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SOME INEQUALITIES RELATED TO M–MATRICES

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(*communicated by B. Opic*)

Abstract. Several different forms of a Euler type inequality are investigated and their relation to M-matrices and doubly stochastic matrices is exhibited. The results are applied to the study of positive biquadratic forms.

Introduction

In connection with positive biquadratic forms, the authors came across an interesting inequality (Theorem 2.). It turned out, however, that this inequality was essentially equivalent to a particular case of a result in [1], as well as to an inequality proved in [4]. The authors present a new simple proof thereof as well as its several consequences. Interesting connections with M-matrices and doubly stochastic matrices are also pointed out.

Let us recall the result posed as a problem by A. Berkes, proved by C. Bindscheidler [1]. In the book of D.S. Mitrinović [5] it appears under 2.41.

Given $n+1$ positive numbers $x_1, \dots, x_{n+1}$ then the following implication holds:

$$\text{If} \quad \sum_1^{n+1} \frac{1}{1+x_k} \geqslant n, \ \text{then} \quad \prod_1^{n+1} \frac{1}{x_k} \geqslant n^{n+1}.$$

For our purposes it will be convenient to restate this implication in the following form:

(*) *Let $a_1, \dots, a_n$, $n \geqslant 3$, be positive numbers such that $\prod a_i \geqslant 1$. Then*

$$\sum \frac{1}{n-1+a_i} \leqslant 1,$$

with equality if and only if $a_i = 1$ for $i = 1, \dots, n$.

Let us recall that a real square matrix is called an M-matrix (in [3], a matrix of class K) if all its off-diagonal entries are nonpositive and all principal minors positive. It is called a *possibly singular* M-matrix (a matrix of class K_0) if all its off-diagonal

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entries are nonpositive and all principal minors nonnegative. A symmetric M-matrix is also called *Stieltjes matrix.*

A real nonnegative matrix is called *doubly stochastic* if all its row-sums as well as all its column-sums equal one.

In the sequel, we shall use the following notation.

If $x_1, \dots, x_n$ are real numbers, we denote by $A(x)$, resp. $A(x^2)$ the arithmetic mean of these numbers, resp. their squares. By $G(y)$ we denote the geometric mean of positive (or, nonnegative) numbers y_i.

Results

We shall first give a simple proof of an inequality stated essentially in ([4], Lemma 5.1).

LEMMA 2.1. *Suppose that $u_1, \dots, u_n$ are real numbers. Then,*

$$A(u)^2 \leqslant \frac{n-1}{n} A(u^2) + \frac{1}{n} G(u^2). \tag{1}$$

For $n \geqslant 3$, equality is attained if and only if $u_1 = \dots = u_n$. (For $n = 2$, equality is attained if and only if $u_1 u_2 \geqslant 0$, for $n = 1$ always.)

Proof. It is immediate that it suffices to prove this for the case that $n \geqslant 3$, all numbers u_i are positive and $G(u)$ equals one. Let then $\mathscr{S}$ denote the set of all n-tuples y_i which satisfy $G(y) = 1$ and

$$\min(u_k) \leqslant y_i \leqslant \max(u_k) \quad \text{for all} \quad i.$$

Since $\mathscr{S}$ is compact, the function $\Phi(y) := A(y)^2 - \frac{n-1}{n} A(y^2)$ attains in $\mathscr{S}$ its maximum. To prove that this maximum is $\frac{1}{n}$, attained for $y_i = 1$ for all i, it suffices to show that for any other $(y) \in \mathscr{S}$ its value can be augmented for some $\tilde{y} \in \mathscr{S}$.

Let thus $y_1 \geqslant y_2 \geqslant \dots \geqslant y_n$ and suppose that $y_1 > y_n$. Denote by $f(x)$ the function

$$f(x) = (xy_1 + \frac{1}{x} y_n + b)^2 - (n-1)(x^2 y_1^2 + \frac{1}{x^2} y_n^2 + B)$$

defined for positive x, where $b = \sum_2^{n-1} y_j$ and $B = \sum_2^{n-1} y_j^2$. Its derivative can be written as

$$f'(x) = 2[-(n-2)(xy_1 + \frac{y_n}{x}) + b](y_1 - \frac{1}{x^2} y_n).$$

Since $y_n > 0$ and $(n-2)y_1 \geqslant b$, this derivative is negative for $x = 1$.

This implies that for $x < 1$ sufficiently close to 1, the point $\tilde{y} = (\tilde{y}_i)$, $\tilde{y}_1 = xy_1$, $\tilde{y}_n = \frac{1}{x} y_n$, $\tilde{y}_k = y_k$ for $2 \leqslant k \leqslant n-1$, belongs to $\mathscr{S}$ and $\Phi(\tilde{y}) > \Phi(y)$. $\square$

THEOREM 2.2. *Suppose* $a_1, \dots, a_n$, $n \geqslant 3$, *are positive numbers satisfying* $\prod a_i = 1$. *Then, for an arbitrary* n*-tuple of real numbers* $u_1, \dots, u_n$,

$$\left(\sum u_i\right)^2 \leqslant (n-1)\sum u_i^2 + \sum u_i^2 a_i.$$

Equality is attained if and only if $a_i = 1$ *for all i and* $u_1 = \dots = u_n$.

Proof. By Lemma 2.1,

$$\begin{aligned} \left(\sum u_i\right)^2 &\leqslant (n-1)\sum u_i^2 + nG(u^2) \\ . &= (n-1)\sum u_i^2 + nG(a_1u_1^2, \dots, a_nu_n^2) \\ &\leqslant (n-1)\sum u_i^2 + nA(a_1u_1^2, \dots, a_nu_n^2) \\ &= (n-1)\sum u_i^2 + \sum a_iu_i^2. \end{aligned}$$

The rest is obvious. $\square$

THEOREM 2.3. *Let* $a_1, \dots, a_n$, $n \geqslant 3$, *be positive numbers such that* $\prod_1^n a_i \geqslant 1$. *Consider the matrix*

$$M(a) = \begin{pmatrix} n-2+a_1 & -1 & \dots & -1 \\ -1 & n-2+a_2 & \dots & -1 \\ \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-2+a_n \end{pmatrix}.$$

Then $M(a)$ *is a possibly singular* M*-matrix.*

If $a_i = 1$ *for all* i, *then* $M(a)$ *is singular with all row-sums equal to zero. In all other cases,* $M(a)$ *is a (nonsingular)* M*-matrix.*

Proof. By Theorem 2.2, we have for all real n-tuples $u_1, \dots, u_n$

$$(n-1)\sum u_i^2 + \sum u_i^2 a_i - \left(\sum u_i\right)^2 \geqslant 0.$$

Since $M(a)$ is the matrix of this quadratic form, it is positive semidefinite. Thus all its principal minors are nonnegative and, at the same time, all its off-diagonal entries are negative, so it is a possibly singular M-matrix as asserted. The last assertion follows from the fact that unless all the a_i's equal one, the last inequality is strict for all non-zero n-tuples which means that $M(a)$ is positive definite.

REMARK 2.4. It is easy to show that the determinant of $M(a)$ is equal to

$$\Pi(n-1+a_i)\left(1 - \sum \frac{1}{n-1+a_i}\right).$$

Therefore, the assertion (*) follows from Theorem 2.3.

COROLLARY 2.5. *Let* $y_1, \dots, y_n$ *be real numbers,* $P = (p(1), p(2), \dots, p(n))$ *a permutation of* $1, 2, \dots, n$, $n \geqslant 2$. *Denote by* $Q(y, P)$ *the matrix*

$$\begin{pmatrix} (n-2)y_1^2 + y_{p(1)}^2 & -y_1y_2 & -y_1y_3 & \dots & -y_1y_n \\ -y_2y_1 & (n-2)y_2^2 + y_{p(2)}^2 & -y_2y_3 & \dots & -y_2y_n \\ \dots & \dots & \dots & \dots & \dots \\ -y_ny_1 & -y_ny_2 & y_ny_3 & \dots & (n-2)y_n^2 + y_{p(n)}^2 \end{pmatrix}.$$

Then, $Q(y, P)$ *is positive semidefinite.*

Proof. It suffices to consider the case that all numbers y_i are different from zero. It is then easy to see that

$$Q(y, P) = D(y)M(a)D(y),$$

where $D(y)$ is the diagonal matrix $\mathrm{diag}\{y_1, \dots, y_n\}$ and $a_i = (y_{P(i)}/y_i)^2$ so that $\prod a_i = 1$. For $n \geqslant 3$, Theorem 2.4 applies. If $n = 2$, the result is also true. □

COROLLARY 2.6. *Let* $P = (p(1), p(2), \dots, p(n))$, $n \geqslant 2$, *be a permutation of* $1, 2, \dots, n$. *Given two n-tuples of real numbers* $x_1, \dots, x_n, y_1, \dots, y_n$, *the following inequality holds:*

$$(n-1)\sum x_i^2 y_i^2 + \sum x_i^2 y_{p(i)}^2 - \left(\sum x_i y_i\right)^2 \geqslant 0. \tag{2}$$

Proof. The expression above equals, in the usual notation, $(Q(y, P)x, x)$. □

COROLLARY 2.7. *Suppose* $D = (d_{ik})$ *is an* n*-by-*n *doubly stochastic matrix,* $n \geqslant 2$. *Then for any pair of* n*-tuples of real numbers* $x_1, \dots, x_n, y_1, \dots, y_n$, *the following inequality holds:*

$$\left(\sum x_i y_i\right)^2 \leqslant (n-1)\sum x_i^2 y_i^2 + \sum_{i,k} d_{ik} x_i^2 y_k^2.$$

Proof. By Birkhoff's theorem, the matrix D may be written in the form of a convex combination of permutation matrices, $D = \sum \lambda_P P$. The inequality then follows from the fact that $\sum \lambda_P Q(y, P)$ is a convex combination of positive semidefinite matrices.

CORROLLARY 2.8. *Let* B, D *be* n*-by-*n *matrices,* $n \geqslant 2$, $D = (d_{ik})$ *doubly stochastic and* $B = (b_{ik})$ *positive semidefinite. Then, for every* n*-tuple of real numbers* $x_1, \dots, x_n$,

$$\sum_{i,k} b_{ik} x_i x_k \leqslant (n-1)\sum_k b_{kk} x_k^2 + \sum_{i,k} d_{ik} b_{kk} x_i^2.$$

Proof. The matrix B may be written in the form $B = \sum y^{(j)} y^{(j)T}$ where $y^{(j)}$ are some real vectors. We have then, using Corollary 2.7,

$$\begin{aligned}\sum_{i,k} b_{ik}x_ix_k &= \sum_j \Big(\sum_i x_i y_i^{(j)}\Big)^2\\ &\leqslant \sum_j (n-1)\sum_i x_i^2 (y_i^{(j)})^2 + \sum_j\sum_{i,k} d_{ik}x_i^2(y_k^{(j)})^2\\ &= (n-1)\sum_i b_{ii}x_i^2 + \sum_{i,k} d_{ik}b_{kk}x_i^2.\end{aligned}$$

COROLLARY 2.9. *Given n real numbers $u_1,\dots,u_n$, the following estimate for the modified dispersion holds:*

$$\frac{1}{n}\sum_{i<k}(u_i-u_k)^2 \leqslant A(u^2)-G(u^2). \tag{3}$$

Proof. The inequality (1) may be rewritten in the form

$$n(n-1)A(u^2)+nG(u^2)-\Big(\sum u_i\Big)^2 \geqslant 0.$$

The expression on the left-hand-side of this inequality equals

$$n^2A(u^2)-\Big(\sum u_i\Big)^2 - n(A(u^2)-G(u^2)) = \sum_{i<k}(u_i-u_k)^2 - n(A(u^2)-G(u^2)).$$

This proves the equivalence of (1) and (3).

REMARK 2.10. The inequality (3) appears, in a slightly different form, also in [4]. Let us conclude by a slight strengthening of a result in [2]:

THEOREM 2.11. *The biquadratic form*

$$2\sum_1^3 (x_iy_i)^2 - \Big(\sum_1^3 x_iy_i\Big)^2 + x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2$$

is positive semidefinite, equal to zero in all points $x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$ and cannot be expressed as a sum of squares of bilinear forms.

Proof. The first assertion follows from Corollary 2.6 for $n = 3$. The second is obvious. The last follows in the manner identical with that in [2], Theorem 1.

REMARK 2.12. An analogous result holds for the case of the biquadratic form in (2) if P is the cyclic permutation ($p(k) \equiv k+1 \bmod n$).

ADDED IN PROOF. The strengthening 2.11 is essentially contained in M. D. Choi, *Positive linear maps*, Proc. Symp. in Pure Math., Amer. Math. Soc. 38(2) (1982), 583–590.

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