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OPERATOR FUNCTIONS IMPLYING GENERALIZED FURUTA INEQUALITY *

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(*communicated by F. Hansen*)

Abstract. As further extensions of the main result in [11], we show the following result.

Let $A \geqslant B \geqslant 0$ *with* $A > 0$. *For each* $t \in [0,1]$ *and* $p \geqslant t$, *the following* (i) *and* (ii) *hold for a fixed real number* q *and they are mutually equivalent:*

(i) *if* $q \geqslant 0$, *then*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ *and* $s \geqslant 1$ *such that* $(p-t)s \geqslant q-t$.

(ii) *if* $p \geqslant q$, *then*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $s \geqslant 1$ *and* $r \geqslant \max\{t, t-q\}$.

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem which asserts: $A \geqslant B \geqslant 0$ ensures $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$.

THEOREM F. (Furuta inequality) [6] *If* $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

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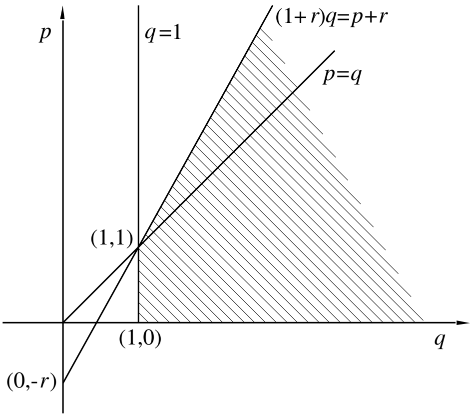
hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.



Figure

We remark that Theorem F yields the Löwner-Heinz theorem when we put $r = 0$. Alternative proofs of Theorem F are given in [3] and [12] and also an elementary one page proof in [7]. It is shown in [13] that the domain drawn for p, q and r in the Figure is the best possible one for Theorem F. We established the following Theorem A as extensions of Theorem F.

THEOREM A. [10] *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for each* $t \in [0,1]$ *and* $p \geqslant 1$,

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ *and* $s \geqslant 1$ *and* $F_{p,t}(A,A,r,s) \geqslant F_{p,t}(A,B,r,s)$, *that is, for each* $t \in [0,1]$ *and* $p \geqslant 1$,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for any $s \geqslant 1$ *and* r *such that* $r \geqslant t$.

Recently a nice mean theoretic proof of Theorem A is shown in [5]. Ando-Hiai [2] established excellent log majorization results and proved the useful inequality equivalent to the main log majorization theorem as follows; If $A \geqslant B \geqslant 0$ with $A > 0$, then

$$A^r \geqslant \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geqslant 1$ and $r \geqslant 1$. Theorem A interpolates the inequality stated above by Ando-Hiai and Theorem F itself and also extends results of [4][8] and [9].

We write $A \gg B$ if $\log A \geqslant \log B$ for invertible positive operator A and B which is called the chaotic order [4] and related results on chaotic order are discussed in [1] and [4].

Very recently the following results are obtained as an extension of Theorem A.

THEOREM B. [11] *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *For each* $t \in [0,1]$, $q \geqslant 0$ *and* $p \geqslant \max\{q,t\}$,

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ *and* $s \geqslant 1$.

In this paper, we show Theorem 1 by using Theorem F and we show Theorem 2 which is an extension of Theorem B and Corollary 3 by using Theorem 1.

2. Results

THEOREM 1. *Let* A *and* B *be positive invertible operators on a Hilbert space satisfying*

$$A \geqslant (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \text{ for fixed } \alpha_0 \geqslant 0 \text{ and } \beta_0 \geqslant 0 \text{ with } \alpha_0+\beta_0 > 0. \tag{2.0}$$

Then the following (i) *and* (ii) *hold and they are mutually equivalent:*

(i) *For any fixed* $\delta \geqslant -\beta_0$,

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}} \tag{2.1}$$

is decreasing for $\mu \geqslant 1$ *and* $\lambda \geqslant 1$ *such that* $\alpha_0\lambda \geqslant \delta$.

(ii) *For any fixed* $\delta \leqslant \alpha_0$,

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}} \tag{2.2}$$

is decreasing for $\lambda \geqslant 1$ *and* $\mu \geqslant 1$ *such that* $\beta_0\mu \geqslant -\delta$.

Applying Theorem 1, we obtain the following extension of Theorem B.

THEOREM 2. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *For each* $t \in [0,1]$ *and* $p \geqslant t$, *the following* (i) *and* (ii) *hold for a fixed real number* q *and they are mutually equivalent:*

(i) *if* $q \geqslant 0$, *then*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ *and* $s \geqslant 1$ *such that* $(p-t)s \geqslant q-t$.

(ii) *if* $p \geqslant q$, *then*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $s \geqslant 1$ *and* $r \geqslant \max\{t, t-q\}$.

Also Theorem 1 implies the following characterization of chaotic order.

COROLLARY 3. *The following assertions are mutually equivalent:*
(i) $A \gg B$ *(i.e.,* $\log A \geqslant \log B$ *).*
(ii) *For any fixed* $q \geqslant 0$,

$$F_q(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{q+r}{p+r}}A^{\frac{-r}{2}}$$

is decreasing for $p \geqslant q$ *and* $r \geqslant 0$.
(iii) *For any fixed* $q \leqslant 0$,

$$F_q(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{q+r}{p+r}}A^{\frac{-r}{2}}$$

is decreasing for $p \geqslant 0$ *and* $r \geqslant -q$.

The equivalence relation between (i) and (ii) is shown in [4, 9].

3. Proofs of results

We need the following lemmas to give proofs of the results in § **2**.

LEMMA 1. [10] *Let* $A > 0$ *and* B *be an invertible operator. For any real number* λ

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*.$$

LEMMA 2. *Let* A *and* B *be positive invertible operators on a Hilbert space satisfying*

$$A \geqslant (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \text{ for fixed } \alpha_0 \geqslant 0 \text{ and } \beta_0 \geqslant 0 \text{ with } \alpha_0+\beta_0 > 0. \tag{3.0}$$

Then the following inequality holds

$$A^{\mu} \geqslant (A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} \text{ for } \lambda \geqslant 1 \text{ and } \mu \geqslant 1. \tag{3.1}$$

Proof. In case $\beta_0 = 0$, (3.0) means $A \geqslant I$, obviously $A^{\mu} \geqslant I$ holds for any $\mu \geqslant 1$, so that (3.1) holds. In case $\alpha_0 = 0$, (3.0) means $I \geqslant B$, obviously $I \geqslant B^{\lambda}$ holds for any $\lambda \geqslant 1$, so that (3.1) holds too. Therefore we have only to consider the case $\alpha_0 > 0$ and $\beta_0 > 0$. Applying Theorem F to (3.0), we have

$$A^{1+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0 p}{\alpha_0+\beta_0}}A^{\frac{r}{2}}\}^{\frac{1+r}{p+r}} \quad \text{for any } p \geqslant 1 \text{ and } r \geqslant 0. \tag{3.2}$$

Putting $p = \frac{\alpha_0+\beta_0}{\beta_0} \geqslant 1$ in (3.2), we have

$$A^{1+r} \geqslant (A^{\frac{1}{2}(1+r)}BA^{\frac{1}{2}(1+r)})^{\frac{(1+r)\beta_0}{\alpha_0+\beta_0+\beta_0 r}}. \tag{3.3}$$

Put $\mu = 1+r \geqslant 1$ in (3.3), then we have

$$A^{\mu} \geqslant (A^{\frac{\mu}{2}}BA^{\frac{\mu}{2}})^{\frac{\beta_0\mu}{\alpha_0+\beta_0\mu}} \quad \text{for } \mu \geqslant 1. \tag{3.4}$$

(3.4) is equivalent to the following (3.5) by Lemma 1

$$(B^{\frac{1}{2}}A^{\mu}B^{\frac{1}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0\mu}} \geqslant B \quad \text{for } \mu \geqslant 1\,. \tag{3.5}$$

Again applying Theorem F to (3.5), we have

$$\{B^{\frac{r}{2}}(B^{\frac{1}{2}}A^{\mu}B^{\frac{1}{2}})^{\frac{\alpha_0 p}{\alpha_0+\beta_0\mu}}B^{\frac{r}{2}}\}^{\frac{1+r}{p+r}} \geqslant B^{1+r} \quad \text{for any } p \geqslant 1 \text{ and } r \geqslant 0\,. \tag{3.6}$$

Putting $p = \frac{\alpha_0+\beta_0\mu}{\alpha_0} \geqslant 1$ in (3.6), we have

$$(B^{\frac{1}{2}(1+r)}A^{\mu}B^{\frac{1}{2}(1+r)})^{\frac{(1+r)\alpha_0}{\alpha_0+\beta_0\mu+\alpha_0 r}} \geqslant B^{1+r} \quad \text{for any } r \geqslant 0\,. \tag{3.7}$$

Put $\lambda = 1+r \geqslant 1$ in (3.7), then we have

$$(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}} \geqslant B^{\lambda} \quad \text{for } \lambda \geqslant 1 \text{ and } \mu \geqslant 1\,, \tag{3.8}$$

hence proof of Lemma 2 is complete since (3.8) is equivalent to (3.1) by lemma 1.

Proof of Theorem 1.

Proof of (i). We recall the following condition on δ, α_0, β_0 and λ in (i):

$$\textit{for any fixed } \delta \geqslant -\beta_0 \textit{ and } \lambda \geqslant 1 \textit{ such that } \alpha_0\lambda \geqslant \delta\,. \tag{3.9}$$

(a) *Proof of the result that* $f(\lambda,\mu)$ *is decreasing for* $\lambda \geqslant 1$ *such that* $\alpha_0\lambda \geqslant \delta$ *for any fixed* $\delta \geqslant -\beta_0$.

The hypothesis in Theorem 1 ensures (3.1) by Lemma 2

$$A^{\mu} \geqslant (A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} \quad \text{for } \lambda \geqslant 1 \text{ and } \mu \geqslant 1 \tag{3.1}$$

and (3.1) is equivalent to the following (3.10) by Lemma 1

$$(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}} \geqslant B^{\lambda} \quad \text{for } \lambda \geqslant 1 \text{ and } \mu \geqslant 1\,. \tag{3.10}$$

(3.10) yields the following (3.11) by Löwner-Heinz theorem

$$(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0 w}{\alpha_0\lambda+\beta_0\mu}} \geqslant B^{w} \quad \text{for } \lambda \geqslant 1, \mu \geqslant 1 \text{ and any } w \text{ such that } \lambda \geqslant w \geqslant 0. \tag{3.11}$$

Then we have

$$\begin{aligned}
g(\lambda) &= (A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} \\
&= \{(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}{\alpha_0\lambda+\beta_0\mu}}\}^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}} \\
&= \{A^{\frac{\mu}{2}}B^{\frac{\lambda}{2}}(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0 w}{\alpha_0\lambda+\beta_0\mu}}B^{\frac{\lambda}{2}}A^{\frac{\mu}{2}}\}^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}} \quad \text{by Lemma 1} \\
&\geqslant (A^{\frac{\mu}{2}}B^{\frac{\lambda}{2}}B^{w}B^{\frac{\lambda}{2}}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0(\lambda+w)+\beta_0\mu}} \\
&= (A^{\frac{\mu}{2}}B^{\lambda+w}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0(\lambda+w)+\beta_0\mu}} = g(\lambda+w)
\end{aligned}$$

and the last inequality holds by (3.11) and Löwner-Heinz theorem since $\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w} \in [0,1]$ holds by (3.9). Hence $f(\lambda,\mu) = A^{\frac{-\mu}{2}} g(\lambda) A^{\frac{-\mu}{2}}$ is decreasing for $\lambda \geqslant 1$ such that $\alpha_0\lambda \geqslant \delta$ for any fixed $\delta \geqslant -\beta_0$.

(b) *Proof of the result that $f(\lambda,\mu)$ is decreasing for $\mu \geqslant 1$.*

(3.1) yield the following (3.12) by Löwner-Heinz theorem

$$A^{\nu} \geqslant (A^{\frac{\mu}{2}} B^{\lambda} A^{\frac{\mu}{2}})^{\frac{\beta_0\nu}{\alpha_0\lambda+\beta_0\mu}} \quad \text{for } \lambda \geqslant 1, \mu \geqslant 1 \text{ and any } \nu \text{ such that } \mu \geqslant \nu \geqslant 0. \tag{3.12}$$

Then we have

$$\begin{aligned}
f(\lambda,\mu) &= A^{\frac{-\mu}{2}} (A^{\frac{\mu}{2}} B^{\lambda} A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} A^{\frac{-\mu}{2}} \\
&= B^{\frac{\lambda}{2}} (B^{\frac{\lambda}{2}} A^{\mu} B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}} B^{\frac{\lambda}{2}} \quad \text{by Lemma 1} \\
&= B^{\frac{\lambda}{2}} \{(B^{\frac{\lambda}{2}} A^{\mu} B^{\frac{\lambda}{2}})^{\frac{\alpha_0\lambda+\beta_0\mu+\beta_0\nu}{\alpha_0\lambda+\beta_0\mu}}\}^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0\nu}} B^{\frac{\lambda}{2}} \\
&= B^{\frac{\lambda}{2}} \{B^{\frac{\lambda}{2}} A^{\frac{\mu}{2}} (A^{\frac{\mu}{2}} B^{\lambda} A^{\frac{\mu}{2}})^{\frac{\beta_0\nu}{\alpha_0\lambda+\beta_0\mu}} A^{\frac{\mu}{2}} B^{\frac{\lambda}{2}}\}^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0\nu}} B^{\frac{\lambda}{2}} \quad \text{by Lemma 1} \\
&\geqslant B^{\frac{\lambda}{2}} (B^{\frac{\lambda}{2}} A^{\frac{\mu}{2}} A^{\nu} A^{\frac{\mu}{2}} B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0(\mu+\nu)}} B^{\frac{\lambda}{2}} \\
&= B^{\frac{\lambda}{2}} (B^{\frac{\lambda}{2}} A^{\mu+\nu} A^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0(\mu+\nu)}} B^{\frac{\lambda}{2}} = f(\lambda,\mu+\nu)
\end{aligned}$$

and the last inequality holds by (3.12) and Löwner-Heinz theorem since $\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0\nu} \in [-1,0]$ by the condition (3.9), and taking inverses. Hence $f(\lambda,\mu)$ is decreasing for $\mu \geqslant 1$.

Proof of (ii). We recall the following condition on δ, α_0, β_0 and μ in (ii):

$$\textit{for any fixed } \delta \leqslant \alpha_0 \textit{ and } \mu \geqslant 1 \textit{ such that } \beta_0\mu \geqslant -\delta. \tag{3.13}$$

(3.0) is equivalent to the following (3.14)

$$B^{-1} \geqslant (B^{\frac{-1}{2}} A^{-1} B^{\frac{-1}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} \textit{ for fixed } \alpha_0 \geqslant 0 \textit{ and } \beta_0 \geqslant 0 \textit{ with } \alpha_0+\beta_0 > 0 \tag{3.14}$$

by Lemma 1 and taking inverses of both sides. We recall that (3.14) just corresponds to (3.0) when replace A by B^{-1} and B by A^{-1} in (3.0) and moreover replace α_0 by β_0 and replace β_0 by α_0 and

$$\begin{aligned}
f(\lambda,\mu) &= A^{\frac{-\mu}{2}} (A^{\frac{\mu}{2}} B^{\lambda} A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} A^{\frac{-\mu}{2}} \\
&= (B^{-1})^{\frac{-\lambda}{2}} \{(B^{-1})^{\frac{\lambda}{2}} (A^{-1})^{\mu} (B^{-1})^{\frac{\lambda}{2}})^{\frac{-\delta+\alpha_0\lambda}{\beta_0\mu+\alpha_0\lambda}} (B^{-1})^{\frac{-\lambda}{2}},
\end{aligned} \tag{3.15}$$

by applying (i), for any fixed $-\delta \geqslant -\alpha_0$, $f(\lambda,\mu)$ is decreasing for $\lambda \geqslant 1$ and $\mu \geqslant 1$ such that $\beta_0\mu \geqslant -\delta$,that is, $f(\lambda,\mu)$ is decreasing for $\lambda \geqslant 1$ and $\mu \geqslant 1$ under the condition (3.13) by (3.15), so the proof of (ii) is complete. The equivalence relation between (i) and (ii) is obvious by scrutinizing the proof of (i) and (ii).

Consequently we have finished a proof of Theorem 1 by (i) and (ii).

Proof of Theorem 2. We may assume that A and B are both invertible in the proof. In case $t = 0$, the result follows by [8, Theorem 3], so we have only to consider the case $p \geqslant t > 0$.

Proof of (i). Put $X = A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}$. Then X is positive invertible and we have $A^{\frac{t}{2}}XA^{\frac{t}{2}} = B^p$ and $A \geqslant (A^{\frac{t}{2}}XA^{\frac{t}{2}})^{\frac{1}{p}}$ by the hypothesis $A \geqslant B \geqslant 0$. Put $\beta_0 = t \in (0,1]$ and $\alpha_0 = p - t \geqslant 0$. Then $A \geqslant (A^{\frac{t}{2}}XA^{\frac{t}{2}})^{\frac{1}{\alpha_0+\beta_0}}$, so that

$$A^t \geqslant (A^{\frac{t}{2}}XA^{\frac{t}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \tag{3.16}$$

holds by Löwner-Heinz theorem. Put $r = \mu\beta_0 = \mu t \geqslant t$ for $\mu \geqslant 1$ and $\delta = q - t$. As $\delta \geqslant -\beta_0$ holds by $q \geqslant 0$, by using Theorem 1,

$$\begin{aligned} f(s,\mu) &= A^{\frac{-\mu t}{2}}(A^{\frac{\mu t}{2}}X^sA^{\frac{\mu t}{2}})^{\frac{\delta+\mu t}{\alpha_0 s+\mu t}}A^{\frac{-\mu t}{2}} \\ &= A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \\ &= G_{p,q,t}(A,B,r,s) \end{aligned} \tag{3.17}$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$ such that $(p-t)s \geqslant q-t$ because $f(s,\mu)$ is decreasing for $\mu \geqslant 1$ and $s \geqslant 1$ such that $\alpha_0 s \geqslant \delta$ by (i) of Theorem 1. Whence the proof of (i) is complete.

Proof of (ii). The condition $p \geqslant q$ and $r \geqslant \max\{t, t-q\}$ in (ii) satisfy $\delta \leqslant \alpha_0$ and $\beta_0\mu \geqslant -\delta$ in the conditions of (ii) in Theorem 1, so that $G_{p,q,t}(A,B,r,s)$ is decreasing for $s \geqslant 1$ and $r \geqslant \max\{t, t-q\}$ by (ii) of Theorem 1 and (3.17). The equivalence relation between (i) and (ii) follows by Theorem 1. Whence the proof of Theorem 2 is complete.

Proof of Theorem B. We have only to put $p \geqslant q$ in (i) of Theorem 2, or put $q \geqslant 0$ in (ii) of Theorem 2.

Proof of Corollary 3. We recall the following (3.18) in [4, 9], which is an extension of [1]:

$$A \gg B \textit{ holds if and only if } A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \textit{ for all } p \geqslant 0 \textit{ and } r \geqslant 0. \tag{3.18}$$

$(i) \implies (ii)$. Assume (i). As (3.18) holds, by (i) of Theorem 1, for any fixed $q \geqslant 0$

$$f(\lambda,\mu) = A^{\frac{-r\mu}{2}}(A^{\frac{r\mu}{2}}B^{p\lambda}A^{\frac{r\mu}{2}})^{\frac{q+r\mu}{p\lambda+r\mu}}A^{\frac{-r\mu}{2}}$$

is decreasing for $\mu \geqslant 1$ and $\lambda \geqslant 1$ such that $p\lambda \geqslant q$, that is, for any fixed $q \geqslant 0$, $F_q(p,r)$ is decreasing for $p \geqslant q$ and $r \geqslant 0$.

$(ii) \implies (i)$. Assume that $F_q(p,r)$ is decreasing for $r \geqslant 0$. Then $F_0(p,0) \geqslant F_0(p,r)$ holds, that is, $I \geqslant A^{-\frac{r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}A^{-\frac{r}{2}}$, so that $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p \geqslant 0$ and $r \geqslant 0$, which is equivalent to $A \gg B$ by (3.18).

$(ii) \iff (iii)$ follows by the equivalence between (i) and (ii) of Theorem 1.

Hence the proof of Corollary 3 is complete.

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