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VALID INEQUALITIES AND CUTTING PLANES FOR SOME POLYTOPES

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Abstract. In this paper we consider multidimensional knapsack polytope. Some important concepts and preliminaries are given at the beginning. Then we give a result connected with valid and dominating inequalities for this polytope, and a modular arithmetic approach for valid inequalities and cutting planes generation for the (one-dimensional) knapsack polytope.

1. Introduction. Basic concepts, formulations and definitions

Denote by $\mathbf{R}$ the set of real numbers, by $\mathbf{R}^n$ the totality of vectors with n real components, by $\mathbf{R}_+$ the set of all real non-negative numbers and by $\mathbf{R}^n_+$ the totality of vectors with n real non-negative components. Analogous notations apply for the set of integers $\mathbf{Z}$.

DEFINITION 1. A *polyhedron* $P \subseteq \mathbf{R}^n$ is the set of points that satisfy a finite number of linear inequalities, i.e. $P = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} \leqslant \mathbf{b}, A_{m\times n}, \mathbf{b} \in \mathbf{R}^m\}$. A bounded polyhedron is called a *polytope*.

DEFINITION 2. [1], [10], [12]. Let $S \subset \mathbf{R}^n$, $\boldsymbol{\pi}$, $\mathbf{x} \in \mathbf{R}^n$. The inequality $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ (i.e. $\sum_{j=1}^n \pi_j x_j \leqslant \pi_0$) is said to be a *valid inequality* for S if it is satisfied by all $\mathbf{x} \in S$. Every inequality is valid for empty set $\emptyset$ by definition.

DEFINITION 3. [12]. The valid inequalities $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ and $\boldsymbol{\gamma}\mathbf{x} \leqslant \gamma_0$ are said to be *equivalent* if $\boldsymbol{\gamma} = \mu\boldsymbol{\pi}$ and $\gamma_0 = \mu\pi_0$ for some $\mu > 0$.

DEFINITION 4. [10], [12]. If the valid inequalities $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ and $\boldsymbol{\gamma}\mathbf{x} \leqslant \gamma_0$ are not equivalent and there exists $\lambda > 0$ such that $\lambda\boldsymbol{\gamma} \geqslant \boldsymbol{\pi}$ and $\lambda\gamma_0 \leqslant \pi_0$ then we say that $\boldsymbol{\gamma}\mathbf{x} \leqslant \gamma_0$ *dominates* or *is stronger than* $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ or that $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is *dominated by* or *is weaker than* $\boldsymbol{\gamma}\mathbf{x} \leqslant \gamma_0$.

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A *maximal valid inequality* is one that is not dominated by any other valid inequality.

DEFINITION 5. [10], [12]. If $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is a valid inequality for polyhedron P and $F = \{\mathbf{x} \in P : \boldsymbol{\pi}\mathbf{x} = \pi_0\}$, F is called a *face* of P and $\boldsymbol{\pi}\mathbf{x} = \pi_0$ is said to *represent* F. A face F is said to be *proper* for P if $F \neq \emptyset$ and $F \neq P$.

$\emptyset$ and P are faces of P by definition and they are called *improper faces*.

Let C be a closed convex set.

DEFINITION 6. [2], [12]. A face F of C such that $0 \leqslant dimF = dimC - 1$ is called a *facet* of C.

Equivalent definition is:

DEFINITION 7. [13], [14]. Let $P \subset \mathbf{R}^n$. The face defined by the inequality $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is called a *facet* of P if:

(i) $\mathbf{x} \in P$ implies $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ (i.e. $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is a valid inequality for P);

(ii) there exist exactly n affinely independent vertices $\mathbf{x}^i$ of P satisfying $\boldsymbol{\pi}\mathbf{x}^i \leqslant \pi_0$, $i = 1, \ldots, n$ with an equality, while all $\mathbf{x} \in P$ satisfy $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$.

We use the term *facet* for the face from definitions 6, 7 as well as for the inequality $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ which produces this facet.

DEFINITION 8. An additional linear constraint to the linear program is said to define a *regular cutting plane* (a *regular cut*) if it is:

a) not satisfied by the “nonintegral” solution to continuous linear problem;

b) satisfied by all integral points of the polyhedron of this problem.

Therefore cutting plane cuts off points from the polyhedron of continuous problem but does not cut off “integral” points.

According to definitions 6 (or 7) and 8 facets of the polytope

$$P_I \stackrel{\text{def}}{=} \operatorname{conv}\{\mathbf{x} \in \mathbf{Z}^n : \mathbf{a}\mathbf{x} \leqslant a_0, \mathbf{d}' \leqslant \mathbf{x} \leqslant \mathbf{d}\}$$

are regular cutting planes of corresponding “continuous” polytope.

M. W. Padberg [14] notices that while cutting planes generally constitute valid inequalities, facets of P_I generate “deepest” cutting planes, and facets belong to the class of inequalities that uniquely determines P_I.

REMARK 1. The inequality which defines a facet of polyhedron is the “strongest” inequality for this polyhedron.

Any maximal valid inequality for P defines a nonempty face of P_I and the set of maximal valid inequalities contains all of the facet-defining inequalities for P_I, where $P \subseteq \mathbf{Z}^n$ [12].

If $S \subseteq P$ and $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is a valid inequality for P then obviously $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is a valid inequality for S as well.

LEMMA 1. *Let variables* $x_1, \dots, x_n$ *be ordered such that*

$$\frac{c_1}{a_1} \geqslant \frac{c_2}{a_2} \geqslant \cdots \geqslant \frac{c_n}{a_n}. \tag{1}$$

Then the solution to the continuous knapsack problem

$$\max \mathbf{cx} = \sum_{j=1}^{n} c_j x_j \tag{2}$$

subject to

$$\sum_{j=1}^{n} a_j x_j \leqslant a_0 \tag{3}$$

$$x_j \in \mathbf{R}_+, \quad j = 1, \dots, n \tag{4}$$

is

$$x_1^* = \frac{a_0}{a_1}, \quad x_j^* = 0, \quad j = 2, \dots, n. \tag{5}$$

Lemma 1 is a corollary (with $x_j \in \mathbf{R}_+, j = 1, \dots, n$) of the following proposition.

PROPOSITION 1. *([3, p. 118–121] — equivalent formulation)*
Let variables $x_1, \dots, x_n$ *be ordered so that (1) holds. Then the solution to the continuous problem (2), (3),*

$$0 \leqslant x_j \leqslant d_j, \quad j = 1, \dots, n \tag{4'}$$

is $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ *with*

$$x_j^* = d_j, \quad j = 1, \dots, r_0,$$

$$x_{r_0+1}^* = \frac{1}{a_{r_0+1}} \left(a_0 - \sum_{j=1}^{r_0} a_j d_j \right),$$

$$x_j^* = 0, \quad j = r_0 + 2, \dots, n$$

where $r_0 = \max\{r : \sum_{j=1}^{r} a_j d_j \leqslant a_0\}$.

When $x_j \in \mathbf{R}_+$, $j = 1, \dots, n$, that is, $d_j = +\infty$, $j = 1, \dots, n$ then $r_0 = 0$.

2. Equivalent, dominating and valid inequalities

Following Theorem 1 is a generalization of Proposition 1.1 [12, p. 208] (which refers to polyhedron $\{\mathbf{x} \in \mathbf{R}_+^n : A\mathbf{x} \leqslant \mathbf{b}\}$) and we use analogous formulation. Theorem 1 relates to so-called multidimensional knapsack problem and to its polytope denoted by $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d})$ below.

THEOREM 1. *Let* A *be an* $m \times n$ *matrix with columns* $\mathbf{a}_j$, $j = 1, \ldots, n$; $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m$; $\mathbf{d}' = (d'_1, \ldots, d'_n)$, $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{R}^n$;

$$P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d}) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} \leqslant \boldsymbol{\alpha}, \quad d'_j \leqslant x_j \leqslant d_j, \quad j = 1, \ldots, n\}$$

and $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ *be a valid inequality for* $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d})$, $\boldsymbol{\pi} \in \mathbf{R}^n$, $\pi_0 \in \mathbf{R}$.

Then $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ *is either equivalent to or is dominated by an inequality of the form*

$$(\boldsymbol{\lambda} A - \mathbf{u} + \mathbf{v})\mathbf{x} \leqslant \boldsymbol{\lambda}\boldsymbol{\alpha} - \mathbf{u}\mathbf{d}' + \mathbf{v}\mathbf{d}, \quad \boldsymbol{\lambda} \in \mathbf{R}^m_+; \ \mathbf{u}, \mathbf{v} \in \mathbf{R}^n_+$$

if any of the following conditions hold:

i) $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d}) \neq \emptyset$;

ii) $D \stackrel{\text{def}}{=} \{\boldsymbol{\lambda} \in \mathbf{R}^m_+; \ \mathbf{u}, \mathbf{v} \in \mathbf{R}^n_+ : \boldsymbol{\lambda} A - \mathbf{u} + \mathbf{v} = \boldsymbol{\pi}\} \neq \emptyset$;

iii) $A = \begin{pmatrix} A' \\ I \end{pmatrix}$, *where* I *is an* $n \times n$ *identity matrix.*

Proof. i) Since $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ is a valid inequality for $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d})$ and $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d}) \neq \emptyset$ then the (primal) linear program:

$$\max \boldsymbol{\pi}\mathbf{x}$$
$$\mathbf{x} \in P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d})$$

has an optimal solution $\mathbf{x}_0$ and $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ for all $\mathbf{x} \in P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d})$. According to Duality theorem, the dual program:

$$\min(\boldsymbol{\lambda}\boldsymbol{\alpha} - \mathbf{u}\mathbf{d}' + \mathbf{v}\mathbf{d})$$
$$\mathbf{y} \equiv (\lambda_1, \ldots, \lambda_m, u_1, \ldots, u_n, v_1, \ldots, v_n) \in D$$

also has an optimal solution $(\boldsymbol{\lambda}^0, \mathbf{u}^0, \mathbf{v}^0) \in \mathbf{R}^{m+2n}_+$ such that $\boldsymbol{\lambda}^0 A - \mathbf{u}^0 + \mathbf{v}^0 = \boldsymbol{\pi}$ and $\boldsymbol{\lambda}^0\boldsymbol{\alpha} - \mathbf{u}^0\mathbf{d}' + \mathbf{v}^0\mathbf{d} = \boldsymbol{\pi}\mathbf{x}_0$. Therefore

$$\begin{aligned} \boldsymbol{\pi}\mathbf{x} &= (\boldsymbol{\lambda}^0 A - \mathbf{u}^0 + \mathbf{v}^0)\mathbf{x} \equiv \boldsymbol{\lambda}^0(A\mathbf{x}) - \mathbf{u}^0\mathbf{x} + \mathbf{v}^0\mathbf{x} \\ &\leqslant \boldsymbol{\lambda}^0\boldsymbol{\alpha} - \mathbf{u}^0\mathbf{d}' + \mathbf{v}^0\mathbf{d} = \boldsymbol{\pi}\mathbf{x}_0 \leqslant \pi_0. \end{aligned}$$

Here we have used the inequalities that define $P(A, \alpha, \mathbf{d}', \mathbf{d})$.

ii) We assume that $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d}) = \emptyset$, otherwise we are in case i). Since $D \neq \emptyset$ then there exist $\widehat{\boldsymbol{\lambda}} \in \mathbf{R}^m_+$; $\widehat{\mathbf{u}}, \widehat{\mathbf{v}} \in \mathbf{R}^n_+$ such that

$$\widehat{\boldsymbol{\lambda}} A - \widehat{\mathbf{u}} + \widehat{\mathbf{v}} = \boldsymbol{\pi}.$$

If $\widehat{\boldsymbol{\lambda}}\boldsymbol{\alpha} - \widehat{\mathbf{u}}\mathbf{d}' + \widehat{\mathbf{v}}\mathbf{d} \leqslant \pi_0$, the statement was proved. Otherwise, since $P(A, \boldsymbol{\alpha}, \mathbf{d}', \mathbf{d}) = \emptyset$ (that is, $\mathbf{x} \in \mathbf{R}^n, \mathbf{d}' \leqslant \mathbf{x} \leqslant \mathbf{d}$ implies $A\mathbf{x} > \boldsymbol{\alpha}$) it follows that there exist $\overline{\boldsymbol{\lambda}} \in \mathbf{R}^m_+; \overline{\mathbf{u}}, \overline{\mathbf{v}} \in \mathbf{R}^n_+$ such that

$$\overline{\boldsymbol{\lambda}} A - \overline{\mathbf{u}} + \overline{\mathbf{v}} = 0, \quad \overline{\boldsymbol{\lambda}}\boldsymbol{\alpha} - \overline{\mathbf{u}}\mathbf{d}' + \overline{\mathbf{v}}\mathbf{d} < 0.$$

Hence, for some $\mu > 0$ we have:

$$(\widehat{\boldsymbol{\lambda}} + \mu\overline{\boldsymbol{\lambda}})A - (\widehat{\mathbf{u}} + \mu\overline{\mathbf{u}}) + (\widehat{\mathbf{v}} + \mu\overline{\mathbf{v}}) \equiv (\widehat{\boldsymbol{\lambda}}A - \widehat{\mathbf{u}} + \widehat{\mathbf{v}}) + \mu(\overline{\boldsymbol{\lambda}}A - \overline{\mathbf{u}} + \overline{\mathbf{v}}) = \boldsymbol{\pi}$$
$$(\widehat{\boldsymbol{\lambda}} + \mu\overline{\boldsymbol{\lambda}})\boldsymbol{\alpha} - (\widehat{\mathbf{u}} + \mu\overline{\mathbf{u}})\mathbf{d}' + (\widehat{\mathbf{v}} + \mu\overline{\mathbf{v}})\mathbf{d} \equiv (\widehat{\boldsymbol{\lambda}}\boldsymbol{\alpha} - \widehat{\mathbf{u}}\mathbf{d}' + \widehat{\mathbf{v}}\mathbf{d})$$
$$+\mu(\overline{\boldsymbol{\lambda}}\boldsymbol{\alpha} - \overline{\mathbf{u}}\mathbf{d}' + \overline{\mathbf{v}}\mathbf{d}) \leqslant \pi_0.$$

For $\widehat{\boldsymbol{\lambda}}$, $\overline{\boldsymbol{\lambda}} \in \mathbf{R}^m_+$; $\widehat{\mathbf{u}}$, $\overline{\mathbf{u}}$, $\widehat{\mathbf{v}}$, $\overline{\mathbf{v}} \in \mathbf{R}^n_+$; $\mu > 0$ denote:

$$\boldsymbol{\lambda}^0 \equiv \widehat{\boldsymbol{\lambda}} + \mu\overline{\boldsymbol{\lambda}} \in \mathbf{R}^m_+; \quad \mathbf{u}^0 \equiv \widehat{\mathbf{u}} + \mu\overline{\mathbf{u}} \in \mathbf{R}^n_+; \quad \mathbf{v}^0 \equiv \widehat{\mathbf{v}} + \mu\overline{\mathbf{v}} \in \mathbf{R}^n_+.$$

Then $\boldsymbol{\lambda}^0 A - \mathbf{u}^0 + \mathbf{v}^0 = \boldsymbol{\pi}$, that is, $(\boldsymbol{\lambda}^0, \mathbf{u}^0, \mathbf{v}^0) \in D$, $\boldsymbol{\lambda}^0\boldsymbol{\alpha} - \mathbf{u}^0\mathbf{d}' + \mathbf{v}^0\mathbf{d} \leqslant \pi_0$ and since we have $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$, the statement of ii) was proved.

iii) Let $A = \begin{pmatrix} A' \\ I \end{pmatrix}$. Therefore $A' = (a'_{ij})$ is an $(m-n) \times n$ matrix. Choose an arbitrary vector $\boldsymbol{\pi} \in \mathbf{R}^n$. From the vector equation $\boldsymbol{\lambda}A - \mathbf{u} + \mathbf{v} = \boldsymbol{\pi}$, $\boldsymbol{\lambda} \in \mathbf{R}^m_+$; $\mathbf{u}$, $\mathbf{v} \in \mathbf{R}^n_+$, which in this case has the form

$$\left(\sum_{i=1}^{m-n} \lambda_i a'_{i1} + \lambda_{m-n+1} - u_1 + v_1, \ \ldots\ , \sum_{i=1}^{m-n} \lambda_i a'_{in} + \lambda_m - u_n + v_n\right)^T$$
$$= (\pi_1, \pi_2, \ldots, \pi_n)^T, \tag{6}$$

if we fix arbitrary non-negative values $\lambda_1 \geqslant 0, \ldots, \lambda_{m-n} \geqslant 0$, we determine the values

$$l_j := \lambda_{m-n+j} - u_j + v_j, \quad j = 1, \ldots, n \tag{7}$$

uniquely in terms of a'_{ij}, π_j, λ_i, $i = 1, \ldots, m-n$; $j = 1, \ldots, n$. Since we have these values, we can choose $\lambda_{m-n+j} \geqslant 0$, $u_j \geqslant 0$, $v_j \geqslant 0$, $j = 1, \ldots, n$ so that (7) holds. Due to the structure of the system (6) (of the matrix A, respectively) we are able to do this.

Hence, $D \neq \emptyset$, and with the use of case ii), it follows the statement of iii).

Therefore in the three cases the inequality $(\boldsymbol{\lambda}^0 A - \mathbf{u}^0 + \mathbf{v}^0)\mathbf{x} \leqslant \boldsymbol{\lambda}^0\boldsymbol{\alpha} - \mathbf{u}^0\mathbf{d}' + \mathbf{v}^0\mathbf{d}$ dominates or is equivalent to $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ by definition. □

When $d_j = +\infty$, $j = 1, \ldots, n$ then the dual variables v_j are equal to zero: $v_j = 0$, $j = 1, \ldots, n$ because the right (upper) inequalities do not exist in this case. Thus, if $d'_j = 0$, $d_j = +\infty$, $j = 1, \ldots, n$ then Theorem 1 implies Proposition 1.1 ([12], p. 208).

COROLLARY 1. *Let* $P_0 \overset{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} \leqslant \boldsymbol{\alpha},\ 0 \leqslant x_j \leqslant d_j,\ j = 1, \ldots, n\}$, A *be* $m \times n$ *matrix,* $\boldsymbol{\alpha} \in \mathbf{R}^m$, $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{R}^n$, $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ *be a valid inequality for* $P_0, \boldsymbol{\pi} \in \mathbf{R}^n$, $\pi_0 \in \mathbf{R}$. *Then* $\boldsymbol{\pi}\mathbf{x} \leqslant \pi_0$ *is either equivalent to or is dominated by the inequality of the form* $(\boldsymbol{\lambda}A + \mathbf{v})\mathbf{x} \leqslant \boldsymbol{\lambda}\boldsymbol{\alpha} + \mathbf{v}\mathbf{d}$, *where* $\boldsymbol{\lambda} \in \mathbf{R}^m_+$; $\mathbf{v} \in \mathbf{R}^n_+$, *if any of the following conditions hold:*

i) $P_0 \neq \emptyset$;

ii) $L_0 \stackrel{\text{def}}{=} \{\boldsymbol{\lambda} \in \mathbf{R}_+^m,\ \mathbf{v} \in \mathbf{R}_+^n : \boldsymbol{\lambda} A + \mathbf{v} \geqslant \boldsymbol{\pi}\} \neq \emptyset$;

iii) $A = \begin{pmatrix} A' \\ I \end{pmatrix}$, *where* I *is an* $n \times n$ *identity matrix.*

This Corollary 1 can also be proved directly using technique similar to that of Proof of Theorem 1.

3. Valid inequalities generation – modular arithmetic approach

Let

$$S = \{\mathbf{x} \in \mathbf{Z}_+^n : \sum_{j=1}^n a_j x_j = a_0\}, \quad a_j \in \mathbf{R}, \quad j = 0, 1, \ldots, n. \tag{8}$$

Assume $q \in \mathbf{Z}, q > 0$. Let us express every a_j, $j = 0, \ldots, n$ as follows:

$$a_j = \alpha_j q + b_j \tag{9}$$

where $0 \leqslant b_j < q$, $\alpha_j \in \mathbf{Z}$, $j = 0, 1, \ldots, n$, i.e. $a_j \equiv b_j (\text{mod } q), b_j$ are the remainders when a_j is devided by q, $j = 0, 1, \ldots, n$.

Consider the (continuous) problem:

$$\max \mathbf{cx} \tag{10}$$

$$\mathbf{x} \in S_0 \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}_+^n : \sum_{j=1}^n a_j x_j = a_0\} \tag{11}$$

where $\mathbf{c}$ is an arbitrary n-vector. S_0 is obtained from S (8) by relaxing integrality requirements $\mathbf{x} \in \mathbf{Z}_+^n$. Without loss of generality we can consider (10) – (11) as a canonical form of the one-dimensional linear knapsack problem.

Hence, $\sum_{j=1}^n (\alpha_j q + b_j) x_j = \alpha_0 q + b_0$ according to (9) and (11), that is,

$$\sum_{j=1}^n b_j x_j - b_0 = \alpha_0 q - q \sum_{j=1}^n \alpha_j x_j. \tag{12}$$

Let $\mathbf{x}$ be an integral feasible solution to (10) – (11), that is, $\mathbf{x} \in S$ (8). Therefore right-hand side of (12) is integer. Hence,

$$\sum_{j=1}^n b_j x_j - b_0 \equiv \gamma \in \mathbf{Z}. \tag{13}$$

Suppose that $\gamma \leqslant -q$. Hence $\sum_{j=1}^n b_j x_j \leqslant b_0 - q$, and $\sum_{j=1}^n b_j x_j + q \leqslant b_0$. For $x_j \in \mathbf{Z}_+, 0 \leqslant b_j < q, j = 1, \ldots, n$ we have : $\sum_{j=1}^n b_j x_j \geqslant 0$. Hence, $b_0 \geqslant q$.

But $0 \leqslant b_0 < q$ - contradiction. It follows that the assumption was wrong. Therefore $\gamma > -q$, and since $\gamma \in Z$, it follows that $\gamma \geqslant -q+1$. From (13) it follows that

$$\sum_{j=1}^{n} b_j x_j \geqslant b_0 + 1 - q \tag{14}$$

for each integral $\mathbf{x} \in S_0$ (11). It means that (14) is a valid inequality for S_0 (11) for each integral $\mathbf{x} \in S_0$, that is, for each point of S (8), and this is *a class* of valid inequalities for S (8) with different $q \in \mathbf{Z}, q > 0$.

When $q = 1$ we have: $\alpha_j = [a_j], 0 \leqslant b_j < 1, j = 0, 1, \ldots, n$, where $[x]$ denotes the greatest integer, not greater than x. From (14) with $q = 1$ we have that $\sum_{j=1}^{n} b_j x_j \geqslant b_0$ is a valid inequality, and it is the strongest inequality among inequalities (14). In this case it has the form:

$$\sum_{j=1}^{n} (a_j - [a_j]) x_j \geqslant a_0 - [a_0] \tag{15}$$

and it is called a Gomory cutting plane. The inequality (15) can also be written in the equivalent form:

$$x_{n+1} \equiv \sum_{j=1}^{n} (a_j - [a_j]) x_j - (a_0 - [a_0]), \quad x_{n+1} \geqslant 0 \tag{15$'$}$$

where x_{n+1} is an additional variable to linear program (10) – (11).

The inequality (15) is well-known. The contribution of the approach suggested consists in the fact that we have obtained a *class* of valid inequalities and (15) is merely an *element* of this class.

Without loss of generality let x_1 be the (single) basic variable to (10) – (11); therefore $x_j = 0, j = 2, \ldots, n$. What is more, according to Lemma 1 the optimal solution to (continuous) linear program (10) – (11) when (1) holds (we can assume that these requirements are satisfied) is namely (5). From (11) it follows that

$$x_1 = \frac{a_0}{a_1} - \sum_{j=2}^{n} \frac{a_j}{a_1} x_j.$$

Therefore

$$x_1 = \left[\frac{a_0}{a_1}\right] + \gamma_0 - \sum_{j=2}^{n} \left(\left[\frac{a_j}{a_1}\right] + \gamma_j \right) x_j$$

where

$$\gamma_j \equiv \frac{a_j}{a_1} - \left[\frac{a_j}{a_1}\right], \quad 0 \leqslant \gamma_j < 1, \quad j = 0, 2, \ldots, n. \tag{16}$$

Then

$$x_1 - \left[\frac{a_0}{a_1}\right] + \sum_{j=2}^{n} \left[\frac{a_j}{a_1}\right] x_j = \gamma_0 - \sum_{j=2}^{n} \gamma_j x_j. \tag{17}$$

i) If $\mathbf{x}$ is an integral feasible solution to (10) – (11) then left-hand side of (17) is integer. Therefore the same is fulfilled for the right-hand side:

$$\gamma_0 - \sum_{j=2}^{n} \gamma_j x_j \equiv \xi \in \mathbf{Z}.$$

Hence, $\gamma_0 - \sum_{j=2}^{n} \gamma_j x_j \leqslant \gamma_0 < 1$ because $\gamma_0 \geqslant 0$, $x_j \geqslant 0$, $j = 1, \ldots, n$ by assumption and according to (16). But integer which is less than 1, is less or equal to 0. Therefore the necessary and sufficient condition for integrality of right-hand side of (17) is: $\gamma_0 - \sum_{j=2}^{n} \gamma_j x_j \leqslant 0$, that is,

$$x_{n+1} \equiv \sum_{j=2}^{n} \gamma_j x_j - \gamma_0 \geqslant 0 \tag{18}$$

where x_1 is the single basic variable for problem (10) – (11), x_{n+1} is an additional variable for (10) – (11). The inequality (18) is a valid inequality for S (8) and it is called a Gomory cutting plane.

Each integral feasible solution to (10) – (11) satisfies (18) with equality because $\gamma_0 = 0$ in this case and $x_j = 0, j = 2, \ldots, n$.

ii) If we write equality $\sum_{j=1}^{n} a_j x_j = a_0$ from (8) in the equivalent form $\sum_{j=1}^{n} \frac{a_j}{a_1} x_j = \frac{a_0}{a_1}$ (assuming that $a_1 \neq 0$) then for $\mathbf{x} \notin \mathbf{Z}^n$ we have: $\frac{a_0}{a_1} \notin \mathbf{Z}$ in general. Therefore $\gamma_0 \equiv \frac{a_0}{a_1} - \left[\frac{a_0}{a_1}\right] > 0$ strictly, and from (18) we obtain that $x_{n+1} \equiv -\gamma_0 < 0$ for an arbitrary "nonintegral" solution to (10) – (11). Hence, this "nonintegral" solution does not satisfy inequality (18).

From i) and ii) it follows that valid inequality (18) for S is a regular cutting plane for S_0 by definition. $\square$

4. Bibliographical notes

The concept of Gomory method is given, e.g. in [7], [8], [9], [17].

The problems considered in this paper are discussed in [1], [3] – [15], [17] – [19], etc.

The relation between facets and Graph Theory (Clique Structure of Graphs) is considered in [11], [13].

The procedures for obtaining classes of facets for $0-1$ knapsack polytope are suggested in [1], [14], [15], etc.

In [16] 0 – 1 knapsack problem is formulated as a paroid optimization problem and in [18] weight inequalities for $0/1$ knapsack polytope are considered.

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