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INEQUALITIES RELATED TO ISOTONICITY OF PROJECTION AND ANTIPROJECTION OPERATORS

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Abstract. A sharp inequality named "the property of four elements" has recently been proved and studied in [6] and [12]. One particular reason for this is that the inequality is closely related to the isotonicity of the projection operator onto a closed convex set in an ordered Hilbert space.

In this paper we prove and study a dual reversed sharp inequality. Moreover, we introduce the concept of antiprojection operator onto a compact non-empty set of a Hilbert space and prove that our new inequality is closely related to the isotonicity of such an operator. Moreover, we prove that both of these inequalities hold also in the reversed direction but of course with other constants.

1. Introduction

The aim of this paper is the study of some inequalities related to the isotonicity of the metric projection operator P_D onto a closed convex subset D of a Banach space $(E, \| \ \|)$. The Banach space E is supposed to satisfy some special properties. It is well known that the metric projection operator P_D is an important tool in Numerical Analysis [1], [3], [18], in Optimization [7], [8], [14], [15], in the study of Complementarity Problems [6], [9], [10], [16], [17], in the study of Variational Inequalities [16], [17], [18] and certainly it has many applications in Functional Analysis [19]. Several authors have studied the metric projection operator from several points of view (cf. the references in [6]). After 1986, the metric projection operator has been considered from the point of view of isotonicity, with respect to an ordering compatible with the vectorial structure on Hilbert spaces, on Banach spaces and on modular spaces [2], [6], [7], [8], [9], [10], [11], [12], [13]. Related to the isotonicity a special inequality was introduced in [6], named "*the property of four elements*", shortly (PFE). This paper now, is related to the study of some inequalities close to *the property of four elements.*

Let $(E, \| \ \|)$ be a Banach space ordered by a closed pointed convex cone $\mathbf{K} \subset E$. Suppose that E with respect to the ordering defined by $\mathbf{K}$ (that is $x \leqslant y \iff y - x \in \mathbf{K}$) is a vector lattice. Denote $x \wedge y = \inf(x, y)$ and $x \vee y = \sup(x, y)$ for any $x, y \in E$.

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We say that E satisfies the property $(PFE)_p$; $1 \leqslant p < \infty$, of four elements if for every $x_1, x_2, x_3, x_4 \in E$ such that $x_1 \geqslant x_3$ we have

$$\|x_1 - x_2\|^p + \|x_3 - x_4\|^p \geqslant \|x_1 - x_2 \vee x_4\|^p + \|x_3 - x_2 \wedge x_4\|^p.$$

This notion for the case when E is a Hilbert space and $p = 2$ was introduced and motivated by Isac [6] and in this generalized form by Isac–Persson [12]. Recently, the property $(PFE)_p$ was considered for modular spaces by Isac–Lewicki [13]. For example it was proved in [12] that the space $L_p = L_p(\Omega, \mu)$ satisfies $(PFE)_p$. If some conditions are satisfied we have on some modular spaces a similar property [13]. The starting point of this investigation was to prove a new sharp inequality (see Theorem 2.1) of independent interest.

In this paper we prove a reversed counterpart of this inequality (see Theorem 2.2). This inequality motivated us to introduce the notion of (upper) property of four elements, $(PFE)^p$, $1 \leqslant p < \infty$; E satisfies $(PFE)^p$ if, for every $x_1, x_2, x_3, x_4 \in E$ such that $x_1 \geqslant x_3$ we have

$$\|x_1 - x_2\|^p + \|x_3 - x_4\|^p \leqslant \|x_1 - x_2 \wedge x_4\|^p + \|x_3 - x_2 \vee x_4\|^p.$$

In particular, our results show that the space L_p satisfies also $(PFE)^p$. The property $(PFE)^p$ will be studied in Hilbert spaces with $p = 2$. In paper [12] the property of four elements for a Lyapunov functional V, $(PFE)_V$ was introduced. This notion is closely related with the isotonicity of the projection operator according to the following theorem:

THEOREM A. [12] *Let* $(E, \|\ \|, \mathbf{K})$ *be an ordered Banach space which is uniformly convex and uniformly smooth. If the cone* $\mathbf{K}$ *satisfies the property* $(PFE)_V$, *then for every latticially closed, convex set* $D \subset E$, *the projection operator* P_D^V *is isotone with respect to the order defined by* $\mathbf{K}$.

Guided by this result and the inequalities proved in *Section 2* we introduce the notion of lower (resp. upper) property of four elements (denoted respectively by $(LPFE)$ and $(UPFE)$) in a general Banach space. Similar properties are also introduced in *Section 6* for Lyapunov functionals. The property $(UPFE)$ is related with antiisotonicity (isotonicity in the reversed direction) of some corresponding projection operator, named the *antiprojection operator*. This operator can not be studied for unbounded sets as in Theorem A and we get problems with the uniqueness. In *Section 5* we prove a variant of Theorem A for the antiprojection operator. Finally, in *Section 6* we consider the properties $(LPFE)$ and $(UPFE)$ in more general spaces and two open problems are defined.

2. Some inequalities related with the properties $(PFE)_p$ and $(PFE)^p$

Let F denote a non-empty set and let $\mathscr{L}$ denote an additive set of real valued functions defined on F (i.e., if $f, g \in \mathscr{L}$, then $f + g \in \mathscr{L}$). We also consider an isotone additive functional $A : \mathscr{L} \to \mathbf{R}$, i.e. A satisfies the following properties:

i) $A(f+g) = A(f) + A(g)$, for all $f, g \in \mathscr{L}$,

ii) $f, g \in \mathscr{L}$ and $f(t) \leqslant g(t)$ for all $t \in F$ imply $A(f) \leqslant A(g)$.

For $0 < p < \infty$ we say that $f \in \mathscr{A}_p$ if $A_p(f) := \Big(A(|f|^p)\Big)^{1/p} < \infty$. First we recall the following result:

THEOREM 2.1. [12] *Let A be an isotone additive functional and let $1 \leqslant p < \infty$. If $f_1, f_2, f_3, f_4 \in \mathscr{A}_p$ and $f_1 \geqslant f_3$, then*

$$\begin{aligned} &A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4) \\ &\qquad \geqslant A_p^p(f_1 - f_2 \vee f_4) + A_p^p(f_3 - f_2 \wedge f_4) + 2A_p^p((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0) \end{aligned} \tag{1}$$

If $0 < p \leqslant 1$, then the inequality (1) holds in the reversed direction.

We shall prove the following reversed counter part of this theorem.

THEOREM 2.2. *Let A be an isotone additive functional and let $1 \leqslant p < \infty$. If $f_1, f_2, f_3, f_4 \in \mathscr{A}_p$ and $f_1 \geqslant f_3$, then*

$$\begin{aligned} &A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4) \\ &\qquad \leqslant A_p^p(f_1 - f_2 \wedge f_4) + A_p^p(f_3 - f_2 \vee f_4) - 2A_p^p((f_1 \wedge f_2 - f_3 \vee f_4) \vee 0) \end{aligned} \tag{2}$$

If $0 < p \leqslant 1$, then the inequality (2) holds in the reversed direction.

The proof of THEOREM 2.2 is based on the following corresponding numerical lemma:

LEMMA 2.3. *Let a_1, a_2, a_3, a_4 be real numbers such that $a_1 \geqslant a_3$. If $p \geqslant 1$, then*

$$\begin{aligned} |a_1 - a_2|^p + |a_3 - a_4|^p \leqslant\ & |a_1 - a_2 \wedge a_4|^p + |a_3 - a_2 \vee a_4|^p \\ & -2|(a_1 \wedge a_2 - a_3 \vee a_4) \vee 0|^p. \end{aligned} \tag{3}$$

Proof. First we remark that if $a_2 \leqslant a_4$, then the third term on the right hand side of (3) is equal to zero so the inequality (3) reduces to an equality. Thus, we may without loss of generality assume that $a_2 \geqslant a_4$. Let $p \geqslant 1$. We need to consider the following six cases:

i) $a_3 \leqslant a_1 \leqslant a_4 \leqslant a_2$. Put $a = a_1 - a_3$, $b = a_4 - a_1$ and $c = a_2 - a_4$ and consider the function $h(a) = (a+b+c)^p - (a+b)^p$. We note that h is nondecreasing so that, in particular, $h(a) \geqslant h(0)$, i.e.

$$(a+b)^p + (b+c)^p \leqslant b^p + (a+b+c)^p. \tag{4}$$

Therefore

$$\begin{aligned} |a_1 - a_2|^p + |a_3 - a_4|^p &= (b+c)^p + (a+b)^p \\ &\leqslant b^p + (a+b+c)^p \\ &= |a_1 - a_4|^p + |a_3 - a_2|^p, \end{aligned}$$

and we have proved that (3) holds.

ii) $a_4 \leqslant a_2 \leqslant a_3 \leqslant a_1$. Put $a = a_2 - a_4$, $b = a_3 - a_2$ and $c = a_1 - a_3$. In this case the third term on the right hand side in (3) is again equal to zero. We have

$$|a_1 - a_2|^p + |a_3 - a_4|^p = (b+c)^p + (a+b)^p,$$
$$|a_1 - a_4|^p + |a_3 - a_2|^p = (a+b+c)^p + b^p$$

and the proof follows by using (4).

iii) $a_3 \leqslant a_4 \leqslant a_1 \leqslant a_2$. Put $a = a_4 - a_3$, $b = a_1 - a_4$ and $c = a_2 - a_1$. In this case we have

$$|(a_1 \wedge a_2 - a_3 \vee a_4) \vee 0|^p = |a_1 - a_4|^p = b^p,$$
$$|a_1 - a_2|^p + |a_3 - a_4|^p = c^p + a^p,$$
$$|a_1 - a_2 \wedge a_4|^p + |a_3 - a_2 \vee a_4|^p = b^p + (a+b+c)^p$$

and the proof follows by using the inequality $a^p + b^p + c^p \leqslant (a+b+c)^p$, where a, b, c are nonnegative real numbers.

iv) $a_4 \leqslant a_3 \leqslant a_2 \leqslant a_1$. The proof follows by arguing as in part (*iii*) taking $a = a_3 - a_4$, $b = a_2 - a_3$, $c = a_1 - a_2$ and using the inequality $a^p + b^p + c^p \leqslant (a+b+c)^p$.

Also the proofs of the remaining cases *v*) $a_4 \leqslant a_3 \leqslant a_1 \leqslant a_2$ and *vi*) $a_3 \leqslant a_4 \leqslant a_2 \leqslant a_1$, are exactly similar, so we omit the details. Finally we note that all inequalities used in the proof presented above hold in the reversed direction when $0 < p \leqslant 1$ and the proof follows similarly in this case. □

Proof of Theorem 2.2. Apply the functional A to

$$f(t) = |f_1(t) - f_2(t)|^p + |f_3(t) - f_4(t)|^p,$$

and

$$g(t) = |f_1(t) - f_2(t) \wedge f_4(t)|^p + |f_3(t) - f_2(t) \vee f_4(t)|^p - 2|(f_1(t) \wedge f_2(t) - f_3(t) \vee f_4(t)) \vee 0|^p$$

and use Lemma 2.3 together with the isotonicity and additivity properties of A (note that $A(|x|^p) = A_p^p(x)$). □

COROLLARY 2.4. *If* $f_1, f_2, f_3, f_4 \in L_p(\Omega, \mu)$, $1 < p < \infty$, *and* $f_1(x) \geqslant f_3(x)$ *a.e., then*

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \geqslant \|f_1 - f_2 \vee f_4\|_p^p + \|f_3 - f_2 \wedge f_4\|_p^p + 2\|(f_4 \wedge f_1 - f_3 \vee f_2) \vee 0\|_p^p, \qquad (\theta)_p$$

and

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \leqslant \|f_1 - f_2 \wedge f_4\|_p^p + \|f_3 - f_2 \vee f_4\|_p^p - 2\|(f_1 \wedge f_2 - f_3 \vee f_4) \vee 0\|_p^p. \qquad (\theta)^p$$

For the case $0 < p < 1$ *both inequalities* $(\theta)_p$ *and* $(\theta)^p$ *hold in the reversed direction and for the case* $p = 1$ *we have equality in both cases.*

Proof. Apply Theorems 2.1 and 2.2 with $A(f) = \int_\Omega f\, d\mu$.

Let $p > 0$ be a real number. Corollary 2.4 suggests us the following definition:

DEFINITION 1. We say that an ordered Banach space $(E, \| \ \|, \leqslant)$ which is a vector lattice satisfies the lower property of four elements of order p (denoted by $(LPFE)(p)$) if, for every $x_1, x_2, x_3, x_4 \in E$ such that $x_1 \geqslant x_3$ we have

$$\|x_1 - x_2\|^p + \|x_3 - x_4\|^p \geqslant \|x_1 - x_2 \vee x_4\|^p + \|x_3 - x_2 \wedge x_4\|^p.$$

DEFINITION 2. We say that an ordered Banach space $(E, \| \ \|, \leqslant)$ which is a vector lattice satisfies the upper property of four elements of order p (denoted by $(UPFE)(p)$) if, for every $x_1, x_2, x_3, x_4 \in E$ such that $x_1 \geqslant x_3$ we have

$$\|x_1 - x_2\|^p + \|x_3 - x_4\|^p \leqslant \|x_1 - x_2 \wedge x_4\|^p + \|x_3 - x_2 \vee x_4\|^p.$$

REMARK 2.5. By Corollary 2.4 we obtain immediately that for any $1 \leqslant p < \infty$ the space $L_p = L_p(\Omega)$, has both the property $(LPFE)(p)$ and the property $(UPFE)(p)$.

In [6] it is proved that in an ordered Hilbert space which is a vector lattice the property $(LPFE)(2)$ is satisfied if and only if the space is Hilbert lattice. Recently, in [13] it is showed that in some modular spaces the property $(LPFE)$ is also satisfied.

3. Digression from theorems 2.1 and 2.2

In this section we shall show that the inequalities in Theorem 2.1 and 2.2 can be reversed up to an absolute constant.

THEOREM 3.1. *Let $A : \mathscr{L} \to \mathbf{R}$ be an isotone linear functional and let $1 \leqslant p < \infty$. If $f_1, f_2, f_3, f_4 \in \mathscr{A}_p$, and $f_1 \geqslant f_3$, then*

$$\begin{aligned} A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4) \leqslant 3^{p-1}\Big\{ & A_p^p(f_1 - f_2 \vee f_4) \\ & + A_p^p(f_3 - f_2 \wedge f_4) + C_p A_p^p((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0)\Big\}, \end{aligned} \tag{5}$$

where $C_p = \max(1 + 3^{1-p}, 2^p \cdot 3^{1-p})$. If $0 < p \leqslant 1$, then (5) holds in the reversed direction with $C_p = \min(1 + 3^{1-p}, 2^p \cdot 3^{1-p})$.

If $T_1(f_1, f_2, f_3, f_4)$ and $T_2(f_1, f_2, f_3, f_4)$ are two real valued expressions defined on $\mathscr{A}_p$ we say that $T_1 \approx T_2$ if and only if there exist two positive constants λ_1 and λ_2 such that

$$\lambda_1 T_2(f_1, f_2, f_3, f_4) \leqslant T_1(f_1, f_2, f_3, f_4) \leqslant \lambda_2 T_2(f_1, f_2, f_3, f_4).$$

REMARK 3.2. We note that in (5), $C_p \leqslant 2$ for all $p \geqslant 1$ and $C_p \geqslant 2$ for all $0 < p < 1$. Therefore Theorems 2.1 and 3.1 imply that the following equivalence holds:

$$\begin{aligned} A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4) \approx {} & A_p^p(f_1 - f_2 \vee f_4) \\ & + A_p^p(f_3 - f_2 \wedge f_4) + 2A_p^p((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0) \end{aligned}$$

for all p, $0 < p < \infty$, and with the embedding constants 1 and 3^{p-1} for all cases.

The proof of Theorem 3.1 follows by arguing as in the proof of Theorems 2.1–2.2 and using the following corresponding numerical lemma.

LEMMA 3.3. *Let a_1, a_2, a_3 and a_4 be real numbers such that $a_1 \geqslant a_3$. If $p \geqslant 1$, then*

$$|a_1 - a_2|^p + |a_3 - a_4|^p \leqslant 3^{p-1}\Big\{|a_1 - a_2 \vee a_4|^p + |a_3 - a_2 \wedge a_4|^p + C_p|(a_4 \wedge a_1 - a_3 \vee a_2) \vee 0|^p\Big\}. \quad (6)$$

If $0 < p \leqslant 1$, then the inequality holds in the reversed direction. (C_p is the constant defined in Theorem 3.1.)

Proof. If $a_2 \geqslant a_4$, then the third term on the right hand side of (6) is equal to zero and (6) reduces to an equality. Hence, we may without loss of generality assume that $a_2 \leqslant a_4$. Let $p \geqslant 1$. We need to consider the following six cases:

(i) $a_3 \leqslant a_1 \leqslant a_2 \leqslant a_4$. Put $a = a_1 - a_3$, $b = a_2 - a_1$ and $c = a_4 - a_2$. Then, according to elementary inequalities, we have

$$\begin{aligned} |a_1 - a_2|^p + |a_3 - a_4|^p &= b^p + (a+b+c)^p \\ &\leqslant b^p + 3^{p-1}(a^p + b^p + c^p) \\ &\leqslant 3^{p-1}(a^p + c^p + 2b^p) \leqslant 3^{p-1}\big[(a+b)^p + (b+c)^p\big] \\ &= 3^{p-1}\big(|a_1 - a_4|^p + |a_3 - a_2|^p\big) \end{aligned}$$

and the inequality is proved.

(ii) $a_2 \leqslant a_4 \leqslant a_3 \leqslant a_1$. Put $a = a_4 - a_2$, $b = a_3 - a_4$ and $c = a_1 - a_3$ and argue exactly as in the proof of case (i).

(iii) $a_3 \leqslant a_2 \leqslant a_1 \leqslant a_4$. Put $a = a_2 - a_3$, $b = a_1 - a_2$ and $c = a_4 - a_1$. Then, by using again elementary inequalities:

$$\begin{aligned} |a_1 - a_2|^p + |a_3 - a_4|^p &= b^p + (a+b+c)^p \\ &\leqslant b^p + 3^{p-1}(a^p + b^p + c^p) \\ &\leqslant 3^{p-1}(a^p + c^p + C_p b^p) \\ &= 3^{p-1}\big(|a_1 - a_4|^p + |a_3 - a_2|^p + C_p|a_4 \wedge a_1 - a_3 \vee a_2|^p\big) \end{aligned}$$

and the proof of this case is also complete.

(iv) $a_2 \leqslant a_3 \leqslant a_4 \leqslant a_1$. Put $a = a_3 - a_2$, $b = a_4 - a_3$ and $c = a_1 - a_4$. Then the proof follows similarly as in part (iii).

(v) $a_2 \leqslant a_3 \leqslant a_1 \leqslant a_4$. Put $a = a_3 - a_2$, $b = a_1 - a_3$ and $c = a_4 - a_1$. Then

$$\begin{aligned} |a_1 - a_2|^p + |a_3 - a_4|^p &= (a+b)^p + (b+c)^p \\ &\leqslant 2^{p-1}(a^p + b^p) + 2^{p-1}(b^p + c^p) \\ &\leqslant 3^{p-1}(a^p + c^p + C_p b^p) \\ &= 3^{p-1}\big(|a_1 - a_4|^p + |a_3 - a_2|^p + C_p|a_4 \wedge a_1 - a_3 \vee a_2|^p\big) \end{aligned}$$

and the inequality is proved.

(vi) $a_3 \leqslant a_2 \leqslant a_4 \leqslant a_1$. The proof is completely similar as that of part (v).

Finally, we note that every inequality we have used, holds in the reversed direction when $0 < p < 1$ and the proof is complete. □

THEOREM 3.4. *Let* $A : \mathscr{L} \to \mathbf{R}$ *be an isotone linear functional and let* $1 \leqslant p < \infty$. *If* $f_1, f_2, f_3, f_4 \in \mathscr{A}_p$, *and* $f_1 \geqslant f_3$, *then*

$$A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4) \geqslant 3^{1-p}\Big\{A_p^p(f_1 - f_2 \wedge f_4) + A_p^p(f_3 - f_2 \vee f_4) - C_p A_p^p\big((f_1 \wedge f_2 - f_3 \vee f_4) \vee 0\big)\Big\}, \quad (7)$$

where $C_p = 1 + 3^{p-1}$. *If* $0 < p \leqslant 1$, *then (7) holds in the reversed direction.*

REMARK 3.5. We note that $C_p \geqslant 2$ for all $p \geqslant 1$ and $C_p \leqslant 2$ for all p, $0 < p \leqslant 1$. Therefore, Theorems 2.2 and 3.4 imply that the following equivalence holds:

$$A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4) \approx A_p^p(f_1 - f_2 \wedge f_4) + A_p^p(f_3 - f_2 \vee f_4) - 2A_p^p\big((f_1 \wedge f_2 - f_3 \vee f_4) \vee 0\big)$$

for all $f_1 \geqslant f_3$ and all p, $0 < p < \infty$, and with the embedding constants 1 and 3^{1-p} for all cases.

The proof of Theorem 3.4 follows by arguing as in the proofs of Theorems 2.1 and 2.2 and using the following corresponding numerical lemma:

LEMMA 3.6. *Let* a_1, a_2, a_3 *and* a_4 *be real numbers such that* $a_1 \geqslant a_3$. *If* $p \geqslant 1$, *then*

$$|a_1 - a_2|^p + |a_3 - a_4|^p \geqslant 3^{1-p}\Big\{|a_1 - a_2 \wedge a_4|^p + |a_3 - a_2 \vee a_4|^p - (1 + 3^{p-1})|(a_1 \wedge a_2 - a_3 \vee a_4) \vee 0|^p\Big\}. \quad (8)$$

If $0 < p \leqslant 1$, *then the inequality (8) holds in the reversed direction.*

Proof. We need to consider the same six cases as in the proof of Lemma 2.3. We omit the details. $\square$

COROLLARY 2.12. *If* $f_1, f_2, f_3, f_4 \in L_p(\Omega, \mu)$, $1 < p < \infty$, *and* $f_1(x) \geqslant f_3(x)$ *a.e., then*

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \geqslant \|f_1 - f_2 \vee f_4\|_p^p + \|f_3 - f_2 \wedge f_4\|_p^p + 2\|(f_4 \wedge f_1 - f_3 \vee f_2) \vee 0\|_p^p, \quad (9)$$

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \leqslant 3^{p-1}\Big\{\|f_1 - f_2 \vee f_4\|_p^p + \|f_3 - f_2 \wedge f_4\|_p^p + C_p\|(f_4 \wedge f_1 - f_3 \vee f_2) \vee 0\|_p^p\Big\}, \quad (10)$$

where $C_p = \max(1 + 3^{1-p}, 2^p \cdot 3^{1-p}) \leqslant 2$,

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \leqslant \|f_1 - f_2 \wedge f_4\|_p^p + \|f_3 - f_2 \vee f_4\|_p^p - 2\|(f_1 \wedge f_2 - f_3 \vee f_4) \vee 0\|_p^p, \quad (11)$$

and

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \geqslant 3^{1-p}\Big\{ \|f_1 - f_2 \wedge f_4\|_p^p + \|f_3 - f_2 \vee f_4\|_p^p - C_p\|(f_1 \wedge f_2 - f_3 \vee f_4) \vee 0\|_p^p \Big\}, \quad (12)$$

where $C_p = 1 + 3^{p-1} \geqslant 2$.

If $0 < p \leqslant 1$, *then all these inequalities* (9)–(12) *hold in the reversed direction with* C_p *in case* (10) *changed to* $C_p = \min(1 + 3^{1-p}, 2^p \cdot 3^{1-p})$.

4. The properties $(LPFE)$ and $(UPFE)$ in Hilbert spaces

It is natural to consider the properties $(LPFE)$ and $(UPFE)$ in Hilbert spaces with $p = 2$. Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space and $\mathbf{K} \subset H$ a closed pointed convex cone, that is (i) $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$; (ii) $\lambda\mathbf{K} \subseteq \mathbf{K}$ for all $\lambda \in \mathbf{R}_+$ and (iii) $\mathbf{K} \cap (-\mathbf{K}) = \{0\}$. Let $\leqslant$ be the ordering defined by $\mathbf{K}$ i.e., $x \leqslant y$ if and only if $y - x \in \mathbf{K}$. We say that H is a *vector lattice* if for every $x, y \in H$, there exists $x \vee y := \sup(x, y)$ and $x \wedge y := \inf(x, y)$. The dual cone of $\mathbf{K}$ is, by definition

$$\mathbf{K}^* = \{y \in H \mid \langle x, y\rangle \geqslant 0, \text{ for all } x \in \mathbf{K}\}.$$

If $(H, \langle\ ,\ \rangle, \mathbf{K})$ is a vector lattice, then in this case we can define $x^+ = 0 \vee x$; $x^- = 0 \vee (-x)$ and $|x| = x^+ + x^-$ for every $x \in H$. In an ordered Hilbert space which is a vector lattice the properties $(LPFE)(2)$ and $(UPFE)(2)$ are also well defined. Finally, we say that an ordered Hilbert space $(H, \langle\ ,\ \rangle, \mathbf{K})$ is a Hilbert lattice if and only if, the following properties are satisfied:

$i)$ H is a vector lattice,
$ii)$ $\|\,|x|\,\| = \|x\|$ for all $x \in H$,
$iii)$ $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$.

THEOREM 4.1. [6] *An ordered Hilbert space* $(H, \langle\ ,\ \rangle, \mathbf{K})$ *which is a vector lattice has the property* $(LPFE)(2)$ *if and only if,* $(H, \langle\ ,\ \rangle, \mathbf{K})$ *is a Hilbert lattice.*

About the property $(UPFE)(2)$ in ordered Hilbert spaces we have the following result:

THEOREM 4.2. *If* $(H, \langle\ ,\ \rangle, \mathbf{K})$ *is an ordered Hilbert space, which is a vector lattice and the cone* $\mathbf{K}$ *is sub-adjoint (i.e.,* $\mathbf{K} \subseteq \mathbf{K}^*$*), then the property* $(UPFE)(2)$ *is satisfied.*

Proof. We must show that the following property is true:

$$(UPFE)(2): \begin{cases} \text{for all } x_1, x_2, x_3, x_4 \in H \text{ such that } x_1 \geqslant x_3 \text{ we have} \\ \|x_1 - x_2\|^2 + \|x_3 - x_4\|^2 \leqslant \|x_1 - x_2 \wedge x_4\|^2 + \|x_3 - x_2 \vee x_4\|^2. \end{cases}$$

Since H is a vector lattice we have:

$$x_2 \vee x_4 - x_2 = x_4 - x_2 \wedge x_4.$$

Using the properties of an inner-product and the properties of the latticial operations $\vee$ and $\wedge$, we have:

$$\begin{aligned}
&\|x_1-x_2\|^2+\|x_3-x_4\|^2-\|x_1-x_2\wedge x_4\|^2-\|x_3-x_2\vee x_4\|^2\\
&\quad=\|(x_1-x_2\wedge x_4)+(x_2\wedge x_4-x_2)\|^2+\|(x_3-x_2\vee x_4)-(x_4-x_2\vee x_4)\|^2\\
&\qquad\quad-\|x_1-x_2\wedge x_4\|^2-\|x_3-x_2\vee x_4\|^2\\
&\quad=\|x_1-x_2\wedge x_4\|^2+\|x_2\wedge x_4-x_2\|^2\\
&\qquad+2\langle x_1-x_2\wedge x_4,x_2\wedge x_4-x_2\rangle+\|x_3-x_2\vee x_4\|^2+\|x_4-x_2\vee x_4\|^2\\
&\qquad-2\langle x_3-x_2\vee x_4,x_4-x_2\vee x_4\rangle-\|x_1-x_2\wedge x_4\|^2-\|x_3-x_2\vee x_4\|^2\\
&\quad=2\|x_2\wedge x_4-x_2\|^2+2\big[\langle x_1-x_2\wedge x_4,x_2\wedge x_4-x_2\rangle\\
&\qquad-\langle x_3-x_2\vee x_4,x_4-x_2\vee x_4\rangle\big]\\
&\quad=2\|x_2\wedge x_4-x_2\|^2+2\big[\langle x_1-x_2\wedge x_4-x_3+x_2\vee x_4,x_2\wedge x_4-x_2\rangle\big]\\
&\quad=2\|x_2\wedge x_4-x_2\|^2+2\big[\langle x_1-x_3,x_2\wedge x_4-x_2\rangle\\
&\qquad-\langle x_2\wedge x_4-x_2,x_2\wedge x_4-x_2\rangle-\langle x_2-x_2\vee x_4,x_2\wedge x_4-x_2\rangle\big]\\
&\quad=2\big[\langle x_1-x_3,x_2\wedge x_4-x_2\rangle-\langle x_2-x_2\wedge x_4,x_4-x_2\wedge x_4\rangle\big].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|x_1-x_2\|^2+\|x_3-x_4\|^2-\|x_1-x_2\wedge x_4\|^2-\|x_3-x_2\vee x_4\|^2\\
&\qquad=2\big[\langle x_1-x_3,x_2\wedge x_4-x_2\rangle-\langle x_2-x_2\wedge x_4,x_4-x_2\wedge x_4\rangle\big].
\end{aligned}$$

We observe that $\mathbf{K}\subseteq\mathbf{K}^*$ implies

$$\langle x_1-x_3,x_2\wedge x_4-x_2\rangle-\langle x_2-x_2\wedge x_4,x_4-x_2\wedge x_4\rangle\leqslant 0$$

and the theorem is proved. □

From Theorems 4.1 and 4.2 we have the following result:

COROLLARY 4.3. *In any Hilbert lattice, both the property* $(LPFE)(2)$ *and* $(UPFE)(2)$ *are satisfied.*

Proof. The Corollary is a consequence of Theorems 4.1 and 4.2, and of the fact that in any Hilbert lattice $(H,\langle\ ,\ \rangle,\mathbf{K})$ the cone $\mathbf{K}$ is self-adjoint, i.e. $\mathbf{K}\subseteq\mathbf{K}^*$. □

5. The properties $(LPFE)$ and $(UPFE)$. Relations with projections and antiprojections

In [6] and [12] it is proved that in any Hilbert lattice $(H,\langle\ ,\ \rangle,\mathbf{K})$ and in any $L_p(\Omega,\mu)$ space, with $1<p<\infty$ ordered by the natural ordering, the projection operator P_D onto a latticially closed convex set is an isotone operator. Moreover, this property of P_D is a consequence of the property $(LPFE)$. A similar result we have for more general spaces [6], [12], [13]. In this section we show that the property $(UPFE)$

is related to the antiisotonicity of the antiprojection operator. We consider the case of Hilbert spaces.

Let $(H,\langle\ ,\ \rangle,\mathbf{K})$ be an ordered Hilbert space which is a vector lattice and such that the cone $\mathbf{K}$ is sub-adjoint. Let $D\subset H$ be a compact non-empty subset. For every $x\in H$ an *antiprojection of* x onto D is an element $P_D^a(x)\in D$ (generally not unique) such that

$$\|x-P_D^a(x)\| = \sup_{y\in D}\|x-y\|.$$

Because D is supposed to be compact and the norm is continuous, by Weierstrass' Theorem we have that $P_D^a(x)$ exists.

THEOREM 5.1. *Let $(H,\langle\ ,\ \rangle,\mathbf{K})$ be an ordered Hilbert space which is a vector lattice and $\mathbf{K}$ is a sub-adjoint cone. Let $D\subset H$ be a non-empty latticially closed, compact subset and $x,y\in H$ such that $y\leqslant x$. If D is latticially closed and $P_D^a(x)$, $P_D^a(y)$ each is a singleton, then $P_D^a(x)\leqslant P_D^a(y)$.*

Proof. By Theorem 4.2 we have that the property $(UPFE)(2)$ is satisfied in H. Since D is latticially closed, then by the definition of P_D^a we have

$$\|x-P_D^a(x)\|\geqslant\|x-P_D^a(x)\wedge P_D^a(y)\|$$

and

$$\|y-P_D^a(y)\|\geqslant\|y-P_D^a(x)\vee P_D^a(y)\|.$$

Since the property $(UPFE)(2)$ is satisfied we have (using the fact that $y\leqslant x$)

$$\|x-P_D^a(x)\|^2+\|y-P_D^a(y)\|^2\leqslant\|x-P_D^a(x)\wedge P_D^a(y)\|^2+\|y-P_D^a(x)\vee P_D^a(y)\|^2$$

which implies

$$\begin{aligned}\|x-P_D^a(x)\|^2 &\geqslant \|x-P_D^a(x)\wedge P_D^a(y)\|^2\\ &\geqslant \|x-P_D^a(x)\|^2+\|y-P_D^a(y)\|^2-\|y-P_D^a(x)\vee P_D^a(y)\|^2\\ &\geqslant \|x-P_D^a(x)\|^2.\end{aligned}$$

Hence, we deduce $\|x-P_D^a(x)\| = \|x-P_D^a(x)\vee P_D^a(y)\|$, and because $P_D^a(x)$ is a singleton it follows $P_D^a(x)=P_D^a(x)\wedge P_D^a(y)$, that is, we have $P_D^a(x)\leqslant P_D^a(y)$ and the theorem is proved. □

REMARK 5.2. Following the proof of Therem 5.1 we can show that Theorem 5.1 is valid in each $L_p(\Omega,\mu)$ space with $1<p<\infty$.

6. On the properties $(LPFE)$ and $(UPFE)$ in other spaces

The properties $(LPFE)$ and $(UPFE)$ can be related to the projection and the antiprojection operators in more general vector spaces.

Let $\mathscr{L}$ be a vector space over $\mathbf{R}$ or $\mathbf{C}$. A function $\rho:\mathscr{L}\to[0,+\infty]$ is called a *modular* if and only if:

1. $\rho(x)=0$, if and only if $x=0$,

2. $\rho(\alpha x) = \rho(x)$ if $\alpha \in \mathbf{C}$, $|\alpha| = 1$,
3. $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ if $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$.

If a modular ρ is given on the vector space $\mathscr{L}$, the set

$$\mathscr{L}_\rho = \{x \in \mathscr{L} \mid \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$$

is called a *modular space*. $\mathscr{L}_\rho$ is a vector subspace of $\mathscr{L}$. If a modular space $\mathscr{L}_\rho$ satisfies some special properties and $D \subset \mathscr{L}_\rho$ is a closed convex subset, then the projection of an arbitrary element x over D, denoted by $P_\rho(x, D)$, is well defined. About this result the reader is referred to [13].

Suppose that $(\mathscr{L}_\rho, \leqslant)$ is an ordered modular space which is a vector lattice.

DEFINITION 3. We say that a latticially closed, closed convex set $D \subset \mathscr{L}_\rho$ *satisfies the property* $(LPFE)_\rho$ if, for every $x_1, x_3 \in \mathscr{L}_\rho$ such that $x_1 \geqslant x_3$ and every $x_2, x_4 \in D$, we have

$$\rho(x_1 - x_2) + \rho(x_3 - x_4) \geqslant \rho(x_1 - x_2 \vee x_4) + \rho(x_3 - x_2 \wedge x_4).$$

DEFINITION 4. We say that a latticially closed, closed convex set $D \subset \mathscr{L}_\rho$ *satisfies the property* $(UPFE)_\rho$ if, for every $x_1, x_3 \in \mathscr{L}_\rho$ such that $x_1 \geqslant x_3$ and every $x_2, x_4 \in D$, we have

$$\rho(x_1 - x_2) + \rho(x_3 - x_4) \leqslant \rho(x_1 - x_2 \wedge x_4) + \rho(x_3 - x_2 \vee x_4).$$

The property $(LPFE)_\rho$ implies the isotonicity of $P_\rho(\cdot, D)$ and it was recently studied in [13]. The property $(UPFE)_\rho$ was never studied, but certainly, when this property is satisfied using a similar proof as for Theorem 5.1 we can show that $(UPFE)_\rho$ implies the antiisotonicity of $P^a_\rho(\cdot, D)$, with D being a latticially closed, compact set ($P^a_\rho(\cdot, D)$ is the antiprojection defined by the modular ρ).

Let $(E, \|\ \ \|)$ be a uniformly convex and uniformly smooth Banach space and let E^* be the topological dual of E. Suppose that E is ordered by a closed pointed convex cone $\mathbf{K} \subset E$. Suppose that E^* is ordered by the dual cone $\mathbf{K}^*$ of $\mathbf{K}$, i.e.,

$$\mathbf{K}^* = \{y \in E^* \mid \langle y, x \rangle \geqslant 0 \text{ for all } x \in \mathbf{K}\}.$$

Denote the ordering defined on E by $\mathbf{K}$, by $\leqslant_{\mathbf{K}}$ and the ordering on E^* defined by $\mathbf{K}^*$, by $\leqslant_{\mathbf{K}^*}$.

As in [6], consider the Lyapunov function $U(\varphi, x) = \|\varphi\|^2_{E^*} - 2\langle \varphi, x \rangle + \|x\|^2_E$, defined for every $x \in E$ and $\varphi \in E^*$, where $\langle\ ,\ \rangle$ is the canonical bilinear form of the duality $\langle E, E^* \rangle$. By definition (see [6] and references), for every $\varphi \in E^*$, the projection of φ onto a closed convex set $D \subset E$ is the unique element $P^U_D(\varphi)$ such that

$$U(\varphi, P^U_D(\varphi)) = \inf_{x \in D} U(\varphi, x).$$

Suppose that $J : E \to E^*$ is a duality mapping. Consider the Lyapunov functional

$$V(J(x), \xi) = \|J(x)\|^2_{E^*} - 2\langle J(x), \xi \rangle + \|\xi\|^2_E$$

defined for every $x, \xi \in E$.

Following the definition given in [12] the projection of an element $x \in E$ onto a closed convex set $D \subset E$ is the element $P_D^V(x) \in D$ such that

$$V(J(x), P_D^V(x)) = \inf_{\xi \in D} V(J(x), \xi).$$

Now, we can introduce the following definitions. Suppose that E (resp. E^*) are vector lattices.

DEFINITION 5. We say that a latticially closed, closed convex set $D \subset E$ *satisfies the property* $(LPFE)_U$ (resp. $(UPFE)_U$) if, for every $\varphi_1, \varphi_3 \in E^*$ such that $\varphi_3 \leqslant_{\mathbf{K}^*} \varphi_1$, and every $x_2, x_4 \in D$, we have

$$U(\varphi_1, x_2) + U(\varphi_3, x_4) \geqslant U(\varphi_1, x_2 \vee x_4) + U(\varphi_3, x_2 \wedge x_4)$$

resp.

$$U(\varphi_1, x_2) + U(\varphi_3, x_4) \leqslant U(\varphi_1, x_2 \wedge x_4) + U(\varphi_3, x_2 \vee x_4).$$

DEFINITION 6. We say that a latticially closed, closed convex set $D \subset E$ *satisfies the property* $(LPFE)_V$ (resp. $(UPFE)_V$) if, for every $x_1, x_3 \in E$ such that $x_1 \geqslant x_3$, and every $x_2, x_4 \in D$, we have

$$V(J(x_1), x_2) + V(J(x_3), x_4) \geqslant V(J(x_1), x_2 \vee x_4) + V(J(x_3), x_2 \wedge x_4)$$

resp.

$$V(J(x_1), x_2) + V(J(x_3), x_4) \leqslant V(J(x_1), x_2 \wedge x_4) + V(J(x_3), x_2 \vee x_4).$$

In [6] it is revealed that the property $(LPFE)_U$ is related to the isotonicity of the projection operator P_D^U and in [12] it is showed that the property $(LPFE)_V$ is related to the isotonicity of the projection operator P_D^V. Certainly the properties $(UPFE)_U$ and $(UPFE)_V$ are related to the antiisotonicity of antiprojections.

REMARK 6.1. When $D = E$ and the property $(LPFE)$ (resp. $(UPFE)$) is satisfied we say that the space E satisfies property $(LPFE)$ (resp. $(UPFE)$).

OPEN PROBLEM 1. It is interesting to study the property $(UPFE)$ in modular spaces or in a general uniformly convex and uniformly smooth Banach space.

OPEN PROBLEM 2. It is interesting to give examples of Banach or modular spaces with the property that, the space does not satisfy properties $(LPFE)$ and $(UPFE)$ but some subsets satisfies these properties.

FINAL REMARK. The inequalities proved in this paper are obviously formally similar to inequalities which can be derived from well known extensions of the original Clarkson inequality [5]. However, these two types of inequalities are not comparable and the cases of equality are completely different. C.f. also Remark 3.3 in [12].

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