

AN UPPER BOUND FOR THE ZEROS OF THE CYLINDER FUNCTION $C_v(x)$

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(communicated by J. Pečarić)

Abstract. For large values of v ($v > 0$) the k -th positive zero of the cylinder function $C_v(x) = J_v(x) \cos \alpha - Y_v(x) \sin \alpha$, $0 \leq \alpha < \pi$, has the asymptotic expansion

$$j_{v\kappa} = v + \gamma_\kappa v^{1/3} + \frac{3}{10} \gamma_\kappa^2 v^{-1/3} + \mathcal{O}(v^{-1})$$

where $\kappa = k - \alpha/\pi$, $\gamma_\kappa = -a_\kappa 2^{-1/3}$ and a_κ is the k -th negative zero of the function $Ai(x) \cos \alpha + Bi(x) \sin \alpha$ and $Ai(x)$, $Bi(x)$ denote the Airy functions of the first and the second kind, respectively [1]. We prove that the sum of the first three terms of the asymptotic expansion gives an upper bound for $j_{v\kappa}$, provided $\gamma_\kappa \geq \sqrt[3]{35/4} = 2.0606427\dots$ or $\kappa \geq \kappa_0 = 1.13019788\dots = 2 - \alpha_0/\pi$ where α_0 is determined by the equation $\cos \alpha_0 Ai(-\sqrt[3]{35/4}) + \sin \alpha_0 Bi(-\sqrt[3]{35/4}) = 0$. This result covers the cases j_{v2}, j_{v3}, \dots and y_{v2}, y_{v3}, \dots , for all $v > 0$. The main tool used is the well-known Watson formula for $dj_{v\kappa}/dv$.

1. Introduction and preliminary results

Let $j_{v\kappa}$ denote the k -th positive zero of the cylinder function $C_v(x) = J_v(x) \cos \alpha - Y_v(x) \sin \alpha$, $0 \leq \alpha < \pi$ where $J_v(x)$ and $Y_v(x)$ are the Bessel functions of the first and the second kind, respectively, and $\kappa = k - \alpha/\pi$, (see [4]). In [3] we proved that $j_{v\kappa}$ has the asymptotic expansion

$$j_{v\kappa} = v + \gamma_\kappa v^{1/3} + \frac{3}{10} \gamma_\kappa^2 v^{-1/3} + \frac{5 - \gamma_\kappa^3}{350} v^{-1} - \frac{479\gamma_\kappa^4 + 20\gamma_\kappa}{63000} v^{-5/3} + \frac{20231\gamma_\kappa^5 - 27550\gamma_\kappa^2}{8085000} v^{-7/3} + \mathcal{O}(v^{-3}) \quad (1.1)$$

where $\gamma_\kappa = -a_\kappa \cdot 2^{-1/3}$ and a_κ is the k -th negative zero of $Ai(x) \cos \alpha + Bi(x) \sin \alpha$ and $Ai(x)$, $Bi(x)$ denote the Airy functions of the first and the second kind, respectively. Recently, L. Lorch and R. Uberti [6] have proved that the sum of the first three terms in

Mathematics subject classification (1991): 41A60, 33C45.

Key words and phrases: Cylinder functions, zeros, asymptotic expansions.

Work sponsored by Consiglio Nazionale delle Ricerche of Italy and by Ministero della Ricerca Scientifica e Tecnologica of Italy and partially by Hungarian Foundation for Scientific Research Grant T 016367.

(1.1) for $\kappa = 1, 2, 3$, gives an upper bound for j_{v1}, j_{v2}, j_{v3} at least for $0 < v \leq 10$. In [5] T. Lang and R. Wong dealt with the case $v > 10$ for $\kappa = 1, 2$.

In this paper we prove the following general result.

THEOREM. *Let $j_{v\kappa}$ be the k -th positive zero of the cylinder function $C_v(x)$ and let γ_κ be defined as above. Then for $\gamma_\kappa \geq \sqrt[3]{35/4} = 2.0606427\dots$ or $\kappa \geq \kappa_0 = 1.13019788\dots$ the following upper bound*

$$j_{v\kappa} < v + \gamma_\kappa v^{1/3} + \frac{3}{10} \gamma_\kappa^2 v^{-1/3}, \quad v > 0 \quad (1.2)$$

holds where $\kappa_0 = 2 - \alpha_0/\pi$, and $\cos \alpha_0 Ai(-\sqrt[3]{35/4}) + \sin \alpha_0 Bi(-\sqrt[3]{35/4}) = 0$.

REMARK. According to the asymptotic formula (1.1) we can not expect an extension of inequality (1.2) to all $\kappa > 0$ because the coefficient of v^{-1} in (1.1) is positive for $\gamma_\kappa < \sqrt[3]{5} = 1.709976\dots$. Thus, for example, we can not extend (1.2) to the first zero y_{v1} of $Y_v(x)$ for every $v > 0$ because in this case $\kappa = 1/2$ and $\gamma_{1/2} = 0.9315768\dots$, [1, p. 371; 9.5.15].

The proof of our Theorem is based on the well-known Watson formula [7, p. 508]

$$\frac{dj}{dv} = 2j \int_0^\infty K_0(2j \sinh t) e^{-2vt} dt \quad (1.3)$$

where $j = j_{v\kappa}$ and $K_0(x)$ denotes the modified Bessel function of the third kind.

The upper bound (1.2) can be thought of as complementing the lower bound

$$j_{v\kappa} > v + \gamma_\kappa \left(v + \frac{A_\kappa^3}{\gamma_\kappa^3} \right)^{1/3}, \quad v \geq 0, \quad \kappa \geq 1/2, \quad (1.4)$$

where $A_\kappa = 2\gamma_\kappa \sqrt{2\gamma_\kappa}/3$, (see Remark in [2, p. 185]).

As in [3] we introduce the functions

$$F_i(\vartheta) = \int_0^\infty K_0(u) u^i e^{-u \cos \vartheta} du, \quad i = 0, 1, 2, \dots \quad (1.5)$$

where

$$F_0(\vartheta) = \int_0^\infty K_0(u) e^{-u \cos \vartheta} du = \frac{\vartheta}{\sin \vartheta} \quad (1.6)$$

and

$$F_{i+1}(\vartheta) = \frac{d}{d\vartheta} \frac{F_i(\vartheta)}{\sin \vartheta}, \quad i = 0, 1, 2, \dots \quad (1.7)$$

In particular we have

$$F_2(\vartheta) = \frac{\vartheta + 2\vartheta \cos^2 \vartheta - 3 \sin \vartheta \cos \vartheta}{\sin^5 \vartheta} = \frac{4}{15} + \frac{6}{35} \vartheta^2 + \dots \quad (1.8)$$

Clearly, these functions are analytic in $|\vartheta| < \pi$.

2. Proof of the Theorem

We introduce the notations

$$X = X(v) = \gamma v^{-2/3} > 0, \quad \gamma = \gamma_\kappa. \quad (2.1)$$

Then (1.2) is equivalent to

$$j < v \left[1 + X + \frac{3}{10} X^2 \right], \quad v > 0, \quad j = j_{v\kappa}. \quad (2.2)$$

Let $\varphi(v)$ and $\psi(v) \in (0, \frac{\pi}{2})$ be defined by

$$\cos \varphi(v) = \frac{v}{j}, \quad \cos \psi(v) = \frac{1}{1 + X + \frac{3}{10} X^2} \quad \text{for } v > 0. \quad (2.3)$$

Concerning the function $\varphi(v)$ we have also the restriction $0 < \varphi(v) < \pi/2$ because in our case $j_{v\kappa} > j_{v1} > v > 0$, [7, p. 485]. So by these notations, the inequality in (2.2) is equivalent to

$$0 < \varphi(v) < \psi(v) < \frac{\pi}{2}. \quad (2.4)$$

By the asymptotic relation (1.1) the inequalities in (2.4) hold at least for sufficiently large v , because the coefficient of v^{-1} in (1.1) is negative. Hence the difference $\psi(v) - \varphi(v)$ is surely positive for sufficiently large v . We have to prove that this difference remains positive for all $v > 0$. Suppose the contrary and define

$$v^* = \inf \{ \bar{v} > 0, \psi(v) - \varphi(v) > 0, v > \bar{v} \}.$$

Then we have the (indirect) relation

$$\psi(v^*) - \varphi(v^*) = 0, \quad \psi'(v^*) - \varphi'(v^*) \geq 0 \quad (2.5)$$

where ' indicates the derivative with respect to v . We are going to show that (2.5) leads to contradiction. By (2.1), (2.3) we have

$$v \frac{d}{dv} \frac{1}{\cos \psi(v)} = v \left(1 + \frac{3}{5} X \right) \frac{dX}{dv} = -\frac{2}{3} X - \frac{2}{5} X^2. \quad (2.6)$$

On the other hand, by (1.3) and (2.3)

$$v \frac{d}{dv} \frac{1}{\cos \varphi(v)} = v \left(\frac{j}{v} \right)' = \frac{j'v - j}{v} = 2j \int_0^\infty K_0(2j \sinh t) e^{-2vt} dt - \frac{1}{\cos \varphi(v)}. \quad (2.7)$$

If we prove the inequality $\varphi'(v) > \psi'(v)$ under the restriction $\psi(v) = \varphi(v)$, then clearly we shall have contradiction with (2.5). Due to (2.6) and (2.7) the inequality $\varphi'(v) > \psi'(v)$ can be written as

$$j' = 2j \int_0^\infty K_0(2j \sinh t) e^{-2vt} dt > 1 + \frac{1}{3} X - \frac{1}{10} X^2 \quad (2.8)$$

under the restriction $j = v\left(1 + X + \frac{3}{10}X^2\right)$. By the substitution $u = 2j \sinh t$ in (2.8), we have

$$\begin{aligned} j' &= 2j \int_0^\infty K_0(2j \sinh t) e^{-2vt} dt = \int_0^\infty K_0(u) e^{-2v \sinh^{-1} \frac{u}{2j}} \frac{du}{\sqrt{1 + \frac{u^2}{4j^2}}} \\ &> \int_0^\infty K_0(u) e^{-\frac{v}{j}u} \frac{du}{\sqrt{1 + \frac{u^2}{4j^2}}}. \end{aligned}$$

Since $(1+z)^{-1/2} > 1 - z/2$, $z > 0$ and $v/j = \cos \varphi$, therefore

$$\begin{aligned} j' &> \int_0^\infty K_0(u) e^{-u \cos \varphi} \left(1 - \frac{u^2}{8j^2}\right) du = \int_0^\infty K_0(u) e^{-u \cos \varphi} du \\ &\quad - \frac{1}{8j^2} \int_0^\infty K_0(u) u^2 e^{-u \cos \varphi} du = F_0(\varphi) - \frac{1}{8j^2} F_2(\varphi), \end{aligned}$$

where the functions F_0 and F_2 have been defined in (1.6), (1.8), respectively. By (2.1) and (2.3) we find

$$\frac{1}{j^2} = \frac{v^2}{j^2} \frac{1}{v^2} = \cos^2 \varphi \cdot \frac{X^3}{\gamma_k^3}$$

hence instead of (2.8) it is sufficient to prove the inequality

$$F_0(\varphi) > 1 + \frac{1}{3}X - \frac{1}{10}X^2 + \frac{1}{8\gamma^3}X^3 \cos^2 \varphi \cdot F_2(\varphi) \quad (2.9)$$

under the condition that

$$\cos \varphi(v) = \frac{1}{1 + X + \frac{3}{10}X^2}. \quad (2.10)$$

We shall prove (2.9) in two steps:

$$F_2(\varphi)(\cos \varphi)^{4/3} \leq F_2(0) = \frac{4}{15}; \quad (2.11a)$$

$$F_0(\varphi) > 1 + \frac{1}{3}X - \frac{1}{10}X^2 + \frac{2}{525}X^3(\cos \varphi)^{2/3}. \quad (2.11b)$$

Clearly (2.11a) and (2.11b) imply inequality (2.8) for $\gamma^3 \geq 35/4$.

For the proof of (2.11a) we use (1.8), and we have to prove that

$$G(\varphi) = \frac{4}{15} \cdot \frac{\sin^5 \varphi}{1 + 2 \cos^2 \varphi} \cdot \cos^{-4/3} \varphi - \varphi + \frac{3 \sin \varphi \cos \varphi}{1 + 2 \cos^2 \varphi} > 0, \quad 0 < \varphi < \frac{\pi}{2}.$$

We get $G(0) = 0$, so it is sufficient to show that

$$G'(\varphi) = \frac{4 \sin^2 \varphi (4 + 31 \cos^2 \varphi + 10 \cos^4 \varphi - 45 \cos^{7/3} \varphi)}{45 \cos^{7/3} \varphi (1 + 2 \cos^2 \varphi)^2} > 0.$$

With the substitution $\cos \varphi = z^3$ the expression in parenthesis of the numerator becomes $(1 - z)(4 + 4z + 4z^2 + 4z^3 + 4z^4 + 4z^5 + 35z^6 - 10z^7 - 10z^8 - 10z^9 - 10z^{10} - 10z^{11})$.

Since $0 < z < 1$, then for $i < j$, $z^i > z^j$ and the new expression in parenthesis is larger than

$$4z^6 + 4z^6 + 4z^6 + 4z^6 + 4z^6 + 4z^6 + 35z^6 - 10z^6 - 10z^6 - 10z^6 - 10z^6 = (59 - 50)z^6 > 0$$

which proves (2.11a).

For the proof of (2.11b) we need the inequality

$$F_0(\varphi) = \frac{\varphi}{\sin \varphi} > A + B \cos \varphi + C \cos^2 \varphi + D \cos^3 \varphi + E \cos^4 \varphi, \quad 0 < \varphi < \frac{\pi}{2} \quad (2.12)$$

where

$$A = \frac{488}{315}, \quad B = -\frac{55}{63}, \quad C = \frac{16}{35}, \quad D = -\frac{10}{63}, \quad E = \frac{8}{315}.$$

To this end let us consider the function

$$\Phi(\varphi) = \varphi - \sin \varphi [A + B \cos \varphi + C \cos^2 \varphi + D \cos^3 \varphi + E \cos^4 \varphi].$$

Since $\Phi(0) = 0$ and $\frac{d}{d\varphi} \Phi(\varphi) = \frac{8}{63}(1 - \cos \varphi)^5 > 0$, this proves (2.12).

Now we define

$$H = 1 + X + \frac{3}{10}X^2 = \frac{1}{\cos \varphi}$$

so we have

$$X^2 = \frac{10}{3}H - \frac{10}{3} - \frac{10}{3}X$$

and

$$X^3 = \frac{10}{3}HX - \frac{100}{9}H + \frac{70}{9}X + \frac{100}{9}.$$

Hence the right-hand side of (2.11b) can be written as

$$\frac{4}{3} - \frac{1}{3}H + X \left[\frac{2}{3} + \frac{4}{315}H^{1/3} + \frac{4}{135}H^{-2/3} \right] + \frac{8}{189}H^{-2/3} - \frac{8}{189}H^{1/3}$$

where the coefficient of X is positive. Therefore we look for an upper bound of X in terms of H , as follows:

$$aH^{2/3} + bH^{1/3} + cH^{-1/3} + d > X, \quad (2.13)$$

where $a = \frac{19}{25}$, $b = \frac{51}{50}$, $c = -\frac{23}{50}$, $d = -\frac{33}{25}$. To show (2.13) we consider the function

$$\Psi(X) = aH^{2/3} + bH^{1/3} + cH^{-1/3} + d - X \quad (2.14)$$

where we recall that $H = H(X) = 1 + X + \frac{3}{10}X^2$. The coefficients a, b, c, d above have been chosen so that

$$\Psi(X) = \frac{79}{16200}X^4 + \mathcal{O}(X^5), \quad X \rightarrow 0.$$

Therefore $\Psi(X) > 0$ at least for small values of X .

Suppose now that the inequality $\Psi(X) > 0$ does not hold for all $X > 0$. Then let $X_1 > 0$ be the first positive zero of $\Psi(X)$: $\Psi(X_1) = 0$. Then clearly we have $\Psi'(X_1) = \frac{d}{dX}\Psi(X_1) \leq 0$. By direct calculations we get $\frac{d}{dX}H(X) = 1 + \frac{3}{5}X$, hence

$$\begin{aligned} \Psi'(X_1) &= \left(\frac{2}{3}aH^{-1/3} + \frac{1}{3}bH^{-2/3} - \frac{1}{3}cH^{-4/3} \right) \left(1 + \frac{3}{5}X_1 \right) - 1 \\ &= \left(\frac{2}{3}aH^{-1/3} + \frac{1}{3}bH^{-2/3} - \frac{1}{3}cH^{-4/3} \right) \left[1 + \frac{3}{5}(aH^{2/3} + bH^{1/3} + cH^{-1/3} + d) \right] - 1 \end{aligned}$$

where the last equality is a consequence of (2.14) taken at $X = X_1$.

Using the notation $z^3 = H$, we get

$$\Psi'(X_1) = \frac{(z-1)^3}{15z^5} \cdot \frac{8664z^3 + 5934z^2 + 3565z + 1587}{2500}$$

which is clearly positive because $z > 1$. This contradicts the existence of a zero of $\Psi(X)$, thus inequality (2.13) is proved.

To complete the proof of (2.11b), we have still to show the inequality

$$\begin{aligned} L &= A + \frac{B}{H} + \frac{C}{H^2} + \frac{D}{H^3} + \frac{E}{H^4} > \frac{4}{3} - \frac{1}{3}H + (aH^{2/3} + bH^{1/3} + cH^{-1/3} + d) \times \\ &\quad \times \left(\frac{2}{3} + \frac{4}{315}H^{1/3} + \frac{28}{945}H^{-2/3} \right) + \frac{8}{189} - \frac{8}{189}H^{1/3} = M. \end{aligned}$$

Using again the substitution $z^3 = H$, we get:

$$\begin{aligned} L - M &= \frac{(z-1)^4}{23625} [600 + 200z + 6000z^2 + 8250z^3 + 6000z^4 - 3900z^5 - 13800z^6 \\ &\quad - 16050z^7 - 3000z^8 + 12697z^9 + 18312z^{10} + 7647z^{11}]. \end{aligned}$$

Since $z > 1$, $z^i > z^j$ if $i > j$, hence the polynomial in the brackets has the lower bound

$$[-3900 - 13800 - 16050 - 3000 + 12697 + 18312 + 7647]z^8 = 1906z^8 > 0.$$

This completes the proof of inequality (2.11b) and of our Theorem as well.

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(Received May 29, 1997)

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