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DISTORTION INEQUALITIES FOR RUSCHEWEYH DERIVATIVES

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Abstract. Let $\mathscr{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathscr{U}$ with $f(0)=0$ and $f'(0)=1$. For $f(z)\in\mathscr{A}$, the Ruscheweyh derivative of order λ is denoted by $\mathscr{D}^{\lambda}f(z)$. The object of the present paper is to derive several distortion inequalities involving $\mathscr{D}^{\lambda}f(z)$ for certain classes of univalent functions $f(z)$ by applying known properties of generalized hypergeometric functions.

1. Introduction

Let $\mathscr{A}$ be the class of functions $f(z)$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathscr{U}:=\{z:z\in\mathbb{C}\quad\text{and}\quad |z|<1\}.$$

Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of all *univalent* functions in $\mathscr{U}$. Further, let $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ be the subclasses of $\mathscr{A}$ consisting, respectively, of functions which are *starlike of order* α $(0\leqslant\alpha<1)$ and *convex of order* α $(0\leqslant\alpha<1)$ in $\mathscr{U}$.

It is well-known (*cf.* Robertson [1]) that

$$f\in\mathscr{S}^*(\alpha)\Rightarrow|a_k|\leqslant\frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!}\qquad(k\in\mathbb{N}\setminus\{1\};\quad\mathbb{N}:=\{1,2,3,\cdots\}) \tag{1.2}$$

and

$$f\in\mathscr{K}(\alpha)\Rightarrow|a_k|\leqslant\frac{\prod_{j=2}^{k}(j-2\alpha)}{k!}\qquad(k\in\mathbb{N}\setminus\{1\}). \tag{1.3}$$

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For $f_j(z) \in \mathscr{A}$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}\, z^k \qquad (j = 1, 2), \tag{1.4}$$

the Hadamard product (or convolution) $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{1.5}$$

Using the convolution (1.5), Ruscheweyh [2] introduced what is now referred to as the Ruscheweyh derivative $\mathscr{D}^{\lambda} f(z)$ of order λ of $f(z) \in \mathscr{A}$ by

$$\mathscr{D}^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1). \tag{1.6}$$

It follows that

$$\mathscr{D}^0 f(z) = f(z), \quad \mathscr{D}^1 f(z) = z f'(z),$$

and, in general,

$$\mathscr{D}^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \qquad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right).$$

Furthermore, we have

$$\mathscr{D}^{\lambda} f(z) = z + \sum_{k=2}^{\infty} C(\lambda, k)\, a_k\, z^k, \tag{1.7}$$

where

$$C(\lambda, k) = \frac{\prod_{j=1}^{k-1} (j + \lambda)}{(k-1)!} \qquad \left(k \in \mathbb{N} \setminus \{1\}\right). \tag{1.8}$$

The generalized hypergeometric function ${}_pF_q(z)$ is given by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; z\right) \\ &= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} \, \frac{z^k}{k!} \qquad (p \leqslant q + 1), \end{aligned} \tag{1.9}$$

where p and q are non-negative integers, a_j $(j = 1, \cdots, p)$ and b_j $(j = 1, \cdots, q)$ are complex numbers with $b_j \neq 0, -1, -2, \cdots$. Here $(\lambda)_k$ denotes the Pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}). \end{cases} \tag{1.10}$$

If we set

$$\omega = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \ , \tag{1.11}$$

we see that the series ${}_pF_q(z)$, with $p = q+1$, is

(i) absolutely convergent for $|z| = 1$, if $\operatorname{Re}(\omega) > 0$,

(ii) conditionally convergent for $|z| = 1$ $(z \neq 1)$ if $-1 < \operatorname{Re}(\omega) \leqslant 0$, and

(iii) divergent for $|z| = 1$ if $\operatorname{Re}(\omega) \leqslant -1$.

If $p < q+1$ and $\operatorname{Re}(\omega) > 0$, then the ${}_pF_q(z)$ series (1.9) is absolutely convergent for $|z| < \infty$.

2. Distortion Inequalities for Starlike Functions

Our first distortion inequality involving Ruscheweyh derivatives is contained in

THEOREM 1. *If a function $f(z)$ given by (1.1) belongs to the class $\mathscr{S}^*(\alpha)$, then*

$$\begin{aligned} |\mathscr{D}^\lambda f(z)| &\leqslant M(n,\lambda,\alpha;\ |z|) \\ &+ \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\, |z|^n \ {}_3F_2\left(\begin{matrix} n+\lambda, n+1-2\alpha, 1; \\ n,\ n; \end{matrix}\ |z|\right), \end{aligned} \tag{2.1}$$

where $n \in \mathbb{N} \setminus \{1,2\}$ and

$$M(n,\lambda,\alpha;\ |z|) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\, |z|^k. \tag{2.2}$$

Proof. We begin by noting that

$$C(\lambda,k) = \frac{\prod_{j=1}^{k-1}(j+\lambda)}{(k-1)!} = \frac{(1+\lambda)_{k-1}}{(k-1)!}\,, \tag{2.3}$$

and

$$|a_k| \leqslant \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} = \frac{(2-2\alpha)_{k-1}}{(k-1)!} \qquad (f \in \mathscr{S}^*(\alpha))\,. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$
\begin{aligned}
|\mathscr{D}^{\lambda} f(z)| &\leqslant |z| + \sum_{k=2}^{\infty} C(\lambda, k)\, |a_k|\, |z|^k \\
&\leqslant |z| + \sum_{k=2}^{\infty} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\, |z|^k \\
&= \left\{ |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\, |z|^k \right\} \\
&\qquad + \sum_{k=n}^{\infty} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\, |z|^k \\
&= M(n, \lambda, \alpha;\, |z|) + \sum_{k=0}^{\infty} \frac{(1+\lambda)_{k+n-1}(2-2\alpha)_{k+n-1}}{\{(k+n-1)!\}^2}\, |z|^{k+n}.
\end{aligned}
\tag{2.5}
$$

Since

$$(1+\lambda)_{k+n-1} = (1+\lambda)_{n-1}(n+\lambda)_k, \tag{2.6}$$

$$(2-2\alpha)_{k+n-1} = (2-2\alpha)_{n-1}(n+1-2\alpha)_k\,, \tag{2.7}$$

and

$$(k+n-1)! = (n-1)!\,(n)_k\ , \tag{2.8}$$

we see that

$$
\begin{aligned}
|\mathscr{D}^{\lambda} f(z)| &\leqslant M(n, \lambda, \alpha;\, |z|) \\
&\quad + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\, |z|^n \sum_{k=0}^{\infty} \frac{(n+\lambda)_k (n+1-2\alpha)_k}{\{(n)_k\}^2}\, |z|^k \\
&= M(n, \lambda, \alpha;\, |z|) \\
&\quad + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\, |z|^n \; {}_3F_2\left(\begin{matrix} n+\lambda,\ n+1-2\alpha,\ 1; \\ n,\ n; \end{matrix}\ |z| \right).
\end{aligned}
\tag{2.9}
$$

COROLLARY 1. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{S}^*(\alpha)$, *then*

$$
\begin{aligned}
|\mathscr{D}^{m} f(z)| &\leqslant M(n, m, \alpha;\, |z|) + \frac{(1+m)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\, |z|^n \\
&\cdot \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(n+1-2\alpha)_k (1)_k}{\{(n)_k\}^2}\ \frac{|z|^k}{(1-|z|)^{k+2-2\alpha}}\ {}_2F_1\left(\begin{matrix} 2\alpha-1,\ n-1; \\ n+k; \end{matrix}\ |z| \right) \right\},
\end{aligned}
\tag{2.10}
$$

where $m \in \mathbb{N}$.

Proof. Note that (*cf.* Srivastava [3])

$$
{}_pF_q\left(\begin{matrix} b_1+m, & a_2,\cdots,a_p; \\ b_1, & b_2,\cdots,b_q; \end{matrix}\ z\right)
= \sum_{k=0}^{m}\binom{m}{k}\frac{\prod_{j=2}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k}\, z^k\ {}_{p-1}F_{q-1}\left(\begin{matrix} a_2+k,\cdots,a_p+k; \\ b_2+k,\cdots,b_q+k; \end{matrix}\ z\right) \tag{2.11}
$$

for $m \in \mathbb{N}$, and

$$
{}_2F_1\left(\begin{matrix} a_1,a_2; \\ b_1; \end{matrix}\ z\right) = (1-z)^{b_1-a_1-a_2}\ {}_2F_1\left(\begin{matrix} b_1-a_1,\ b_1-a_2; \\ b_1; \end{matrix}\ z\right). \tag{2.12}
$$

Therefore, we have

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} n+m, & n+1-2\alpha, & 1; \\ & n, & n; \end{matrix}\ |z|\right) \\
&= \sum_{k=0}^{m}\binom{m}{k}\frac{(n+1-2\alpha)_k(1)_k}{\{(n)_k\}^2}\,|z|^k\ {}_2F_1\left(\begin{matrix} n+1+k-2\alpha, & 1+k; \\ & n+k; \end{matrix}\ |z|\right) \\
&= \sum_{k=0}^{m}\binom{m}{k}\frac{(n+1-2\alpha)_k(1)_k}{\{(n)_k\}^2}\ \frac{|z|^k}{(1-|z|)^{k+2-2\alpha}}\ {}_2F_1\left(\begin{matrix} 2\alpha-1, & n-1; \\ & n+k; \end{matrix}\ |z|\right).
\end{aligned} \tag{2.13}
$$

The assertion of Corollary 1 follows from (2.13).

COROLLARY 2. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{S}^*\left(\frac{1}{2}\right)$, *then*

$$
\begin{aligned}
|\mathscr{D}^\lambda f(z)| \leqslant\ & M\left(n,\lambda,\frac{1}{2};\ |z|\right) \\
&+ \frac{(1+\lambda)_{n-1}}{(n-1)!}\ \frac{|z|^n}{(1-|z|)^{1+\lambda}}\ {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n; \end{matrix}\ |z|\right),
\end{aligned} \tag{2.14}
$$

where

$$
M\left(n,\lambda,\tfrac{1}{2};\ |z|\right) = |z| + \sum_{k=2}^{n-1}\frac{(1+\lambda)_{k-1}}{(k-1)!}\,|z|^k. \tag{2.15}
$$

Further, taking $\alpha = 0$ in Theorem 1, we have

COROLLARY 3. *If a function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}^*$, then*

$$\begin{aligned}
\left|\mathcal{D}^{\lambda} f(z)\right| &\leqslant M(n,\lambda,0;\,|z|) \\
&\quad + \frac{n(1+\lambda)_{n-1}}{(n-1)!}\,|z|^n \left\{ \frac{1}{(1-|z|)^{1+\lambda}}\; {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n; \end{matrix}\;|z|\right)\right. \qquad (2.16)\\
&\qquad \left. + \frac{n+\lambda}{n^2}\,\frac{|z|}{(1-|z|)^{2+\lambda}}\; {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n+1; \end{matrix}\;|z|\right)\right\},
\end{aligned}$$

where

$$M(n,\lambda,0;\,|z|) = |z| + \sum_{k=1}^{n-1} \frac{k(1+\lambda)_{k-1}}{(k-1)!}\,|z|^k. \tag{2.17}$$

3. Distortion Inequalities for Convex Functions

For the Ruscheweyh derivatives of convex functions belonging to the class $\mathcal{K}(\alpha)$, we have

THEOREM 2. *If a function $f(z)$ given by (1.1) belongs to the class $\mathcal{K}(\alpha)$, then*

$$\begin{aligned}
\left|\mathcal{D}^{\lambda} f(z)\right| &\leqslant N(n,\lambda,\alpha;\,|z|) \\
&\quad + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{n!\,(n-1)!}\,|z|^n\; {}_3F_2\left(\begin{matrix} \lambda+n, & n+1-2\alpha, & 1; \\ & n, & n+1; \end{matrix}\;|z|\right),
\end{aligned} \tag{3.1}$$

where $n \in \mathbb{N}\setminus\{1,2\}$ and

$$N(n,\lambda,\alpha;\,|z|) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{k!\,(k-1)!}\,|z|^k. \tag{3.2}$$

Proof. Using the fact that

$$\begin{aligned}
|a_k| &\leqslant \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} \\
&= \frac{(2-2\alpha)_{k-1}}{k!} \qquad (k\in\mathbb{N}\setminus\{1\};\quad f\in\mathcal{K}(\alpha)),
\end{aligned} \tag{3.3}$$

we readily arrive at the inequality (3.1) by applying the proof of Theorem 1 *mutatis mutandis*.

COROLLARY 4. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{K}(\alpha)$, *then*

$$|\mathscr{D}^m f(z)| \leqslant N(n,m,\alpha;\ |z|) + \frac{(1+m)_{n-1}(2-2\alpha)_{n-1}}{n!\ (n-1)!} \cdot \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(n+1-2\alpha)_k (1)_k}{(n)_k\,(n+1)_k}\ \frac{|z|^k}{(1-|z|)^{k+1-2\alpha}}\ {}_2F_1\left(\begin{matrix} 2\alpha,\ n; \\ n+k+1; \end{matrix}\ |z|\right)\right\}, \tag{3.4}$$

where $m \in \mathbb{N}$.

Setting $\alpha = \frac{1}{2}$ in Theorem 2, we have

COROLLARY 5. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{K}\left(\frac{1}{2}\right)$, *then*

$$\left|\mathscr{D}^\lambda f(z)\right| \leqslant N\left(n,\ \lambda,\ \frac{1}{2};\ |z|\right) + \frac{(1+\lambda)_{n-1}}{n!}\ \frac{|z|^n}{(1-|z|)^\lambda}\ {}_2F_1\left(\begin{matrix} 1-\lambda,\ n; \\ n+1; \end{matrix}\ |z|\right), \tag{3.5}$$

where

$$N\left(n,\lambda,\tfrac{1}{2};\ |z|\right) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}}{k!}\ |z|^k. \tag{3.6}$$

Finally, letting $\alpha = 0$ in Theorem 2, we have

COROLLARY 6. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{K}$, *then*

$$\left|\mathscr{D}^\lambda f(z)\right| \leqslant N(n,\lambda,0;\ |z|) + \frac{(1+\lambda)_{n-1}}{(n-1)!}\ \frac{|z|^n}{(1-|z|)^{1+\lambda}}\ {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n; \end{matrix}\ |z|\right), \tag{3.7}$$

where

$$N(n,\lambda,0;\ |z|) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}}{(k-1)!}\ |z|^k. \tag{3.8}$$

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