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A BEST POSSIBLE HADAMARD INEQUALITY

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Abstract. The classical Hadamard-Hermite inequality requires that the measure be a symmetric and positive. We prove versions which require neither of these conditions. Furthermore, we prove that no such theorems exist with less restrictions than ours, ie. they are best possible.

Introduction. The Hadamard inequality, [5], sometimes denoted the Hermite-Hadamard inequality, is

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1}$$

which holds for f convex. This inequality is a special case of a result of Fejér [1]

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)dt \leqslant \int_a^b f(t)p(t)dt \leqslant \frac{f(a)+f(b)}{2}\int_a^b p(t)dt, \tag{2}$$

which holds when f is convex and p is a nonnegative function whose graph is symmetric with respect to the center $(a+b)/2$. One wonders what the symmetry has to do with this result and if such an inequality holds for other functions. In particular, one would like to have a result which cannot be generalized by being a 'best possible inequality', see [2], [3], and [4]. Here it would mean being able to prove the two statements.

(A) The inequality (2) holds for all functions $p \in M$ if and only if f is convex; and

(B) The inequality (2) holds for all convex f if and only if $p \in M$.

The problem is to find the correct class of functions or measures M. It turns out that the class M will not be a subset of the positive measures.

The Lower Bound. For convenience, we will take the interval to be $[-1,1]$ and concentrate on the left hand inequality of (2) first. To see that symmetry is not essential in Fejér's result, we first see how one might establish a result in this direction. To do this we will replace $p(x)dx$ by a nonnegative regular Borel measure μ with the requirement that $\int_{-1}^{1} d\mu(x) > 0$.

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Since f is convex, its graph lies above its tangent lines. Let y be an arbitrary number in $[-1,1]$ and write down this condition

$$f(x) \geqslant f(y) + f'(y)(x-y). \tag{3}$$

Let the moments of the measure be defined by $P_k = \int_{-1}^{1} x^k d\mu(x)$. If we integrate the inequality (3) we arrive at

$$\int_{-1}^{1} f(x)d\mu(x) \geqslant f(y)P_0 + f'(y)(P_1 - yP_0). \tag{4}$$

This inequality holds for any y in $[-1,1]$ so we may choose any y we please. But we get the best one by maximizing the right hand side of this inequality. Assuming that f has two derivatives one gets the derivative of this quantity to be $f''(y)(P_1 - yP_0)$. Since $P_1 - P_0 \leqslant 0$ and $P_1 + P_0 > 0$, the maximum is at $y_0 = P_1/P_0$. So we arrive at

$$\int_{-1}^{1} f(x)d\mu(x) \geqslant P_0 f(P_1/P_0). \tag{5}$$

Of course, if μ is an even measure we have $P_1 = 0$ and Fejer's result. At this stage we are able to prove statement (A) (with (2) replaced by (5)) if we take M to be the nonnegative regular Borel measures. For the sufficiency is the above argument and the necessity is obtained by taking the measure

$$d\mu = \alpha\delta_x + (1-\alpha)\delta_y \tag{6}$$

for δ_z the unit mass at z and $0 \leqslant \alpha \leqslant 1$. Then (5) becomes the convexity of f. Of course the sufficiency in the statement (B) also obtains from the above argument. It is the necessity that fails. That is, we cannot prove that if (5) holds for all convex f then the measure must be nonnegative. This turns out to be false. If we allow μ to be a signed measure, the above proof fails since we may not integrate an inequality. But here is what we can prove. Let (EP for end positive)

$$\int_{-1}^{t} (t-x)d\mu(x) \geqslant 0 \text{ and } \int_{t}^{1} (x-t)d\mu(x) \geqslant 0 \text{ for } t \in [-1,1]. \tag{EP}$$

This condition will be revisited in the section on upper bounds.

THEOREM 1. *Let f be continuous on $[-1,1]$ and μ a regular Borel measure such that $\mu[-1,1] > 0$. Then*

i) the inequality (5) holds for all measures μ satisfying (EP) if and only if f is convex; and

ii) The inequality (5) holds for all convex f if and only if μ satisfies (EP).

Equality holds in (5) for linear f.

Proof. We first argue the sufficiency. For f convex, we can write

$$f(x) = a + b(x-y) + \int_y^x (x-t)d\sigma(t). \tag{7}$$

The non-negative measure σ, a and b depend on the choice of y. The reader may take $d\sigma$ to be $f''(t)dt$ for first understanding. For the general case, a bounded convex function f has a derivative f' a.e. and f' is an increasing function. So f' can be written as $b + \int_y^x d\sigma$ where $d\sigma$ may contain point masses and b is the slope of some supporting line at y. Then

$$\int_{-1}^{1} f(x)d\mu(x) = (a-yb)P_0 + bP_1 + R \tag{8}$$

where $R = \int_{-1}^{1}\int_y^x (x-t)d\sigma(t)d\mu(x)$ which can be written as

$$\int_{-1}^{y}\left(\int_{-1}^{t}(t-x)d\mu(x)\right)d\sigma(t) + \int_{y}^{1}\left(\int_{t}^{1}(x-t)d\mu(x)\right)d\sigma(t).$$

It is now obvious that $\sigma \geqslant 0$ and $\mu \in \text{(EP)}$ make $R \geqslant 0$, and again we may choose $y = y_0 = P_1/P_0$ to get (5) (since $a = f(y_0)$). To prove the converse in i) we observe that the measure defined in (6) is in (EP). To prove that if (5) holds for all convex f then $\mu \in \text{(EP)}$ we take for $f(x)$ the function $f(x) = (x-t)_+$ for $t \in [-1,1]$, Then

$$\int_{t}^{1}(x-t)d\mu(x) \geqslant \int_{-1}^{1} d\mu\left(\frac{P_1}{P_0} - t\right)_+ . \tag{9}$$

Since the right hand side is nonnegative we get the second condition in (EP). Note that for $t=-1$, (9) reads $P_1 + P_0 \geqslant 0$ so that $y_0 = P_1/P_0 \geqslant -1$. If $t \leqslant y_0$ then (9) becomes

$$\int_{t}^{1}(x-t)d\mu(x) \geqslant \int_{-1}^{1} x d\mu(x) - t\int_{-1}^{1} d\mu(x)$$

which becomes $\int_{-1}^{t}(t-x)d\mu(x) \geqslant 0$, the first of (EP) for $t \leqslant y_0$. If $t > y_0$ we have $tP_0 > P_1$. The identity

$$\int_{-1}^{t}(t-x)d\mu(x) = (tP_0 - P_1) + \int_{t}^{1}(x-t)d\mu(x)$$

gives the first term in (EP) as the sum of two positive terms. Note that at $t = 1$ this condition gives $P_0 - P_1 \geqslant 0$ or $y_0 \leqslant 1$.

EXAMPLE 1. Let $d\mu_a(x) = (x^2 - a)dx, 0 < a < \frac{1}{3}$. Then

$$\int_{-1}^{t} (t-x)d\mu(x) = \int_{t}^{1} (x-t)d\mu(x) = \frac{1}{12}(t^2-1)^2 \text{ and } P_0 = P_1 = 0.$$

Thus for $0 < a < \frac{1}{3}$ $d\mu$ is not a nonnegative measure and

$$\int_{-1}^{1} (x^2 - a)f(x)dx \geqslant 2\left(\frac{1}{3} - a\right)f(0) \tag{10}$$

for any convex f. For $a = \frac{1}{3}$ we have (11).

$$\int_{-1}^{1} x^2 f(x)dx \geqslant \frac{1}{3}\int_{-1}^{1} f(x)dx \tag{11}$$

for any convex f.

REMARK 1. If f is concave, all of the above inequalities are reversed.

The nth Order Case. One can obtain inequalities with $f'' \geqslant 0$ replaced by $f^{(n+1)} \geqslant 0$. For then, we look at the simple case when $\mu \geqslant 0$.

$$f(x) \geqslant \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k. \tag{12}$$

If $\mu \geqslant 0$ we have

$$\int_{-1}^{1} f(x)d\mu(x) \geqslant \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}\int_{-1}^{1} (x-y)^k d\mu(x) \equiv g(y). \tag{13}$$

It follows that $g'(y) = \dfrac{f^{(n+1)}(y)}{n!} \int_{-1}^{1} (x-y)^n d\mu(x)$.

THEOREM 2. *If n is even, $f^{(n+1)} \geqslant 0$ on $[-1,1]$ and $\mu \geqslant 0$, then*

$$\int_{-1}^{1} f(x)d\mu(x) \geqslant \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}\int_{-1}^{1} (x-1)^k d\mu(x) \tag{14}$$

with equality if f is a polynomial of degree n.

Proof. If n is even $g' \geqslant 0$ so we get the best inequality by taking $g(1)$.

THEOREM 3. *If n is odd, $f^{(n+1)} \geqslant 0$ on $[-1,1]$ and $\mu \geqslant 0$ then*

$$\int_{-1}^{1} f(x)d\mu(x) \geqslant \sum_{k=0}^{n-1} \frac{f^{(k)}(y_0)}{k!} \int_{-1}^{1} (x-y_0)^k d\mu(x) \tag{15}$$

where $\int_{-1}^{1} (x-y_0)^n d\mu(x) = 0$.

Proof. If n is odd then g' has a factor $\int_{-1}^{1} (x-y)^n d\mu(x)$ which has opposite signs at ± 1 and this factor has a derivative which is less than zero, so it has a unique zero. This zero maximizes $g(y)$.

This result is reversed if $f^{(n+1)} \leqslant 0$. Moreover if μ is even than $\int_{-1}^{1} x^n d\mu(x) = 0$ if n is odd, so that $\mu_0 = 0$.

REMARK 2. If $n = 1$ then (15) becomes (5) since $y_0 = P_1/P_0$ in this case.

EXAMPLE 2. If n is odd, μ an even measure and $f^{(n+1)} \geqslant 0$ then

$$\int_{-1}^{1} f(x)d\mu(x) \geqslant \sum_{k=0}^{\frac{n-1}{2}} \frac{f^{(2k)}(0)}{(2k)!} \int_{-1}^{1} x^{2k} d\mu(x).$$

Having disposed of the easy case we look at the replacement of nonnegative measures by signed measures. Here (12) is replaced by

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k + R^1 \tag{16}$$

where

$$R^1 = \int_{y}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

THEOREM 4. *Let $f \in C^{n+1}[-1,1]$ and μ a regular Borel measure. Then*

(i) *The inequality (13) holds for all $y \in [-1,1]$ and all f with $f^{(n+1)} \geqslant 0$ if and only if μ satisfies $(EP)_n$.*

(ii) *The inequality (13) holds for all $y \in [-1,1]$ and measures μ satisfying $(EP)_n$ if and only if $f^{(n+1)} \geqslant 0$.*

Proof. Proceeding as in the proof of Theorem 1, we arrive at

$$R = \int_{-1}^{y} \frac{f^{(n+1)}(t)}{n!} \int_{-1}^{t} (-1)(x-t)^n d\mu(x)dt + \int_{y}^{1} \frac{f^{(n+1)}(t)}{n!} \int_{t}^{1} (x-t)^n d\mu(x)dt. \tag{17}$$

So the sufficiency of (13) is that

$$\int_t^1 (x-t)^n d\mu(x) \geqslant 0 \text{ and } \int_{-1}^t (x-t)^n d\mu(x) \geqslant 0;\ t \in [-1,1]. \qquad (\mathrm{EP}_n)$$

The necessity is gotten by taking $f^{(n+1)}(t) = \delta_{t_0}$ i.e. $f(t) = \dfrac{(t-t_0)_+^n}{n!}$. For then $R \geqslant 0$ gives $(\mathrm{EP})_n$.

To show that (13) holding implies that $f^{(n+1)} \geqslant 0$ one needs to assume the existence of $f^{(n+1)}$. The $(n+1)^{st}$ divided difference $g[x_1,\ldots,x_{n+2}]$ at distinct points of a function g is a linear combination of the values of g at these points. That is, $g[x_1,\ldots,x_{n+2}] = \sum_1^{n+2} \alpha_i x_i$ the coefficients being determined by the $(n+2)$ points $x_1,\ldots,x_{n+2}$. Consequently, the measure $d\mu = \sum_1^{n+2} \alpha_i \delta_{x_i}$ has the property that

$$g[x_1,\ldots,x_{n+2}] = \int_{-1}^1 g(x)d\mu(x).$$

We take this measure in (13). Now $g[x_1,\ldots,x_{n+2}] = \frac{g^{(n+1)}(s)}{(n+1)!}$ by a generalized mean value theorem. Consequently $\int_{-1}^1 (x-y)^k d\mu(x) = 0$ for $k = 0,\ldots,n$ and (13) becomes $f[x_1,\ldots,x_{n+2}] = \int_{-1}^1 f(x)d\mu(x) \geqslant 0$. Now it is known that $\lim f[x_1\ldots,x_{n+2}] = \dfrac{f^{(n+1)}(x)}{(n+1)!}$ where the limit has all the $x_i \to x$. To complete the proof we must argue that this measure μ satisfies $(\mathrm{EP})_n$. Since we are assuming (13) for all $y \in [-1,1]$ we may take $y = 1$ so that $(\mathrm{EP})_n$ reduces to the simple condition $\int_{-1}^1 (x-t)_t^n d\mu(x) \geqslant 0$, see (17).

Now $\int_{-1}^1 (x-t)_+^n d\mu(x)$ is the $(n+1)^{st}$ divided difference of the function $(x-t)_+^n$ as a function of x. This is the classical B-spline $M(t,x_0,\ldots,x_{n+2})$, which is known to be nonnegative. See [6, page 2]. This completes the proof.

Upper Bounds. One could begin a study of the upper bound by using (7) to compute

$$f(x) - f(1)\frac{x+1}{2} - f(-1)\frac{1-x}{2} = h(x)$$

and then $\int_{-1}^1 h d\mu(x)$ as a linear combination of $f(y), f'(y)$ as in formula 8 and an integral which is generally like R in (8). When one does this, the integral term turns out to be independent of y, and the coefficients of $f(y)$ and $f'(y)$ are zero. If $R \leqslant 0$, one gets

$$\int_{-1}^{1} f(x)d\mu(x) - \frac{f(1)}{2}(P_0 + P_1) - f(-1)\left(\frac{P_0 - P_1}{2}\right) \leqslant 0$$

which replicates Fejer's upper bound if μ is nonnegative and even. The general condition on μ obtained in this way suggests a much easier proof and statement of the theorem.

THEOREM 5. *Let f be a twice differentiable convex function and let μ be a measure such that the solution to the boundary value problem $y'' = d\mu;\ y(-1) = y(1) = 0$, is $\leqslant 0$ on $[-1,1]$, then*

$$\int_{-1}^{1} f\,d\mu \leqslant P_0\frac{f(-1)+f(1)}{2} + P_1\frac{f(1)-f(-1)}{2}.$$

REMARK 3. The meaning of the boundary value problem is this. Let $G(x,t)$ be the Green's function for the problem $Ly = y''$, $y(1) = y(-1) = 0$ (note the change in sign in L) then $y(x) = \int_{-1}^{1} G(x,t)d\mu(t)$ is a C' function satisfying the boundary conditions and if $d\mu(t) = p(t)dt, y'' = p$ a.e.. The boundary value problem is self adjoint so $G(x,y) = G(y,x)$.

Proof of the theorem.

Let $y(x) = \int_{-1}^{1} G(x,t)d\mu(t)$, then

$$\int_{-1}^{1} f''(x)y(x)dx = \int_{-1}^{1} f''(x)\int_{-1}^{1} G(x,t)d\mu(t)dx$$
$$= \int_{-1}^{1}\int_{-1}^{1} G(x,t)f''(x)dx\,d\mu(t) = \int_{-1}^{1}\left(\int_{-1}^{1} G(t,x)f''(x)dx\right)d\mu(t).$$

Now $\int_{-1}^{1} G(t,x)f''(x)dx$ is a function whose second derivative is f'' and whose values at ± 1 is zero since G is the Green's function. This function is $f(x) - f(-1)\frac{1-x}{2} - f(1)\frac{1+x}{2}$, a.e. So we have

$$\int_{-1}^{1} f(x)\mu(x) - \frac{f(-1)}{2}(P_0 - P_1) - \frac{f(1)}{2}(P_1 + P_0) = \int_{-1}^{1} f''(x)y(x)dx.$$

Now $f'' \geqslant 0$ and $y \leqslant 0$ by hypothesis. This completes the proof.

Note that it can be verified that $G \leqslant 0$ so that if $\mu \geqslant 0$, then $y \leqslant 0$.

At this point it is instructive to look at the condition (EP). The functions in (EP) $\int_{-1}^{t} (t-x)d\mu(x)$ and $\int_{t}^{1}(x-t)d\mu(x)$ are solutions of initial value problems (respectively)

$$y'' = d\mu;\ y(-1) = y'(-1) = 0;\ y(1) = y'(1) = 0.$$

For $(\mathrm{EP})_n$ the initial value problem is $y^{(n+1)} = \frac{(-1)^{n+1}}{n!}d\mu$ and $y^{(k)}(-1) = 0,\ k = 0,\dots,n$; $y^{(k)}(1) = 0,\ k = 0,\dots,n$ respectively.

EXAMPLE 3. Let $p(x) = x^2 - \frac{1}{6}$ then y in Theorem 5 is $y(x) = \frac{x^2}{12}(x^2-1) \leqslant 0$. Moreover $P_0 = \frac{1}{3}$ and $P_1 = 0$ so we get $\int_{-1}^{1} f(x)\left(x^2 - \frac{1}{6}\right)dx \leqslant \frac{f(1)+f(-1)}{6}$ for f convex. For a non-symmetric example, let $p(x) = x^2 - x$ so that $P_0 = -P_1 = \frac{2}{3}$. Then $y(x) = \frac{1}{12}(x^2-1)(x-1)^2 \leqslant 0$ and we have

$$\int_{-1}^{1} f(x)(x^2-x)dx \leqslant \frac{2}{3}f(1) \text{ for } f \text{ convex.}$$

We cannot expect the result in Theorem 5 to be best possible. Convexity from an inequality which is an upper bound on f seems impossible.

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