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A GEOMETRIC MEAN INEQUALITY AND SOME MONOTONICITY RESULTS FOR THE q–GAMMA FUNCTION

N. ELEZOVIĆ, C. GIORDANO AND J. PEČARIĆ

(*communicated by A. Laforgia*)

Abstract. A geometric mean inequality and some monotonicity results for the q-Gamma function are proved.

1. Introduction and preliminary results

A geometric mean inequality for the Gamma function was proved by A. Laforgia and S. Sismondi [1]:

THEOREM 1.1. *Let $\Gamma(x)$ be the Gamma function. Then for $x > 0$ and $0 < \lambda < 1$ the following inequality*

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+\lambda)} \cdot \frac{\Gamma(1/x+1)}{\Gamma(1/x+\lambda)}\right]^{1/2} \geqslant \frac{1}{\Gamma(\lambda+1)} \tag{1.1}$$

holds. Equality is assumed when $x = 1$ and, of course, when $\lambda = 0$ and $\lambda = 1$. In the case $\lambda > 1$ (1.1) must be reversed.

Note that $\Gamma(x+1)\Gamma(1/x+1) = \Gamma(x)\Gamma(1/x)$, so that (1.1) can be written in the following form

$$\left[\frac{\Gamma(x)}{\Gamma(x+\lambda)} \cdot \frac{\Gamma(1/x)}{\Gamma(1/x+\lambda)}\right]^{1/2} \geqslant \frac{1}{\Gamma(\lambda+1)}. \tag{1.2}$$

The following monotonicity theorem was proved by D. Kershaw and A. Laforgia [2]:

THEOREM 1.2. *We have that*

$$\left[\Gamma\left(1+\frac{1}{x}\right)\right]^x \quad \textit{decreases with } x > 0 \tag{1.3}$$

and

$$x\left[\Gamma\left(1+\frac{1}{x}\right)\right]^x \quad \textit{increases with } x > 0. \tag{1.4}$$

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In this paper we shall prove some analoguous results for the q-Gamma function Γ_q:

$$\Gamma_q(x) := (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \qquad (0<q<1). \tag{1.5}$$

Note that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^-$, and that

$$\begin{aligned} \Psi_q(x) &:= \Gamma_q'(x)/\Gamma_q(x) \\ &= -\log(1-q) + \log q \sum_{n=0}^{\infty} q^{n+x}/(1-q^{n+x}) \\ &= -\log(1-q) + \log q \sum_{k=1}^{\infty} q^{kx}/(1-q^k). \end{aligned} \tag{1.6}$$

The following Stieltjes integral representation of Ψ_q is valid:

$$\Psi_q(x) = -\log(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t), \qquad 0<q<1,\ x>0, \tag{1.7}$$

where γ_q is a discrete measure with positive masses $-\log q$ at the positive points $-k\log q$, $k=1,2,\ldots$, i.e.

$$\gamma_q(t) = -\log q \sum_{k=1}^{\infty} \delta(t + k\log q), \qquad 0<q<1. \tag{1.8}$$

Note that ([3])

$$\int_0^{\infty} e^{-xt} d\gamma_q(t) = \frac{-q^x \log q}{1-q^x}, \qquad 0<q<1,\ x>0. \tag{1.9}$$

2. Results

THEOREM 2.1. *Let $\Gamma_q(x)$ be the q-Gamma function. Then for $x>0$ and $0<\lambda<1$ the following inequality*

$$\left[\frac{\Gamma_q(x)}{\Gamma_q(x+\lambda)} \cdot \frac{\Gamma_q(1/x)}{\Gamma_q(1/x+\lambda)}\right]^{1/2} \geqslant \frac{1}{\Gamma_q(\lambda+1)}. \tag{2.1}$$

holds. Equality is assumed when $x=1$ and, of course, when $\lambda=0$ and $\lambda=1$. In the case $\lambda>1$ (2.1) must be reversed.

Proof. We shall use idea of proof from [1] (with some simplifications).

In view of the invariance of inequality (2.1) under the substitution $x \mapsto 1/x$, we can consider only the case $0<x<1$.

Let us define functions

$$F(x) := \frac{\Gamma_q(x)}{\Gamma_q(x+\lambda)} \cdot \frac{\Gamma_q(1/x)}{\Gamma_q(1/x+\lambda)} \tag{2.2}$$

and

$$g(x) = \frac{d}{dx} \log F(x). \tag{2.3}$$

By (1.7) we can obtain

$$g(x) = \int_0^\infty \frac{1-\exp[-\lambda t]}{1-\exp[-t]} \left(\frac{1}{x^2} \exp\left[-\frac{1}{x}t\right] - \exp[-xt] \right) dt. \tag{2.4}$$

It is interesting that same function appears in [1], so we can deal as there. Let us define functions

$$\varphi_1(t) := \begin{cases} (1-e^{-\lambda t})/(1-e^{-t}), & t > 0, \\ \lambda, & t = 0; \end{cases} \tag{2.5}$$

and

$$\varphi_2(t) := \frac{1}{x^2} \exp\left[-\frac{1}{x}t\right] - \exp[-xt], \quad t \geqslant 0. \tag{2.6}$$

Concerning the function φ_2 we have that under our hypothesis $0 < x < 1$, there exists one and only one value $t_0 > 0$ such that $\varphi_2(t_0) = 0$. Moreover, $\varphi_2(t) > 0$ if $t < t_0$ and $\varphi_2(t) < 0$ if $t > t_0$ (see [1]). From (2.5) we have

$$\varphi_1'(t) = \frac{e^{-\lambda t}}{(1-e^{-t})^2} \Big[(1-\lambda)\exp[-t] + \lambda - \exp[-t(1-\lambda)] \Big] \tag{2.7}$$

Thus, φ_1' is nonnegative if and only if

$$(1-\lambda)\exp[-t] + \lambda - \exp[-t(1-\lambda)] \geqslant 0. \tag{2.8}$$

For $e^{-t} = z$, (2.8) becomes

$$(1-\lambda)z + \lambda z \geqslant z^{1-\lambda}. \tag{2.9}$$

what is the well-known arithmetic–geometric mean inequality for numbers z and 1. Therefore (2.9) (and so (2.8)) is true for $\lambda \in (0,1)$, while for $\lambda > 1$ we have reverse inequalities in (2.9) and (2.8). So, for $0 < \lambda < 1$, φ_1' is nonnegative and for $\lambda > 1$ nonpositive, i.e., φ_1 is increasing and decreasing respectively.

Now application of Theorem of the mean leads to

$$\begin{aligned} q(x) &= \int_0^\infty \varphi_1(t)\varphi_2(t)dt = \int_0^{t_0} \varphi_1(t)\varphi_2(t)dt + \int_{t_0}^\infty \varphi_1(t)\varphi_2(t)dt \\ &= \varphi_1(t_1)\int_0^{t_0} \varphi_2(t)dt + \varphi_1(t_2)\int_{t_0}^\infty \varphi_2(t)dt \\ &= \frac{\exp[-xt_0] - \exp[-t_0/x]}{x}[\varphi_1(t_1) - \varphi_1(t_2)], \\ &\qquad 0 < t_1 < t_0, \quad t_0 < t_2 < \infty, \quad 0 < x < 1. \end{aligned}$$

So, the sign of q depends only of $\varphi_1(t_1) - \varphi_1(t_2)$ what is negative for $0 < \lambda < 1$ and positive for $\lambda > 1$. Therefore, for $0 < x < 1$ the function $x \mapsto F(x)$ defined by (2.2) decreases for $0 < \lambda < 1$ and increases for $\lambda > 1$. Since $F(1) = [\Gamma(\lambda+1)]^{-2}$, we have

$$F(x) \geqslant [\Gamma(\lambda+1)]^{-2}, \quad \text{for } 0 < \lambda < 1,$$

and

$$F(x) \leqslant [\Gamma(\lambda+1)]^{-2}, \quad \text{for } \lambda > 1.$$

When $x = 1$ and λ is an arbitrary positive number, (2.1) reduces to an inequality. Similarly, when $\lambda = 0$ or $\lambda = 1$ the equality sign holds for arbitrary positive x. This can be checked by direct substitution in (2.1).

REMARK. For $q \to 1^-$ we get (1.2).

THEOREM 2.2. *We have that*

$$\left[\Gamma_q\left(1+\frac{1}{x}\right)\right]^x \quad \textit{decreases with } x > 0 \tag{2.10}$$

and

$$x\left[\Gamma_q\left(1+\frac{1}{x}\right)\right]^x \quad \textit{increases with } x > 0 \tag{2.11}$$

Proof. Let, for $x > 0$, $f(x)$ be defined as in [2], by

$$f(x) = x^\alpha \left[\Gamma_q\left(1+\frac{1}{x}\right)\right]^x$$

where $\alpha = 0$ or $\alpha = 1$.

To obtain monotonicity results for f, we need to know the sign of the function

$$q(x) := \frac{f'(x)}{f(x)} = \frac{\alpha}{x} + \log\Gamma_q\left(1+\frac{1}{x}\right) - \frac{1}{x}\Psi_q\left(1+\frac{1}{x}\right). \tag{2.12}$$

With $h(y) = q(1/x)$, we get

$$h(y) = \alpha y + \log\Gamma_q(1+y) - y\Psi_q(1+y). \tag{2.13}$$

Clearly, we have

$$h(0) = 0, \qquad h'(y) = \alpha - y\Psi_q'(1+y). \tag{2.14}$$

For $\alpha = 0$ we have at once $h'(y) < 0$, and since $h(0) = 0$ we have $h(y) < 0$, i.e. $g(1/x) < 0$, which gives (2.10).

When $\alpha = 1$, (2.14) gives

$$h'(y) = 1 - y\Psi_q'(1+y)$$

and by (1.7)

$$\begin{aligned} h'(y) &= 1 - y\int_0^\infty \frac{t}{1-e^{-t}} e^{-t(1+y)}\,d\gamma_q(t) \\ &= 1 - y\int_0^\infty \frac{t}{e^t - 1} e^{-yt}\,d\gamma_q(t). \end{aligned} \tag{2.15}$$

Note that

$$\frac{t}{e^t-1} = \frac{t}{t + t^2/2 + \ldots} < 1, \qquad (t > 0),$$

so (2.15) gives

$$h'(y) > 1 - y\int_0^\infty e^{-yt}d\gamma_q(t) = 1 + y\frac{q^y \log q}{1-q^y}. \tag{2.16}$$

Now, we shall prove that the last expression is positive, that is

$$-y\frac{q^y \log q}{1-q^y} \leqslant 1 \tag{2.17}$$

i.e. with $a = 1/q$

$$y\frac{\log a}{a^y - 1} \leqslant 1$$

or

$$y \log a \leqslant a^y - 1. \tag{2.18}$$

Note that the function $f(y) = a^y - 1$ is convex, so its graph is above the tangent at $y = 0$. This gives (2.18). So, now we have $h(y) > 0$ which gives (2.11).

Moreover we can give the following generalization of the above result for function (2.11).

THEOREM 2.3. *Let*

$$\alpha \geqslant \frac{q\log q}{q-1}. \tag{2.19}$$

Then

$$x^\alpha\left[\Gamma_q\left(1+\frac{1}{x}\right)\right]^x \quad \textit{increases with } x > 0. \tag{2.20}$$

REMARK. It is easy to see that we can always take $\alpha < 1$ since $q\log q/(q-1) < 1$ for $0 < q < 1$ is equivalent to $\log u < u - 1$ for $u > 1$.

Proof. With the same notation as in Theorem 2.2, we shall prove

$$\begin{aligned} h'(y) &= \alpha - y\int_0^\infty \frac{t}{e^t - 1}e^{-yt}d\gamma_q(t) \\ &= \alpha - y\int_{-\log q}^\infty \frac{t}{e^t - 1}e^{-yt}d\gamma_q(t). \end{aligned}$$

It is sufficient to prove

$$\frac{t}{e^t - 1} < \alpha \quad \text{for} \quad t > -\log q.$$

Denote $g(t) = \alpha(e^t - 1) - t$. For $0 < \alpha < 1$, g has exactly two real roots, $t = 0$ and $t = t_1 > 0$, and it holds $g(t) > 0$ for $t > t_1$. But,

$$g(-\log q) = \alpha\frac{1-q}{q} + \log q \geqslant 0$$

because of (2.19). Hence, we have

$$
\begin{aligned}
h'(y) &> \alpha\left(1 - y\int_0^\infty e^{-yt}\,d\gamma_q(t)\right) \\
&= \alpha\left(1 + y\frac{q^y \log q}{1-q^y}\right)
\end{aligned}
$$

and we can proceed as in Theorem 2.2.

REMARK. Note that Theorem 2.2 and Theorem 2.3 give further extension of some results from [4].

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Neven Elezović
Department of Applied Mathematics
Faculty of Electrical Engineering and Computing
Unska 3, 10 000 Zagreb, Croatia
e-mail: neven.elezovic@fer.hr

Carla Giordano
Department of Mathematics
University of Torino
Via Carlo Alberto 10
10123 Torino, Italy
e-mail: giordano@dm.unito.it

Josip Pečarić
Faculty of Textile Technology,
University of Zagreb,
Pierottijeva 6,
10 000 Zagreb, Croatia
pecaric@hazu.hr