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NORMS AND DETERMINANTS OF PRODUCTS OF LOGARITHMIC FUNCTIONS OF POSITIVE SEMI–DEFINITE OPERATORS

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Abstract. Let A, B be bounded positive semi-definite operators (matrices) on a Hilbert space. We will show

$$||\log(1+A)\log(1+B)|| \leqslant \{\log(1+||AB||^{\frac{1}{2}})\}^2,$$

and

$$||\log(1+B)\log(1+A)\log(1+B)|| \leqslant \{\log(1+||BAB||^{\frac{1}{3}})\}^3.$$

Further we will prove the corresponding determinantal inequalities.

Let A and B be bounded positive semi-definite operators (or matrices) on a Hilbert space. The following inequalities are known ([3], [2, Section 9.2]):

$$||A^aB^a|| \leqslant ||AB||^a \text{ and } ||B^aA^aB^a|| \leqslant ||BAB||^a \qquad (0 \leqslant a \leqslant 1).$$

These mean

$$||f(A)f(B)|| \leqslant f(||AB||) \text{ and } ||f(B)f(A)f(B)|| \leqslant f(||BAB||),$$

where $f(t) = t^a$. We want to get similar inequalities for $f(t) = \log(1+t)$. But since there is a positive number b such that $\{\log(1+b)\}^2 > \log(1+b^2)$, the inequality is not valid even for the case $A = B = bI$. We can write, however, the above inequalities also in the following forms:

$$||f(A)f(B)|| \leqslant \{f(||AB||^{\frac{1}{2}})\}^2 \text{ and } ||f(B)f(A)f(B)|| \leqslant \{f(||BAB||^{\frac{1}{3}})\}^3,$$

where $f(t) = t^a$.

The aim of this short note is to show that the inequalities of these types hold for $f(t) = \log(1+t)$.

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THEOREM 1. *If A, B are bounded positive semi-definite operators, then*

$$|| \log(1+A)\log(1+B)|| \leqslant \{\log(1+||AB||^{\frac{1}{2}})\}^2, \tag{1}$$

and

$$|| \log(1+B)\log(1+A)\log(1+B)|| \leqslant \{\log(1+||BAB||^{\frac{1}{3}}\}^3. \tag{2}$$

Actually we will establish more general inequalities for a class of functions. Further we will show

THEOREM 2. *If A, B are positive semi-definite matrices, then*

$$\det[\log(1+A)\log(1+B)] \leqslant \det[\log(1+|AB|^{\frac{1}{2}})]^2, \tag{3}$$

and

$$\det[\log(1+B)\log(1+A)\log(1+B)] \leqslant \det[\log(1+(BAB)^{\frac{1}{3}})]^3. \tag{4}$$

Recall that a real-valued continuous function $f(t)$ on $[0,\infty)$ is said to be *operator monotone* if $f(A) \geqslant f(B)$ whenever $A \geqslant B \geqslant 0$. Löwner's theorem says that $f(t)$ is operator monotone on $[0,\infty)$ if and only if it has an analytic extension $f(z)$ to the upper half plane such that $Im\, f(z) \geqslant 0$ for $Im\, z > 0$.

A consequence of this characterization is that if $f(t) \geqslant 0$ is operator monotone, so is $f(t^p)^{\frac{1}{p}}$ $(0 < p \leqslant 1)$.

Examples of operator monotone functions are t^p $(0 < p \leqslant 1)$, $\frac{t}{s+t}$ with $s > 0$ and $\log(1+t)$.

As one of the special properties common to these functions $f(t)$ we single out the property that $\log f(e^x)$ is concave on $(-\infty,\infty)$.

We can easily check this property for t^p $(0 < p \leqslant 1)$, $\frac{t}{s+t}$ with $s > 0$ and $\log(1+t)$ by calculating the second derivatives. But not all non-negative operator monotone function has this property as seen in $1+t^{\frac{1}{2}}$.

In the following theorems, (1) and (2) of Theorem 1 will be shown in much generalized forms.

THEOREM 3. *Let $0 \leqslant f(t)$ be an operator monotone function on $[0,\infty)$ such that $\log f(e^x)$ is concave on $(-\infty,\infty)$. Then for every $A, B \geqslant 0$, and for every $p, q > 0$,*

$$||f(B^{\frac{1}{q}})^q f(A^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q|| \leqslant f(||BAB||^{\frac{1}{2p+2q}})^{2p+2q}, \tag{5}$$

$$||f(A^{\frac{1}{p}})^p f(B^{\frac{1}{q}})^q|| \leqslant f(||AB||^{\frac{1}{p+q}})^{p+q}. \tag{6}$$

Proof. Since $\log f(e^x)$ is concave on $(-\infty,\infty)$ we can see

$$f(s^{\frac{1}{p}})^p f(t^{\frac{1}{q}})^q \leqslant f((st)^{\frac{1}{p+q}})^{p+q} \quad (s,t > 0;\ p,q > 0). \tag{7}$$

This implies

$$f(t^{\frac{1}{q}})^q f((\frac{s}{t^2})^{\frac{1}{2p}})^{2p} f(t^{\frac{1}{q}})^q \leqslant f(s^{\frac{1}{2p+2q}})^{2p+2q} \quad (s,t > 0\,;\ p,q > 0).$$

First, we assume $p \geqslant \frac{1}{2}$. If B is invertible, by the above, we have

$$f(B^{\frac{1}{q}})^q f((sB^{-2})^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q \leqslant f(s^{\frac{1}{2p+2q}})^{2p+2q}.$$

Here we set $s = ||BAB||$. Then since $A \leqslant sB^{-2}$ and $f(t^{\frac{1}{2p}})^{2p}$ is an operator monotone function for $p \geqslant \frac{1}{2}$, we have

$$f(A^{\frac{1}{2p}})^{2p} \leqslant (f(sB^{-2})^{\frac{1}{2p}})^{2p}.$$

Therefore, it follows from the above inequality that

$$f(B^{\frac{1}{q}})^q f(A^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q \leqslant f(||BAB||^{\frac{1}{2p+2q}})^{2p+2q}.$$

Thus we obtain (5) for invertible B. For general $B \geqslant 0$, approximate it by $B + \delta I$ and let $\delta \downarrow 0$. Consequently we have shown (5) for $p \geqslant \frac{1}{2}$.

Next, we assume $0 < p < \frac{1}{2}$. By (7) we have

$$f(s^{\frac{1}{2p}})^{2p} f(t^{\frac{1}{1-2p}})^{1-2p} \leqslant f(st) \quad (s,t > 0),$$

from which it follows that

$$f(A^{\frac{1}{2p}})^{2p} \leqslant f(t^{\frac{1}{1-2p}})^{-1+2p} f(tA) \quad (t > 0).$$

Using (5) for $p = \frac{1}{2}$, we get

$$\begin{aligned} ||f(B^{\frac{1}{q}})^q f(A^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q|| &\leqslant f(t^{\frac{1}{1-2p}})^{-1+2p} ||f(B^{\frac{1}{q}})^q f(tA) f(B^{\frac{1}{q}})^q|| \\ &\leqslant f(t^{\frac{1}{1-2p}})^{-1+2p} f(||tBAB||^{\frac{1}{1+2q}})^{1+2q}. \end{aligned}$$

By setting $t = ||BAB||^{\frac{1-2p}{2p+2q}}$, we can get (5) for $0 < p < \frac{1}{2}$.

Observe that (6) is equivalent to

$$||f(B^{\frac{1}{q}})^q f(A^{\frac{1}{p}})^{2p} f(B^{\frac{1}{q}})^q|| \leqslant f(||BA^2B||^{\frac{1}{2p+2q}})^{2p+2q},$$

which follows from (5) with A replaced by A^2. Thus we conclude the proof. $\square$

The special cases of (5) with $p = \frac{1}{2}, q = 1$ and of (6) with $p = q = 1$ have simpler forms:

$$||f(B)f(A)f(B)|| \leqslant f(||BAB||^{\frac{1}{3}})^3, \tag{8}$$

$$||f(A)f(B)|| \leqslant f(||AB||^{\frac{1}{2}})^2. \tag{9}$$

With $f(t) = \log(1+t)$ (8) and (9) reduce to (2) and (1) of Theorem 1, respectively. Further applying (9) to $f(t) = \frac{t}{t+1}$ we have

COROLLARY 4.

$$||(A+I)^{-1}AB(I+B)^{-1}|| \leqslant \frac{||AB||}{(1+||AB||^{\frac{1}{2}})^2}.$$

If $0 \leqslant f(t)$ is a non-constant operator monotone function on $[0,\infty)$ such that $\log f(e^x)$ is concave on $(-\infty,\infty)$, then necessarily $f(0) = 0$ and the inverse function f^{-1} is defined on $[0, ||f||_\infty)$. Therefore the following is an immediate consequence of (8) and (9).

COROLLARY 5. *Let* $0 \leqslant f(t)$ *be a non-constant operator monotone function on* $[0,\infty)$ *such that* $\log f(e^x)$ *is concave on* $(-\infty,\infty)$. *Then for* $A,B \geqslant 0$

$$f^{-1}(||AB||^{1/2}) \leqslant ||f^{-1}(A)f^{-1}(B)||^{1/2},$$

and

$$f^{-1}(||BAB||^{1/3}) \leqslant ||f^{-1}(B)f^{-1}(A)f^{-1}(B)||^{1/3}.$$

Since the inverse function of $\log(1+t)$ is $\exp t - 1$, this implies

COROLLARY 6. *For* $A,B \geqslant 0$

$$\exp(||AB||^{1/2}) - 1 \leqslant ||(\exp A - 1)(\exp B - 1)||^{1/2},$$

and

$$\exp(||BAB||^{1/3}) - 1 \leqslant ||(\exp B - 1)(\exp A - 1)(\exp B - 1)||^{1/3}.$$

Remark here that the following inequality is known for Hermitian A,B (see [2, p. 261])

$$||\exp(A+B)|| \leqslant ||\exp A \exp B||.$$

In the sequel we consider $N \times N$ positive semi-definite matrices. In the following theorem, (3) of Theorem 2 will be shown in much generalized forms. Then (4) will follow as a corollary.

THEOREM 7. *Let* $f(t)$ *be a (not necessarily operator monotone) non-negative function on* $[0,\infty)$ *such that* $\log f(e^x)$ *is concave on* $(-\infty,\infty)$. *Then for every* $N \times N$ *matrices* $A,B \geqslant 0$ *and for every* $p,q > 0$

$$\det\{f(A^{\frac{1}{p}})^p f(B^{\frac{1}{q}})^q\} \leqslant \det f(|AB|^{\frac{1}{p+q}})^{p+q}, \tag{10}$$

where $|X| = (X^*X)^{\frac{1}{2}}$.

Proof. Arrange the eigenvalue of each matrix $X \geqslant 0$ in decreasing order

$$\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_N(X).$$

First assume that both A,B are positive definite. A. Horn's theorem (see [2, p.72]) says that

$$\prod_{i=1}^{k} \lambda_i(|AB|) \leqslant \prod_{i=1}^{k} \lambda_i(A)\lambda_i(B) \quad (k = 1,2,\ldots,N-1)$$

and

$$\prod_{i=1}^{N} \lambda_i(|AB|) = \det(A)\det(B) = \prod_{i=1}^{N} \lambda_i(A)\lambda_i(B).$$

These mean the sequence $\{\log \lambda_i(|AB|)\}_{i=1}^N$ is *majorized* by $\{\log(\lambda_i(A)\lambda_i(B))\}_{i=1}^N$. Then according to a general theorem on majorization (see [1], [4], [2, Section 2.3]) if $g(e^x)$ is convex on $(-\infty,\infty)$, then

$$\sum_{i=1}^{N} g(\lambda_i(|AB|)) \leqslant \sum_{i=1}^{N} g(\lambda_i(A)\lambda_i(B)).$$

Since for any $\mu > 0$ the function $-\log f(e^{\frac{x}{\mu}})^{\mu}$ is convex on $(-\infty, \infty)$ by assumption, we have

$$-\sum_{i=1}^{N} \log f((\lambda_i(|AB|))^{\frac{1}{\mu}})^{\mu} \leqslant -\sum_{i=1}^{N} \log f((\lambda_i(A)\lambda_i(B))^{\frac{1}{\mu}})^{\mu},$$

from which it follows that

$$\prod_{i=1}^{N} f((\lambda_i(A)\lambda_i(B))^{\frac{1}{\mu}})^{\mu} \leqslant \prod_{i=1}^{N} f((\lambda_i(|AB|))^{\frac{1}{\mu}})^{\mu}.$$

Here we set $\mu = p + q$. Then by (7) we get

$$\begin{aligned}\prod_{i=1}^{N} f(\lambda_i(A)^{\frac{1}{p}})^{p} f(\lambda_i(B)^{\frac{1}{q}})^{q} &\leqslant \prod_{i=1}^{N} f((\lambda_i(A)\lambda_i(B))^{\frac{1}{p+q}})^{p+q} \\ &\leqslant \prod_{i=1}^{N} f(\lambda_i(|AB|)^{\frac{1}{p+q}})^{p+q}.\end{aligned}$$

This implies

$$\prod_{i=1}^{N} \lambda_i(f(A^{\frac{1}{p}})^{p})\lambda_i(f(B^{\frac{1}{q}})^{q}) \leqslant \prod_{i=1}^{N} \lambda_i(f(|AB|^{\frac{1}{p+q}}))^{p+q},$$

from which it follows that

$$\det f(A^{\frac{1}{p}})^{p} \cdot \det f(B^{\frac{1}{q}})^{q} \leqslant \det f(|AB|^{\frac{1}{p+q}})^{p+q}.$$

Thus we have proved (10) for invertible A, B. For general $A, B \geqslant 0$, approximate them by $A + \delta I, B + \delta I$ respectively and let $\delta \downarrow 0$. $\square$

Now (3) of Theorem 2 follows from (10) with $f(t) = \log(1+t)$ and $p = q = 1$.

COROLLARY 8. *Let $f(t)$ be a (not necessarily operator monotone) non-negative function on $[0, \infty)$ such that $\log f(e^x)$ is concave on $(-\infty, \infty)$. Then for matrices $A_1, \ldots, A_n \geqslant 0$ and for real numbers $p_1, \ldots, p_n > 0$*

$$\prod_{i=1}^{n} \det f(A_i^{\frac{1}{p_i}})^{p_i} \leqslant \det f(|\prod_{i=1}^{n} A_i|^{\frac{1}{p_1+\cdots+p_n}})^{p_1+\cdots+p_n}. \tag{11}$$

Proof. Since $||A_1A_2|A_3| = |A_1A_2A_3|$, by using Theorem 7, we can prove the assertion by induction on n. $\square$

(4) of Theorem 2 follows from (11) with $n = 3, A_1 = A_3 = B, A_2 = A$ and $p_1 = p_2 = p_3 = 1$.

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