(f_* w_0)^{p_0} &= \left(k^{\frac{1}{p_1}} \frac{w_0}{w_1} \right)^{r_1} \\
&= \left(k^{\frac{1}{p_1}} \frac{w_0}{w_1} \right)^{\xi r_1} \left((1-k)^{\frac{1}{p_2}} \frac{w_0}{w_2} \right)^{(1-\xi)r_2} \\
&\leq \frac{w_0^{\xi r_1 + (1-\xi)r_2}}{w_1^{\xi r_1} w_2^{(1-\xi)r_2}} = \left(\frac{w_0}{w_1^\theta w_2^{1-\theta}} \right)^q.
\end{aligned}$$

(Note that $0 < k < 1$.) Furthermore

$$(f_* w_1)^{p_1} = \left(k^{\frac{1}{p_0}} \frac{w_0}{w_1} \right)^{r_1} = k \left(k^{\frac{1}{p_1}} \frac{w_0}{w_1} \right)^{r_1} \leq (f_* w_0)^{p_0}$$

and similarly we find that $(f_* w_2)^{p_2} \leq (f_* w_0)^{p_0}$.

Now taking $f = f_\varepsilon$ in (6), we get

$$A \geq \frac{\|f_\varepsilon w_0\|_{L_{p_0}(\Omega)}}{\|f_\varepsilon w_1\|_{L_{p_1}(\Omega)}^\theta \|f_\varepsilon w_2\|_{L_{p_2}(\Omega)}^{1-\theta}} = C_0 C_1^{-\theta} C_2^{-(1-\theta)} \varepsilon^{-\frac{1}{q}}.$$

Passing to the limit as $\varepsilon \rightarrow 0^+$, we see that for any $A > 0$ inequality (6) cannot be valid, and we have got a contradiction so that we also have $d_1 \neq d_2$. The proof is complete.

4. Concluding remarks and results

Our proof of Theorem 2 shows that the multiplicative inequality (6) can in fact be replaced by some corresponding additive ones. This fact is described in the following remark:

REMARK 2. We note that the inequality

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq A_* \|f w_1\|_{L_{p_1}(\Omega)}^{\theta} \|f w_2\|_{L_{p_2}(\Omega)}^{1-\theta} \quad (33)$$

where A_* is defined by (10), the inequality

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \left(\theta^{\theta} (1-\theta)^{1-\theta} \right)^{1/p_0} A_* \left(\|f w_1\|_{L_{p_1}(\Omega)}^{p_0} + \|f w_2\|_{L_{p_2}(\Omega)}^{p_0} \right)^{1/p_0} \quad (34)$$

and the inequality

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \left((\varepsilon^{d_1-d_0} B_{1,*} \|f w_1\|_{L_{p_1}(\Omega)})^{p_0} + (\varepsilon^{d_2-d_0} B_{2,*} \|f w_2\|_{L_{p_2}(\Omega)})^{p_0} \right)^{1/p_0} \quad (35)$$

for arbitrary $\varepsilon > 0$, where

$$B_{1,*} = \left(\theta^{\frac{\theta}{p_1}} (1-\theta)^{-\frac{1-\theta}{p_2}} A_*^q \right)^{1/r_1},$$

$$B_{2,*} = \left(\theta^{\frac{\theta}{p_1}} (1-\theta)^{-\frac{1-\theta}{p_2}} A_*^q \right)^{1/r_2}$$

are equivalent. Indeed, taking into consideration (29), (31) and (32), the implication (35) \implies (33) was established in the proof of Theorem 2. Moreover, by (25) it follows that (33) implies (34). Finally, applying (34) to $f\left(\frac{x}{\delta}\right)$, $\delta > 0$, using the relation

$$\left\| f\left(\frac{x}{\delta}\right) w_i(x) \right\|_{L_{p_i}(\Omega)} = \delta^{d_i} \|f w_i\|_{L_{p_i}(\Omega)}$$

and choosing appropriate δ one gets (35).

Moreover, similar argument shows that (33) is also equivalent to

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \theta^{\theta} (1-\theta)^{1-\theta} A_* (\|f w_1\|_{L_{p_1}(\Omega)} + \|f w_2\|_{L_{p_2}(\Omega)}) \quad (36)$$

and to

$$\|f w_0\|_{L_{p_0}(\Omega)} \leq \left(\theta^{\theta} (1-\theta)^{1-\theta} \right)^{1/p_0'} (\varepsilon^{d_1-d_0} B_{1,*} \|f w_1\|_{L_{p_1}(\Omega)} + \varepsilon^{d_2-d_0} B_{2,*} \|f w_2\|_{L_{p_2}(\Omega)}) . \quad (37)$$

Thus all the inequalities (33)–(37) are sharp.

COROLLARY 1. Let Ω be defined by (4), $\alpha_i \in \mathbb{R}$, and let (7) and (9) be satisfied. Then, for each function f which is measurable on Ω and satisfying $\|f(x)|x|^{\alpha_i}\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$, it yields that

$$\|f(x)|x|^{\alpha_0}\|_{L_{p_0}(\Omega)} \leq C_1 \|f(x)|x|^{\alpha_1}\|_{L_{p_1}(\Omega)}^{\theta} \|f(x)|x|^{\alpha_2}\|_{L_{p_2}(\Omega)}^{1-\theta}, \quad (38)$$

where

$$C_1 = \theta^{-\frac{\theta}{p_1}} (1-\theta)^{-\frac{1-\theta}{p_2}} \left(\frac{B(\theta \frac{q}{p_1}, (1-\theta) \frac{q}{p_2})}{|d_1 - d_2|(\theta p_2 + (1-\theta)p_1)} \right)^{1/q} (\text{meas}_{n-1} S)^{1/q}.$$

The constant C_1 is the best possible. Moreover, equality in (38) holds if and only if, for some $B \geq 0$ and $\gamma > 0$,

$$|f(x)| = B (\varphi(\gamma|x|)|x|^{\alpha_0 p_0 - \alpha_1 p_1})^{\frac{1}{p_1 - p_0}},$$

where the function φ is defined by (30).

REMARK 3. Let Ff denote the Fourier transform of a function f on \mathbb{R}^n , i.e.,

$$Ff(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

We apply Corollary 1 with $p_0 = 1$, $p_1 = p_2 = 2$, $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = l$, where l is an integer. Then $d_0 = n$, $d_1 = \frac{n}{2}$, $d_2 = l + \frac{n}{2}$, $\theta = 1 - \frac{n}{2l}$ and $q = 2$, and, also by using Parseval's relation, we find that

$$\begin{aligned} \|f\|_{L_\infty(\mathbb{R}^n)} &= \|F^{-1}Ff\|_{L_\infty(\mathbb{R}^n)} \\ &= (2\pi)^{-n/2} \left\| \int_{\mathbb{R}^n} e^{ix \cdot \xi} (Ff)(\xi) d\xi \right\|_{L_\infty(\mathbb{R}^n)} \\ &\leq (2\pi)^{-n/2} \|Ff\|_{L_1(\mathbb{R}^n)} \\ &\leq C \|Ff\|_{L_2(\mathbb{R}^n)}^{1-\frac{n}{2l}} \|\xi|^l (Ff)(\xi)\|_{L_2(\mathbb{R}^n)}^{\frac{n}{2l}} \\ &= C \|Ff\|_{L_2(\mathbb{R}^n)}^{1-\frac{n}{2l}} \|F(\nabla^l f)\|_{L_2(\mathbb{R}^n)}^{\frac{n}{2l}} \\ &= C \|f\|_{L_2(\mathbb{R}^n)}^{1-\frac{n}{2l}} \|\nabla^l f\|_{L_2(\mathbb{R}^n)}^{\frac{n}{2l}}, \end{aligned}$$

where

$$C = \left((2\pi)^{-n/2} \left(1 - \frac{n}{2l}\right)^{-(1-\frac{n}{2l})} \left(\frac{n}{2l}\right)^{-\frac{n}{2l}} \frac{B(1-\frac{n}{2l}, \frac{n}{2l})}{2l} \sigma_n \right)^{1/2}$$

and σ_n is the surface area of the unit sphere in \mathbb{R}^n .

Here $\nabla^l f$ is the weak gradient of order l of the function f :

$$\nabla^l f = \left(\frac{\partial^l f}{\partial x_{i_1} \dots \partial x_{i_l}} \right)_{i_1, \dots, i_l=1}^n.$$

Moreover, since $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ and $B(1 - \frac{n}{2l}, \frac{n}{2l}) = \Gamma(1 - \frac{n}{2l})\Gamma(\frac{n}{2l}) = \frac{\pi}{\sin \frac{n\pi}{2l}}$ we have

$$C = \left(\frac{2^{-n}\pi^{1-n/2}}{\Gamma(\frac{n}{2})\sin \frac{n\pi}{2l}} \left(1 - \frac{n}{2l}\right)^{-(1-\frac{n}{2l})} \left(\frac{n}{2l}\right)^{-\frac{n}{2l}} \right)^{1/2}. \quad (39)$$

Thus Corollary 1 implies the following precise version of Sobolev's inequality: If $l > n/2$, then

$$\|f\|_{L_\infty(\mathbb{R}^n)} \leq C \|f\|_{L_2(\mathbb{R}^n)}^{1-n/2l} \|\nabla^l f\|_{L_2(\mathbb{R}^n)}^{n/2l}, \quad (40)$$

where C is defined by (39).

Moreover, the inequality (40) is sharp and equality occurs if and only if $f = Ff(x)$, where

$$Ff(\xi) = B \frac{1}{b + |\xi|^{2n}}, \quad \xi \in \mathbb{R}^n,$$

for some $B \in \mathbb{C}$ and $b > 0$.

In particular, for the case $n = l$ (40) has the form

$$\|f\|_{L_\infty(\mathbb{R}^n)} \leq \left(\frac{2^{-n}\pi^{1-n/2}}{\Gamma(\frac{n}{2} + 1)} \right)^{\frac{1}{2}} \|f\|_{L_2(\mathbb{R}^n)}^{1/2} \|\nabla^n f\|_{L_2(\mathbb{R}^n)}^{1/2}.$$

REMARK 4. Inequality (6) is a multidimensional generalization of both of the inequalities (2) and (3): In fact, let $p_0 = 1$ and $\alpha_0 = 0$. If $n = 1$ and $\Omega = [0, \infty)$, then $\text{meas}_0 S = 1$ and we get (2). If $n = 1$ and $\Omega = (-\infty, \infty)$, then $\text{meas}_0 S = 2$ and we get (3).

REMARK 5. If in Theorem 2, $p_1 = p_2 = p$, then the optimal constant A_* has the following forms:

$$\begin{aligned} A_* &= \left(\theta^{-\theta} (1 - \theta)^{-(1-\theta)} \right)^{1/p} \left(\frac{B(\theta \frac{r}{p}, (1 - \theta) \frac{r}{p})}{|d_1 - d_2|p} \right)^{1/r} \left\| \frac{w_0}{w_1^\theta w_2^{1-\theta}} \right\|_{L_r(S)} \\ &= \left(\theta^{-\theta} (1 - \theta)^{-(1-\theta)} \right)^{1/p} \left\| \frac{w_0}{(w_1^p + w_2^p)^{1/p}} \right\|_{L_r(\Omega)}, \end{aligned}$$

where

$$\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p}.$$

In fact, the first equality is just (10) for $p_1 = p_2 = p$, while the second one follows from (29) since in this case (see (13))

$$k = \frac{w_1^p}{w_1^p + w_2^p}.$$

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