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DELAY–DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS: SHARP RESULTS IN OSCILLATION THEORY

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Abstract. The following aspects of oscillation theory for Delay-difference equation

$$x(n+1) - x(n) + b(n)x(n-k) = 0, \quad k \in \mathbf{N}, b(n) \geqslant 0, n \geqslant n_0 \tag{1}$$

are investigated:

1. Two following facts *are equivalent*:
   a) There exists at least one non-oscillatory solution of Eq. (1).
   b) There is no regularly (slowly) oscillatory solution of Eq. (1).
2. An explicit condition for oscillation of all solutions of Eq. (1), in which $b(n) \geqslant a(n) \geqslant 0$, $a(n+r) = a(n)\ \forall n$, is elaborated. In the particular case $b(n) = a(n),\ \forall n$ it turns to the *sufficient and necessary condition*. For some particular cases $[\{r=k\}; \{r=k+1\}; \{k=1, r=3 \text{ or } 4\}; \{r=2\}]$ this condition is formulated in a recognisable form *in terms of coefficients*.

1. Introduction

A new method for investigation of oscillation properties of discrete difference equations was elaborated in our papers [1], [2], [3]. It can be defined as Discrete version of Sturmian Comparison Method. In this work a further development of the method is presented.

In order that the presented results should be recognisable, we will be considering the simplest $\mathrm{D}\Delta\mathrm{E}$ only: so called (see [5]) "Delay-difference equation" ($\mathrm{D}\Delta\mathrm{E}$)

$$x(n+1) - x(n) + b(n)x(n-k) = 0, \quad k \in \mathbf{N},\ b(n) \geqslant 0,\ n \geqslant n_0 \tag{1}$$

with both general (non-periodic) and periodic coefficients.

We would like that the principal idea should be as clear as possible. And so, we don't touch here on such important problems in oscillation theory as

— oscillation properties of Eq. (1) with an *oscillating* coefficient $b(n)$,

— obtaining the explicit upper estimates for lengths of sign-preservation intervals of solutions.

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We will be considering the equation

$$y(n-1)-y(n)+\tilde{b}(n+k)y(n+k)=0, \quad n \geqslant n_0, \tilde{b}(n) \geqslant 0 \tag{2}$$

which is closely connected with Eq. (1).

We list here some results, proved in [1]–[3] which are adapted for Eq. (1) and formulated in convenient form.

THEOREM A. [1] *Let* $p,\ q \in \mathbf{N}$, $q-p \geqslant k+1$, $\{b_0(n)\}_1^\infty$, $0 \leqslant b_0(n) \leqslant b(n)$, $\sum_{j=1}^k b_0(n+j) > 0\ \forall n$ *and* $\{z(n)\}_1^\infty$, $z(n) > 0$ *be arbitrary sequences and the following conditions hold:*

1°

$$b(i) \geqslant \frac{\sin(\nu z(i) b_0(i))}{\sin(\nu \sum_{i+1}^{i+k} z(n) b_0(n))} \prod_{l=i-k}^{i} \frac{\sin(\nu \sum_{l+k+1}^{l+2k} z(n) b_0(n))}{\sin(\nu \sum_{l+k}^{l+2k} z(n) b_0(n))} \tag{3}$$

for $\forall i \in \langle p-k, q \rangle$ *;*

2°

$$\sum_{p+1+k}^{q+k} z(n) b_0(n) \leqslant \frac{\pi}{\nu}, \quad \frac{\pi}{\nu} \leqslant \sum_{p+1+k}^{i+k} z(n) b_0(n) \leqslant \frac{2\pi}{\nu}, \qquad i \in \langle q+1, q+k \rangle .$$

Then any solution of Eq. (1) *has at least one change of sign on the interval* $\langle p-k, q \rangle$.

Define a sequence $\{d_0(i)\}_1^\infty$ by the following expression:

$$d_0(i) := \frac{1}{z(i)} \prod_{l=i}^{i+k} \left(\sum_{l}^{l+k} z(n) b_0(n) \right) \left[\prod_{l=i+1}^{i+k} \left(\sum_{l+1}^{l+k} z(n) b_0(n) \right) \right]^{-1} . \tag{4}$$

By the limiting transition $\nu \to 0$ in (3) we have the following:

THEOREM B. [1] *Let the sequences* $\{b_0(i)\}$ *and* $\{z(n)\}$ *be as above and* $\sum^\infty z(n) b_0(n) = \infty$. *If*

$$\liminf_{n\to\infty} d_0(n) > 1 \tag{5}$$

then all solutions of Eq. (1) *are oscillatory.*

Remark 1. Let $b_0(n) > 0\ \forall n$ and put $z(n) := \dfrac{1}{b_0(n)}$ in (4). Then Cond.(5) reduces to the well known Ladas's condition

$$\liminf_{n\to\infty} b(n) > \frac{k^k}{(k+1)^{k+1}} . \tag{6}$$

Definition 1. We call an interval $\langle p,q\rangle$ as *regular half-cycle* of Eq. (1) if there exists a solution $\{x(n)\}$ such that

$$x(n) \leqslant 0,\ n \in \langle p-k,p\rangle\,;\ x(n) > 0,\ n \in \langle p+1,q\rangle\,;\ x(q+1) \leqslant 0. \tag{7}$$

Definition 1$'$. We call an interval $\langle p,q\rangle$ as *regular half-cycle* of Eq. (2) if there exists a solution $\{y(n)\}$ such that $q-p \geqslant k+1$ and

$$y(p) \leqslant 0;\ y(n) > 0,\ n \in \langle p+1,q\rangle\,;\ y(n) \leqslant 0,\ n \in \langle q+1,q+k\rangle\,. \tag{8}$$

Definition 2. We call Eq. (1) or Eq. (2) *regularly (*or *slowly) oscillatory* if for $\forall n_0$ there exists a regular half-cycle $\langle p,q\rangle$ such that $p > n_0$.

Definition 3. An oscillatory solution of Eq. (1) or Eq. (2) for which *all* its half-cycles $\langle p,q\rangle$ are such that $q-p \leqslant k$, called *quickly* oscillatory solution.

THEOREM C [C$'$]. (Sturmian Oscillation Theorem.) *If*

$$b(n) \geqslant \tilde b(n) \geqslant 0\ [\tilde b(n) \geqslant b(n) \geqslant 0],\ \forall n \geqslant n_0 \tag{9}$$

and Eq. (2) *[Eq.* (1)*] has at least one regularly oscillatory solution, then all solutions of Eq.* (1) *[Eq.* (2)*] are oscillatory.*

An equivalent formulation of Th. C[C$'$] is the following

THEOREM D[D$'$]. (Sturmian Non-Oscillation Theorem.) *If*

$$b(n) \geqslant \tilde b(n) \geqslant 0\ \ [\tilde b(n) \geqslant b(n)]\ \forall n \geqslant n_0$$

and Eq. (1) [Eq. (2)] has at least one non-oscillatory solution, then Eq. (2) [Eq. (1)] has no regularly oscillatory solution $\square$

Note that Theorems C, C$'$, D and D$'$ are also true for more general difference equations rather than Eq. (1).

2. Existence of a non-oscillatory solution

The main result of this section is following:

THEOREM 2.1. *Let* $\{\gamma_n\}_k^\infty,\ 0 < \gamma_n \leqslant 1,\ \gamma_n \equiv 1,\ n = \overline{-k,0}$ *be an arbitrary sequence and*

$$0 \leqslant b(n) \leqslant (1-\gamma_{n+1}) \prod_{n-k+1}^{n} \gamma_j,\ n \geqslant 0. \tag{10}$$

Then Eq. (1) has at least one solution $\{x(n)\}_k^\infty$ *such that*

$$\prod_1^n \gamma_j \leqslant x(n) \leqslant 1,\ x(n) \geqslant x(n-k) \cdot \prod_{n-k+1}^n \gamma_j,\ n \geqslant 0. \tag{11}$$

The proof of Th. 2.1 based on the well known *Schauder-Tichonov Fixed Point Theorem:* Consider the Banach space $l_\infty := \{\{u_n\}_{-k}^\infty : \sup_n \|u_n\| < \infty\}$ and an operator $\Phi : l_\infty \to l_\infty$ which is continuous in W^*-topology of l_∞. If S_γ is a convex set belonging to l_∞ and $\Phi S_\gamma \subset S_\gamma$ then there exists $x \in S_\gamma$ such that $\Phi x = x$.

Proof of Th. 2.1. We define the set $S_\gamma \subset l_\infty$ as follows:

$$S_\gamma := \left\{ u \in l_\infty : u_n = 1, n = \overline{-k,0}; \prod_1^n \gamma_i \leqslant u_n \leqslant 1 \text{ and } u_n \geqslant u_{n-k} \prod_{n-k+1}^n \gamma_i \text{ for } n \geqslant 1 \right\} \tag{12}$$

1° The set S_γ is convex. (We omit the proof of this simple fact).
2° Define the operator $\Phi : l_\infty \to l_\infty$ by the following equality:

$$v_n = (\Phi u)_n := \begin{cases} 1, & n = \overline{-k,0} \\ \prod_{j=0}^{n-1} [1 - b(j)\frac{u_{j-k}}{u_j}], & n \geqslant 1 \end{cases} \tag{13}$$

3° Show that the operator Φ maps of S_γ inself. Indeed, $v_n = 1$ for $\forall n = \overline{-k,0}$. For $n \geqslant 1$ we have

$$v_n = \prod_0^{n-1} [1 - b(j)\frac{u_{j-k}}{u_j}] \geqslant \prod_0^{n-1} \left[1 - (1 - \gamma_{j+1}) \prod_{j-k+1}^{j} \gamma_l \cdot \frac{1}{\prod_{j-k+1}^{j} \gamma_l} \right]$$
$$= \prod_0^{n-1} \gamma_{j+1} = \prod_1^n \gamma_j \text{ , and } v_n \leqslant 1 . \tag{14}$$

Further,

$$\frac{v_n}{v_{n-k}} = \prod_{j=n-k}^{n-1} (1 - b(j)\frac{u_{j-k}}{u_j}) \geqslant \prod_{j=n-k}^{n-1} [1 - (1 - \gamma_{j+1}) \cdot \prod_{j-k+1}^{j} \gamma_l \cdot \prod_{j-k+1}^{\gamma} \gamma_l^{-1}]$$
$$= \prod_{n-k+1}^{n} \gamma_j. \tag{15}$$

Thus, $\Phi S_\gamma \subset S_\gamma$.

4° Now consider the Banach space $l_1 := \{\{w_n\}_{-k}^\infty : \sum^\infty |w_n| < \infty\}$ and define the following neighborhoods system of $u^{(0)} \in l_\infty$ constructed by the elements $\{w^{(1)}, w^{(2)}, \ldots, w^{(N)}\} \subset l_1$:

$$\Omega_{w^{(1)},\ldots,w^{(N)}}(u^{(0)}) := \{u \in l_\infty : \sum_{n=1}^\infty |u_n - u_n^{(0)}||w_n^{(i)}| < \infty, i = \overline{1,N}\}. \tag{16}$$

This Ω-system defines a W^*-topology in l_∞. This topology is equivalent to the convergence in l_∞^{loc}:

$$u^{(m)} \overset{l_\infty^{loc}}{\to} u^{(0)} \Longleftrightarrow \{\|u^{(m)}\|_\infty < \infty \text{ and } \lim_{m\to\infty} u_n^{(m)} = u_n^{(0)} \ \ \forall n\}.$$

5° We omit the proof of the following evident fact: the operator Φ is continuous with respect to the topology of l_∞^{loc}. So, there exists $\overline{x} \in S_\gamma$ such that $\Phi\overline{x} = \overline{x}$, i.e. $\overline{x}_{n+1} = \prod_0^n (1 - b(j)\dfrac{\overline{x}_{j-k}}{\overline{x}_j}) \Longleftrightarrow \dfrac{\overline{x}_{n+1}}{\overline{x}_n} = 1 - b(n)\dfrac{\overline{x}_{n-k}}{\overline{x}_n} \Longleftrightarrow$ Eq. (1) with the conditions (11). □

REMARK. For the particular case $\gamma_n = \frac{k}{k+1}$ $\forall n$ this result was proved in [4]. The Cond. (10) for such $\{\gamma_n\}$ turns to be

$$b(n) \leqslant \frac{k^k}{(k+1)^{k+1}}, \quad \forall n \geqslant n_0. \tag{17}$$

3. Delay-difference equations which are not possessing regular oscillatory solutions

We define the following properties of any Delay-difference equation (DΔE):

The AOS-*property:* All solutions of DΔE are oscillatory.

The ENO-*property:* There is at least one non-oscillatory solution of DΔE .

The NRO-*property:* DΔE has no regularly oscillation solution.

The ANO-*property:* All solutions of DΔE are non-oscillatory.

When the question about the negation of AOS-property is investigated, efforts are directed to proving that ENO-property is present. Doing this, all the information concerning other solutions is lost. Our method allows to single out several classes of difference equations possessing the NRO-property.

Consider again Eqs. (1) and (2).

Suppose that $y(i) > 0$ on $i \geqslant i_0$ is a solution of Eq. (2). Denote $\psi(n) := \dfrac{y(n-1)}{y(n)}$. Then $y(n) = \prod_{j=1}^n \psi^{-1}(j)$ and

$$\tilde{b}(n) \leqslant [1 - \psi(n-k)] \prod_{l=n-k+1}^{n} \psi(l). \tag{18}$$

From Th. D$'$ it follows:

THEOREM 3.1. *Let* $\{\psi(n)\}_1^\infty,\ 0 < \psi(n) \leqslant 1$, *be an arbitrary sequence and*

$$b(n) \leqslant \prod_{n-k+1}^{n} \psi(l) \cdot [1 - \psi(n-k)],\ n \geqslant n_0. \tag{19}$$

Then Eq. (1) possesses the NRO-property.

Compare the conditions (13) and (10) in Ths 3.1 and 2.1. The following Lemma shows that they are equivalent.

LEMMA 3.2. *Let* $\{\gamma(n)\}_{-k}^{\infty}$ *be an arbitrary sequence with* $0 < \gamma(n) \leqslant 1$ *and*

$$C(n) := [1 - \gamma(n+k)]\cdot \prod_{n}^{n+k-1} \gamma(j),\ n \geqslant -k. \tag{20}$$

Then there exists a sequence $\{\psi(n)\}$, $0 < \psi(n) \leqslant 1$, *such that* $\{C(n)\}$ *can be represented in the form*

$$C(n) = [1 - \psi(n-k)]\cdot \prod_{n-k+1}^{n} \psi(j) \tag{21}$$

as well and vice versa.

Proof. Consider a set S of *double-sided* sequences:

$$S := \{(\dots, \psi(-1), \psi(0), \psi(1), \dots),\ 0 < \psi(n) \leqslant 1, -\infty < n < \infty\}$$

with the following identification:

$$u, v, \in S:\ u \equiv v \Longleftrightarrow \exists m_0 \in \mathbf{Z} \text{ such that } u(n) = v(n+m_0),\ -\infty < n < \infty.$$

Denote by R the following “turn-over”-operator on S:

$$\tilde{u} = Ru \ \ , \ \ \tilde{u}(n) = u(-n) \ \ , \ \ -\infty < n < \infty$$

and define two operators T and T^* on S as follows:

$$g(n) = (T\psi)(n) := \psi(n)\dots\psi(n+k-1)\cdot[1 - \psi(n+k)]$$

$$h(n) = (T^*\psi)(n) := [1 - \psi(n-k)]\cdot \psi(n-k+1)\dots\psi(n),\ n \in \langle -\infty, +\infty\rangle\,.$$

It is obvious, that operators T and T^* are connected by the equalities $T = T^*R$ and $T^* = RT$ since $R^{-1} = R$.

Let us show that the ranges of the operators T and T^* are the same, i.e. $TS = T^*S$. Indeed, let $g^{(0)} \in TS \Longrightarrow \exists \psi^{(0)} \in S$ be such that $g^{(0)} = T\psi^{(0)}$. Then

$$g^{(0)} = T\psi^{(0)} = T^*R\psi^{(0)} = T^*\tilde{\psi}^{(0)} \in T^*S.$$

The inverse inclusion will be proved analogously. So, the sets TS and T^*S are identical, and the equation $T^*\psi = T\gamma$ will be solvable with respect to ψ, if γ is given and vice versa. $\square$

From the recent Lemma the following important statement holds:

THEOREM 3.3. *Eq.* (1) *has at least one non-oscillatory solution if and only if there is no regular oscillatory solution of this equation.*

REMARK. The analogous statement for the second order *differential* equation is well known more than 100 years. But for a *difference* equation it is established here by the first time. However, this fact is not universal at all. In many cases (for example, for the four-term difference equation) such statement may be not true. There exists a suitable counterexample.

Now we define a sequence $\{d^0(n)\}$ by

$$d^0(n) := \frac{1}{z(n)} \cdot \prod_{l=n}^{n+k}\Big(\sum_{l}^{l+k} z(j)b^0(j)\Big) \cdot \left[\prod_{l=n+1}^{n+k}\Big(\sum_{l+1}^{l+k} z(j)b^0(j)\Big)\right]^{-1}, \ n \geqslant n_0 , \qquad (22)$$

in which $z(n)$ will be an arbitrary positive sequence and $b^0(n)$ be such that is formulated as below in Th. 3.4.

Note the following important fact: the last expression and the formula (4) for $d_0(n)$ *are identical* (with changing $b_0(n)$ to $b^0(n)$, of course).

THEOREM 3.4. *Let* $b^0(n) \geqslant b(n) \geqslant 0,\ n \geqslant n_0,\ \{z(n)\}$ *be an arbitrary positive sequence and*

$$d^0(i) \leqslant 1,\ \forall i . \qquad (23)$$

Then Eq. (1) possesses ENO- and NRO-properties.

Proof. Since $b^0(n) \geqslant b(n)$, from (22) follows

$$b(n) \leqslant z_0(n)b^0(n)\cdot \prod_{n+1}^{n+k}\Big(\sum_{l+1}^{l+k} z(j)b^0(j)\Big)\cdot \left[\prod_{n}^{n+1}\Big(\sum_{l+1}^{l+k} z(j)b^0(j)\Big)\right]^{-1}, \ n \geqslant n_0. \qquad (24)$$

One can check that (24) turns to the inequality (13) with

$$\psi(n) := \sum_{n+k+1}^{n+2k} z(j)b^0(j)\cdot \left(\sum_{n+k}^{n+2k} z(j)b^0(j)\right)^{-1}, \qquad (25)$$

and so, the condition (23) implies the ENO- and NRO-properties. □

It's useful to formulate the following statement as well:

THEOREM 3.5. *Let* $\{z(n)\}$ *will be an arbitrary sequence and* $\{d(n)\}$ *defined by*

$$d(n) := \frac{1}{z(n)}\cdot \prod_{n}^{n+k}\Big(\sum_{l}^{l+k} z(j)b(j)\Big)\cdot \left[\prod_{n+1}^{n+k}\Big(\sum_{l+1}^{l+k} z(j)b(j)\Big)\right]^{-1}. \qquad (26)$$

Then two following statements are true:

a) If

$$\liminf_{n\to\infty} d(n) > 1, \tag{27}$$

then Eq. (1) *possesses AOS-property;*

b) If

$$d(n) \leqslant 1 \textit{ for } n \geqslant n_0, \tag{28}$$

then Eq. (1) *possesses ENO- and NRO-properties.*

Proof. Proof follows from Ths. 2.1 and 3.4. □

Let us expose some observations related to statements a), b) in Th. 3.5.

Conditions (27) and (28) do not exhaust, of course, all arising possibilities. In other words, there exist sequences $\{b(n)\}$ for which neither (27) nor (28) are satisfied. Therefore (27) is a *sufficient* condition for Eq. (1) only to have the AOS-property. However, if it is possible to choose a sequence $\{z(n)\}$ that the corresponding sequence $\{d(n)\}$ will be *constant* $[d(n) \equiv d,\ \forall n]$, then two sets of equations of type (1) for which, respectively, conditions (27) and (28) are satisfied, are *exact complement* to each other. In other words, the following statement is true:

THEOREM 3.6. *Let $\{z(n)\}$ be a positive sequence such that $d(n) \equiv d = const\ \forall n$, in which $d(n)$ is defined by (26). Then Eq. (1) possesses AOS-property* if and only if $d > 1$. *Otherwise, if $d \leqslant 1$, then Eq. (1) possesses ENO- and NRO-properties.*

Unfortunately, Th. 3.6 stated in the above form is as much elegant as absolutely non-effective. It looks that for the *general* case it is impossible to state the condition $\{d > 1\}$ *in terms of coefficients* $b(n)$, since the corresponding sequence $\{z(n)\}$ must be expressed explicitly via $\{b(n)\}$. Definitely it is not sufficient to prove that such a sequence does exist.

In the next paragraph *we construct directly* a sequence $\{z(n)\}$ from Th. 3.6 in the following important special cases:

1° The sequence $\{b(n)\}$ has a r-periodic positive *minorant* $\{a(n)\}$, $a(n+r) = a(n),\ \forall n$

2° The sequence $\{b(n)\}$ has a r-periodic positive *majorant* $\{a(n)\}$, $a(n+r) = a(n),\ \forall n$.

REMARK. Consider Eq. (1) for, in particular, $k = 1$:

$$x(n+1) - x(n) + b(n)x(n-1) = 0,\ b(n) \geqslant 0, n \geqslant n_0. \tag{29}$$

Let $\langle p, p+1\rangle$ will be a half-cycle of a non-trivial solution $\{x(n)\}$ of Eq. (29). Then

$$x(p) \leqslant 0,\ x(p+1) > 0,\ x(p+2) \leqslant 0.$$

But this is impossible, because $x(p+2) - x(p+1) + b(n)x(p) = 0$. And so, Eq. (29) has no quick oscillatory solution. Therefore, for the case $k = 1$ all statements "Eq. (1) possesses ENO- and NRO-properties" must be replaced by the statements "Eq. (28) possesses ANO-property".

4. Oscillation properties of delay-difference equation with periodic coefficients

In this paragraph we consider DΔE

$$x(n+1)-x(n)+p(n)x(n-k)=0 \ , \quad n \geqslant n_0 \tag{30}$$

where a non-negative sequence $\{p(n)\}$ is r-periodic.

Oscillation properties for such DΔE some times can be investigated directly. Indeed, consider the most simplest case $\{k=1, r=2\}$:

$$x(n+1)-x(n)+p(n)x(n-1)=0 \ , \quad n \geqslant n_0 \tag{31}$$

with

$$p(n) := \begin{cases} a & , \ n=2m \\ b & , \ n=2m+1 \end{cases} \quad , a \geqslant 0 \ , \quad b \geqslant 0.$$

Rewrite this equation in the following equivalent form:

$$\begin{cases} x(2m+2)-(1-a-b)x(2m)+abx(2m-2)=0, \\ x(2m+1)=x(2m+2)+bx(2m). \end{cases}$$

Oscillation properties of Eq. (31), it is clear, depends on the roots of its characteristic equation

$$\lambda^2-(1-a-b)\lambda+ab=0 \ . \tag{32}$$

All solutions of (31) are oscillatory *if and only if* Eq. (32) has no positive root:

$$\begin{cases} (1-a-b)^2-4ab<0 \\ a+b \leqslant 1 \end{cases} \iff 1-a-b<2\sqrt{a}\sqrt{b} \iff \sqrt{a}+\sqrt{b}>1. \tag{33}$$

However, if we consider on this way the more complicated cases than the such simplest case, this will bring us to very intricated algebraic problems. Therefore, we offer below another method which is based on the Sturmian Comparison Method, as stated above.

Here we are discussing the principles of the choozing the $\{z(n)\}$-sequences in Ths. B, 3.4 and 3.5.

Such choosing will be as better as Conditions (27) and (28) of AOS-property and ENO-property correspondingly, will be *close one to other*.

The *best* choozing of the sequence $\{z(n)\}$ will be such (if it is possible, of course) that the corresponding sequence $\{d(n)\}$ will be *constant*. Then the each of two classes of Eqs.(1) with (27) and (28) will be *the exact complement* of the other. Such *best* choozing of the $\{z(n)\}$-sequence is possible sometimes for the *periodic* case.

Let $\{b_0(n)\}$ in Th. B will be r-*periodic*: $b_0(n+r)=b_0(n) \quad \forall n$ with $0 \leqslant b_0(n) \leqslant b(n) \quad \forall n$ and $\sum_{j=1}^{m} b_0(n+j)>0$, $m=\min\{r;k\}$. We chooze $\{z(n)\}_1^\infty$ as a r-*periodic* positive solution of the system $d_0(1)=d_0(2)=\ldots=d_0(r)$ (then the sequence $\{d_0(n)\}$ will be r-*periodic* as well).

If $\{z(n)\}$ is a solution of this system then the sequence $\{cz(n)\}$ will be its solution as well. Therefore we can *normalize* the solution $\{z(n)\}$, for example, by the equality

$$z(1)b_0(1)+\ldots+z(r)b_0(r)=1$$

and find the r*-periodic* positive solution of the following system

$$\begin{cases} d_0(1)=\ldots=d_0(r), \\ z(n+r)=z(n) \quad \forall n, \\ z(1)b_0(1)+\ldots+z(r)b_0(r)=1. \end{cases} \tag{34}$$

The *existence problem* of the such solution for a general case is an independent and, may be, a very difficult. We will solve it directly for some particular cases in the next points.

Let $\{z(n)\}$ be a r*-periodic* positive solution of the system (34). Then Cond.(27) is equivalent to the following:

$$\Phi[z,b_0]:=\sum_{n=1}^{r}\left\{b_0(n)\cdot\prod_{l=n}^{n+k}\left(\sum_{l}^{l+k}z(j)b_0(n)\right)\cdot\left[\prod_{l=n+1}^{n+k}\left(\sum_{l+1}^{l+k}z(j)b_0(j)\right)\right]^{-1}\right\}>1. \tag{35}$$

Indeed,

$$\begin{aligned} &\liminf_{n\to\infty} d_0(n)>1 \iff d_o(n)>1 \ \ \forall n=\overline{1,r} \iff \\ &\iff \textstyle\prod_n^{n+k}\left(\sum_l^{l+k}z(j)b_0(j)\right)\cdot\left[\prod_{n+1}^{n+k}\left(\sum_{l+1}^{l+k}z(j)b_0(j)\right)\right]^{-1}>z(n) \ \ \forall n=\overline{1,r} \iff \\ &\iff \Phi[z,b_0]>1. \end{aligned}$$

Analogiously, Cond. (28) may be formulated as

$$\Phi[z,b^0]\leqslant 1 \ . \tag{36}$$

Thus, Ths. B, 3.4 and 3.5 are implying three following statements:

THEOREM 4.1. *Let* $\{b(n)\}$ *in Eq.* (1) *has a* r*-periodic minorant* $\{b_0(n)\}$*:*

$$0\leqslant b_0(n)\leqslant b(n)\ ,\ \ b_0(n+r)=b_0(n)\ ,\ \ \sum_{j=n+1}^{m}b_0(n+j)>0\ ,\ \ m=min\{r,k\}$$

for which the system (34) *has a positive solution* $\{z(n)\}$.

If Cond. (35) *holds then Eq.* (1) *possesses AOS-property.*

THEOREM 4.2. *Let* $\{b(n)\}$ *has a* r*-periodic majorant* $\{b^0(n)\}$*:*

$$0\leqslant b(n)\leqslant b^0(n)\ ,\ \ b^0(n+r)=b^0(n)\ \ \forall n\ ,$$

for which the system

$$\begin{cases} d^0(1)=\ldots=d^0(r) \\ z(n+r)=z(n) \quad \forall n \\ z(1)b^0(1)+\ldots+z(r)b^0(r)=1 \end{cases} \tag{37}$$

has a positive solution $\{z(n)\}$.

If Cond. (36) *holds then Eq.* (1) *possesses ENO- and NRO-properties.*

THEOREM 4.3. *Let* $\{b(n)\}$ *in Eq.* (1) *be a* r*-periodic non-negative sequence with*
$\sum_{j=1}^{m} b(n+j) > 0$*, and the system*

$$\begin{cases} d(1) = \cdots = d(r) \\ z(1)b(1) + \cdots + z(r)b(r) = 1 \end{cases} \tag{38}$$

has a r*-periodic positive solution* $\{z(n)\}$*. Then Eq.* (1) *possesses ASO-property if and only if*

$$\Phi[z,b] > 1 \ . \tag{39}$$

Otherwise $(\Phi[z,b] \leqslant 1)$*, Eq.* (1) *possesses ENO- and NRO-properties.*

5. Some particular cases

In this item a *direct immediate solving* of the system (38) will be executed for some particular cases and the conditions (35) or (36) will be written *in terms of coefficients* $b(n)$.

A: $r = k+1$.
We denote here $z(n) := z_n$, $p(n) := p_n$, $p_{n+k+1} = p_n$, $p(n) \geqslant 0 \ \forall n$:

$$\begin{aligned} &\{d(1) = \cdots = d(r)\} \Leftrightarrow \tfrac{1-z_1p_1}{z_1} = \cdots = \frac{1 - z_{k+1}p_{k+1}}{z_{k+1}} := t \Longleftrightarrow \\ &z_j = \tfrac{1}{t+p_j},\ j = \overline{1,k+1}; \quad \{z_1p_1 + \cdots + z_{k+1}p_{k+1} = 1\} \leftrightarrow F_{k+1}(t) = 0, \end{aligned} \tag{40}$$

where $F_{k+1}(t) := \dfrac{p_1}{t+p_1} + \cdots + \dfrac{p_{k+1}}{t+p_{k+1}} - 1$. It is clear, $F_{k+1}(0) = k > 0$, $F_k(t)$ is continuous for $t > 0$, $\lim_{t\to\infty} F_{k+1}(t) = -1$ and $F'(t) < 0$ for $t > 0$. Therefore, Eq. (40) has unique positive root t_0.

Condition (39) is equivalent to

$$H_{k+1}(t_0) := \prod_{j=1}^{k+1}(t_0 + p_j) - t_0^k > 0. \tag{41}$$

Thus, the following statements are true:

COROLLARY 5.1. *Let the sequence* $\{b(n)\}$ *has a* $(k+1)$*-periodic minorant* $\{b_0(n)\}$*:*

$$\forall n: \quad 0 \leqslant b_0(n) \leqslant b(n)\ , \quad b_0(n+k+1) = b_0(n)\ , \ \sum_{j=0}^{k} b_0(j+n) > 0$$

and t_0 *be a (unique) positive root of the equation*

$$F_{k+1}(t) := \sum_{j=1}^{k+1} \frac{b_0(j)}{t + b_0(j)} - 1 = 0. \tag{42}$$

If

$$H_{k+1}(t_0) := \prod_{j=1}^{k+1}[t_0 + b_0(j)] - t_0^k > 0, \tag{43}$$

then the statement of Th. 4.1 is true.

COROLLARY 5.2. *Let the non-negative sequence* $\{b(n)\}$ *has a* $(k+1)$*-periodic majorant* $\{b^0(n)\}$*:*

$$b^0(n) \geqslant b(n) \geqslant 0, \quad b^0(n+k+1) = b^0(n), \quad \forall n$$

and t_0 *is a (unique) positive root of Eq.* (42) *(by changing* b_0 *to* b^0*). If the condition* $H_{k+1}(t_0) \leqslant 0$ *(with the changing* b_0 *to* b^0*) holds then the statement of Th. 4.2 is true.*

COROLLARY 5.3. *Let the sequence* $\{b(n)\}$ *in Eq.* (1) *be* $(k+1)$*-periodic. Then the condition* (43) *will be sufficiently and necessary for AOS-property of Eq.* (1)*. If* $H_{k+1}(t_0) \leqslant 0$ *then Eq.* (1) *possesses ENO- and NRO-property.*

A1: $k=1,\ r=2$.

Consider, in particular, the *second order* DΔE

$$x(n+1) - x(n) + b(n)x(n-1) = 0\ , \quad b(n) > 0\ . \tag{44}$$

Then

$$F_2(t) := \frac{\lambda}{t+\lambda} + \frac{\mu}{t+\mu} - 1\ ; \quad t_0 = \sqrt{\lambda\mu}\ , \quad \lambda, \mu \geqslant 0$$

$$H_2(t_0) > 0 \Longleftrightarrow \sqrt{\lambda} + \sqrt{\mu} > 1. \tag{45}$$

(Compare with (33)!).

Cors. 5.1, 5.2, 5.3 and Remark 3. are implying the following statement:

THEOREM 5.4. *Let* $p(n) := \{\lambda, \mu, \lambda, \mu, \ldots\}$*,* $\lambda, \mu \geqslant 0$*,* $\lambda + \mu > 0$.

a) If $b(n) \geqslant p(n) \quad \forall n$ *and* $\sqrt{\lambda} + \sqrt{\mu} > 1$ *then all solutions of Eq.* (44) *are oscillatory;*

b) If $b(n) \leqslant p(n) \quad \forall n$ *and* $\sqrt{\lambda} + \sqrt{\mu} \leqslant 1$ *then all solutions of Eq.* (44) *are non-oscillatory;*

c) Either all solutions of the 2-periodic equation

$$y(n+1) - y(n) + p(n)y(n-1) = 0\ , \quad n \geqslant 0$$

are oscillatory (in the case $\sqrt{\lambda} + \sqrt{\mu} > 1$*) or are non-oscillatory (in the case* $\sqrt{\lambda} + \sqrt{\mu} \leqslant 1$*).*

B: $r = k$;

$$d(n) = \frac{1 + z_n p_n}{z_n} \prod_{j=1}^{k}(1 + z_j p_j).$$

So, $\{d(1)=\ldots=d(r)\} \iff \dfrac{1+z_1p_1}{z_1}=\cdots=\dfrac{1+z_kp_k}{z_k}:=s \iff z_j=\dfrac{1}{s-p_j}$, $j=\overline{1,\ k}$ and the system (34) turns to the following:

$$\begin{cases} \dfrac{1+z_1p_1}{z_1}=\cdots=\dfrac{1+z_kp_k}{z_k}(:=s>0) \\ z_1p_1+\ldots+z_kp_k=1. \end{cases} \tag{46}$$

Substitute $z_j=\dfrac{1}{s-p_j}$ to (46). We obtain

$$G_k(s):=\frac{p_1}{s-p_1}+\ldots+\frac{p_k}{s-p_k}-1=0\ ,\quad s>0. \tag{47}$$

It is clear, the function $G_k(s)$ is not continuous in the points $s=p_j$. But it is continuous for $s>\max_{1\leqslant j\leqslant k}p_j:=\bar{p}$. Since $G_k'(s)<0$ for $s>\bar{p}$, $\lim_{s\to+\infty}G_k(s)=-1$ and $\lim_{s\to\bar{p}+0}G_k(s)=+\infty$, it follows that there exists a (unique) root $s_0>\bar{p}$ of the equation (47). The condition (35) turns to

$$\Gamma_k(s_0):=s_0^{k+1}-\prod_{j=1}^{k}(s_0-p_j)>0 \tag{48}$$

and the condition (36) turns to $\Gamma_k(s_0)\leqslant 0$. Thus, the following statements hold:

COROLLARY 5.5. *Let the sequence $\{b(n)\}$ has a k-periodic minorant $\{b_0(n)\}$:*

$$\forall n:\ \ 0\leqslant b_0(n)\leqslant b(n)\ \ ,\quad b_0(n+k)=b_0(n)\ ,\ \sum_{j=1}^{k}b_0(j+n)>0$$

and s_0 be a (unique) root on $\{s>max_{1\leqslant j\leqslant k}b_0(j)\}$ of the equation (47).
If the condition (48) holds then the statement of Th. 4.1 will be true.

COROLLARY 5.6. *Let the sequence $\{b(n)\}$ has a k-periodic majorant $\{b^0(n)\}$:*

$$b^0(n)\geqslant b(n)\geqslant 0\ \ ,\ \ b^0(n+k)=b^0(n)\ \ \forall n$$

and s_0 be as stated above.
If $\Gamma_k(s_0)\leqslant 0$, then the statement of Th. 4.2 will be true.

COROLLARY 5.7. *Let the non-negative sequence $\{b(n)\}$ in Eq. (1) will be k-periodic. Then the condition (48) will be sufficiently and necessary for the AOS-property of Eq. (1). Otherwise, if $\Gamma_k(s_0)\leqslant 0$ then Eq. (1) possesses ENO- and NRO-properties.*

B1: $r=k=2$.
Consider, in particular, DΔE

$$x(n+1)-x(n)+b(n)x(n-2)=0\ \ ,\quad n\geqslant n_0 \tag{49}$$

$$p(n) := \{\lambda, \mu, \lambda, \mu, \ldots\}\ ,\quad \lambda, \mu \geqslant 0\quad ,\qquad \lambda + \mu > 0 \tag{50}$$

$$G_2(s) := \frac{\lambda}{s-\lambda} + \frac{\mu}{s-\mu} - 1 = 0 \Longrightarrow s_0 = \lambda + \mu + \sqrt{\lambda^2 - \lambda\mu + \mu^2},$$

$$\begin{aligned} \Gamma_2(s_0) := h(\lambda, \mu) \;&=\; \left(\lambda + \mu + \sqrt{\lambda^2 - \lambda\mu + \mu^2}\right)^3 \\ &-\; \left(\lambda + \sqrt{\lambda^2 - \lambda\mu + \mu^2}\right)\left(\mu + \sqrt{\lambda^2 - \lambda\mu + \mu^2}\right). \end{aligned} \tag{51}$$

Cors 5.5, 5.6, 5.7 imply the following statements:

THEOREM 5.8. *Let* $\{p(n)\}$ *be defined by* (50).

a) If $b(n) \geqslant p(n)\quad \forall n$ *and* $h(\lambda, \mu) > 0$, *then all solutions of Eq.* (49) *are oscillatory.*

b) If $b(n) \leqslant p(n)\ \ \forall n$ *and* $h(\lambda, \mu) \leqslant 0$, *then Eq.* (49) *has no regularly oscillatory solution and at least one from its solutions will be non-oscillatory.*

c) All solutions of the 2-periodic equation

$$y(n+1) - y(n) + p(n)y(n-2) = 0\quad ,\quad n \geqslant n_0 \tag{52}$$

are oscillatory if and only if $h(\lambda, \mu) > 0$. *If* $h(\lambda, \mu) \leqslant 0$ *then there is no regularly oscillatory solution of Eq.* (52) *and at least one its solution will be non-oscillatory.*

REMARK. Note that the statement c) of Th. 5.8 was proved in [6] for $\lambda\mu < 0$. And so, it is true for *every* $2-$periodic sequence $\{p(n)\}$ *without any limitation* on the sign of λ, μ. To all appearence, the statements a) and b) will be true for the case $\lambda\mu < 0$ as well. But of course, this problem requires an additional investigation.

C: $k = 1,\ r = 3$ or 4 . These cases are not particular cases of **A** or **B** and must be considered independently.

C1: $k = 1,\ r = 3$.

$$p(n) := \{\lambda, \mu, , \nu, \lambda, \mu, \nu, \ldots\}\ ,\quad \lambda, \mu, \nu > 0. \tag{53}$$

Denoting $y_1 := \lambda z_1,\ y_2 := \mu z_2,\ y_3 := \nu z_3$, we obtain

$$d_1 = \lambda\frac{(y_1+y_2)(y_2+y_3)}{y_1y_3};\ \ d_2 = \mu\frac{(y_2+y_3)(y_3+y_1)}{y_1y_2};\ \ d_2 = \nu\frac{(y_3+y_1)(y_1+y_2)}{y_3y_2}. \tag{54}$$

The system (34) will be turning to

$$\begin{cases} d_1 = d_2 = d_3 \\ y_1 + y_2 + y_3 = 1. \end{cases} \tag{55}$$

Define a new variable p by the following equality: $d_1 = d_2 = d_3 := \lambda\mu\nu \cdot p^{-2}$. Then we obtain

$$p^3 \cdot \sqrt{d_1d_2d_3} = (\lambda\mu\nu)^{\frac{2}{3}} \Longleftrightarrow p^3(y_1+y_2)(y_2+y_3)(y_3+y_1) = \lambda\mu\nu \cdot y_1y_2y_3 \tag{56}$$

Therefore the system (56) is equivalent to the following "linear homogenious" system

$$\begin{cases} y_1 - \frac{\lambda}{p}y_2 + y_3 = 0 \\ y_1 + y_2 - \frac{\mu}{p}y_3 = 0 \\ -\frac{\nu}{p}y_1 + y_2 + y_3 = 0. \end{cases} \tag{57}$$

This system has a non-trivial solution *if and only if*

$$\begin{vmatrix} 1 & -\frac{\lambda}{p} & 1 \\ 1 & 1 & -\frac{\mu}{p} \\ -\frac{\nu}{p} & 1 & 1 \end{vmatrix} = 0 \Longleftrightarrow g(p) := 2p^3 + (\lambda+\mu+\nu)p^2 - \lambda\mu\nu = 0. \tag{58}$$

The equation (58) has (unique) positive root $p_0 > 0$ because $g(0) < 0$, $g'(p) > 0$ on $p > 0$. The condition (35) in terms of p_0 will be following: $p_0 < \sqrt{\lambda\mu\nu}$. Indeed,

$$\{d_1 = d_2 = d_3 > 1\} \Longleftrightarrow \sqrt{d_1 d_2 d_3} > 1 \Longleftrightarrow (\lambda\mu\nu)^{\frac{3}{2}} > p_0^3 \Longleftrightarrow p_0 < \sqrt{\lambda\mu\nu}.$$

Since $g(0) = -\lambda\mu\nu < 0$ and $g'(0) = 0$; $g''(0) = 0$ we obtain

$$p_0 < \sqrt{\lambda\mu\nu} \Longleftrightarrow g\left(\sqrt{\lambda\mu\nu}\right) > 0 \Longleftrightarrow f_3(\lambda,\mu,\nu) := \lambda+\mu+\nu+2\sqrt{\lambda\mu\nu}-1 > 0.$$

Of course, the condition (36) holds *if and only if* $f_3(\lambda,\mu,\nu) \leqslant 0$.

So, the following statement is true:

THEOREM 5.9. *Let* $p(n) = \{\lambda,\mu,\nu,\lambda,\mu,\nu\ldots\}, \lambda,\mu,\nu > 0$,

a) If $b(n) \geqslant p(n)$ $\forall n$ *and* $f_3(\lambda,\mu,\nu) > 0$ *then all solutions of Eq.* (44) *are oscillatory.*

b) If $b(n) \leqslant p(n)$ $\forall n$ *and* $f_3(\lambda,\mu,\nu) \leqslant 0$ *then all solutions of Eq.* (44) *are non-oscillatory.*

c) Either all solutions of the 3-periodic equation

$$y(n+1) - y(n) + p(n)y(n-1) = 0 \quad , \qquad n \geqslant n_0$$

are oscillatory (in the case $f_3(\lambda,\mu,\nu) > 0$*) or are non-oscillatory (in the case* $f_3(\lambda,\mu,\nu) \leqslant 0$*).*

C2: $k = 1,\ r = 4$.

$$p(n) := \{p,q,r,s,p,q,r,s,\ldots\},\ p,q,r,s > 0. \tag{59}$$

By denoting $y_j := z_j a_j, j = \overline{1,4}$ we obtain

$$d_1 = p\frac{(y_1+y_2)(y_2+y_3)}{y_1 y_3},\ \ d_2 = q\frac{(y_2+y_3)(y_3+y_4)}{y_2 y_4},$$

$$d_3 = r\frac{(y_3+y_4)(y_4+y_1)}{y_1 y_3},\ \ d_4 = s\frac{(y_4+y_1)(y_1+y_2)}{y_2 y_4}$$

$$y_{n+4} = y_n,\ d_{n+4} = d_n\ \forall n.$$

The system (34) will be turning to

$$\begin{cases} d_1 = d_2 = d_3 = d_4 \\ y_1 + y_2 + y_3 + y_4 = 1, \end{cases} \tag{60}$$

Cond. (39) turns to

$$p\frac{(y_1+y_2)(y_2+y_3)}{y_3} + q\frac{(y_2+y_3)(y_3+y_4)}{y_4} + r\frac{(y_3+y_4)(y_4+y_1)}{y_1} + s\frac{(y_4+y_1)(y_1+y_2)}{y_2} > 1. \tag{61}$$

The system (60):

$$\begin{cases} d_1 = d_2 \\ d_2 = d_3 \\ d_3 = d_4 \\ y_1+y_2+y_3+y_4 = 1 \end{cases} \iff \begin{cases} y_1y_3 = y_2y_4\sqrt{\frac{pr}{qs}} \\ y_1 + y_2 = \dfrac{\sqrt{qr}}{\sqrt{qr}+\sqrt{sp}} \\ y_2 + y_3 = \dfrac{\sqrt{sr}}{\sqrt{rs}+\sqrt{pq}} \\ y_3 + y_4 = \dfrac{\sqrt{sp}}{\sqrt{sp}+\sqrt{qr}} \end{cases} \tag{62}$$

From (62) we obtain:

$$\text{Cond. } (61) \iff (\frac{1}{y_1}+\frac{1}{y_3})\sqrt{pr}+(\frac{1}{y_2}+\frac{1}{y_4})\sqrt{qs} > \frac{(\sqrt{qr}+\sqrt{sp})(\sqrt{rs}+\sqrt{pq})}{\sqrt{pqrs}}. \tag{63}$$

Futher, from (62) it follows

$$qrs\left(\frac{1}{y_2}\right)^2 - [(p+q-r+s)\sqrt{pqrs}+2qrs]\cdot\frac{1}{y_2} - (\sqrt{sp}+\sqrt{qr})(\sqrt{rs}+\sqrt{pq})(\sqrt{pr}-\sqrt{qs}) = 0 \Longrightarrow$$

$$\frac{1}{y_2} = 1 + \frac{p+q-r+s}{\sqrt{pqrs}}\cdot p + \frac{p}{2\sqrt{pqrs}}\cdot\sqrt{M(p,q,r,s)}, \tag{64}$$

in which

$$M(p,q,r,s) := (p+q+r+s)^2 - 4(\sqrt{pr}-\sqrt{qs})^2.$$

Analogously,

$$\frac{1}{y_4} = 1 + \frac{r+s-p+q}{\sqrt{pqrs}}\cdot r + \frac{r}{2\sqrt{pqrs}}\cdot\sqrt{M(p,q,r,s)},$$

$$\frac{1}{y_2}+\frac{1}{y_4}=2+\frac{(q+s)(p+r)+(p-r)^2}{2\sqrt{pqrs}}+\frac{p+r}{2\sqrt{pqrs}}\cdot\sqrt{M(p,q,r,s)},$$
$$\frac{1}{y_3}+\frac{1}{y_1}=2+\frac{(q+s)(p+r)+(q-s)^2}{2\sqrt{pqrs}}+\frac{q+s}{2\sqrt{pqrs}}\cdot\sqrt{M(p,q,r,s)}.$$

Substituting to (63) we obtain:

$$\text{Cond. (61)} \Longleftrightarrow \sqrt{M(p,q,r,s)} > 2-p-q-r-s \Longleftrightarrow$$

$$\sqrt{M(p,q,r,s)} > [2-p-q-r-s]^2 \bigvee p+q+r+s>2 \Longleftrightarrow$$

$$f_4(p,q,r,s) := p+q+r+s-\min\{2; 1+(\sqrt{pr}-\sqrt{qs})^2\} > 0. \tag{65}$$

So, the following statement is true:

THEOREM 5.10. *Let $\{a(n)\}$ be defined by (59).*

a) If $b(n) \geqslant a(n)$ $\forall n$ and $f_4(p,q,r,s) > 0$, then all solutions of Eq. (44) are oscillatory.

b) If $b(n) \leqslant a(n)$ $\forall n$ and $f_4(p,q,r,s) \leqslant 0$, then all solutions of Eq. (44) are non-oscillatory.

c) Either all solutions of the 4-periodic equation

$$y(n+1)-y(n)+a(n)y(n-1)=0, \quad n \geqslant n_0$$

are oscillatory (if $f_4(p,q,r,s) > 0$) or are non-oscillatory (if $f_4(p,q,r,s) \leqslant 0$).

REMARK. If, in particular, $p=r$ and $q=s$, the 4-periodic case turns to 2-periodic case and Cond. (65) turns to $\sqrt{p}+\sqrt{q}>1$:

$$\text{Cond. (65)} \Longleftrightarrow 2p+2q > \min\{2; 1+(p-q)^2\} \Longleftrightarrow$$

$$\left\{\begin{array}{l} 1+p^2+q^2-2pq-2p-2q<0 \\ p+q \leqslant 1 \end{array}\right. \bigvee \{p+q \geqslant 1\} \Longleftrightarrow$$

$$\{0 \leqslant 1-p-q < 4pq\} \forall \{p+q \geqslant 1\} \Longleftrightarrow \sqrt{p}+\sqrt{q} > 1$$

D: $r=2$.

$$\begin{array}{l} x_{n+1}-x_n+p_n x_{n-k}=0\ , \quad n>n_0\ , \quad k\in \mathbf{N} \\ p_n=\{a,b,a,b,\ldots\}, a,b \geqslant 0\ , \quad a\neq b\ , \quad a+b>0\ . \end{array} \tag{66}$$

The system (39) turns to the system

$$\left\{\begin{array}{l} d_1=d_2 \\ z_1 a+z_2 b=1\ . \end{array}\right. \tag{67}$$

The cases $k=2m$ and $k=2m-1$, $m \geqslant 1$ will be considering separately.

D1: $r = 2\ ,\ \ k = 2m-1\ ,\ \ m \geqslant 1$.

$$\begin{cases} d_1 = \dfrac{1}{z_1} \cdot \dfrac{\left[mz_1a + mz_2b\right]^{2m}}{\left[(m-1)z_1a + mz_2b\right]^m \cdot \left[mz_1a + (m-1)z_2b\right]^{m-1}} \\ d_2 = \dfrac{1}{z_2} \cdot \dfrac{\left[mz_1a + mz_2b\right]^{2m}}{\left[(m-1)z_1a + mz_2b\right]^m \cdot \left[mz_1a + (m-1)z_2b\right]^{m-1}} \end{cases}$$

From (67) we obtain $\{d_1 = d_2\} \Longleftrightarrow \dfrac{1}{z_1(m - z_2b)} = \dfrac{1}{z_2(m - z_1a)} \Longleftrightarrow \dfrac{z_1}{m - z_1a} = \dfrac{z_2}{m - z_2b} := \dfrac{1}{t} > 0$, $z_1 = \dfrac{m}{t+a}$, $z_2 = \dfrac{m}{t+b}$ and

$$z_1a + z_2b = 1 \Longleftrightarrow F_m(t) := \frac{ma}{t+a} + \frac{mb}{t+b} - 1 = 0 \tag{68}$$

It is clear, Eq. (68) has a (unique) positive root $t_0 > 0$ defined by

$$\left.\begin{array}{l} t_0 = \sqrt{ab} \quad \text{for } m = 1 \text{ (see above the case } \{r = 2,\ \ k = 1\} \text{),} \\ t_0 = \dfrac{m-1}{2}\left[a + b + \sqrt{(a+b)^2 + \dfrac{4(2m-1)}{(m-1)^2}ab}\right] \text{ for } m \geqslant 2\ . \end{array}\right\} \tag{69}$$

Therefore, the condition $\{d_1 = d_2 > 1\}$ turns to

$$\begin{aligned} &\frac{1}{z_1z_2} \cdot \frac{m^{4m}}{(m - z_1a)^{2m-1} \cdot (m - z_2b)^{2m-1}} > 1 \Longleftrightarrow \\ &\Longleftrightarrow t_0^{2-\frac{1}{m}} < (t_0 + a)(t_0 + b) \Longleftrightarrow t_0^{\frac{1}{m}} > \frac{t_0^2}{(t_0+a)(t_0+b)}\ . \end{aligned}$$

One can check that the following equality is true:

$$\frac{t_0^2}{(t_0+a)(t_0+b)} = \frac{m-1}{m} + \frac{ab}{m\left[t_0(a+b) + 2ab\right]}.$$

So, Cond. (64) is equivalent to

$$A(t_0) := t_0^{\frac{1}{m}} - \frac{m-1}{m} - \frac{ab}{m\left[t_0(a+b) + 2ab\right]} > 0$$

and the following statement holds:

COROLLARY 5.11. *Let* $k = 2m-1$, $m \geqslant 1$, *and* t_0 *is defined by* (69). *Then the following statements hold:*

1° *If* $A(t_0) > 0$ *then all solutions of Eq.* (66) *are oscillatory.*

2° *If* $A(t_0) \leqslant 0$ *then there exists at least one non-oscillatory solution of Eq.* (66) *and this equation has no regularly oscillatory solution.*

D2: $r = 2,\ k = 2m$

$$\begin{cases} d_1 = \dfrac{1}{z_1} \cdot \dfrac{[(m+1)z_1 a + m z_2 b]^{m+1} \cdot [m z_1 a + (m+1) z_2 b]^m}{(m z_1 a + m z_2 b)^{2m}} \\ d_2 = \dfrac{1}{z_2} \cdot \dfrac{[(m+1)z_1 a + m z_2 b]^{m} \cdot [m z_1 a + (m+1) z_2 b]^{m+1}}{(m z_1 a + m z_2 b)^{2m}} \end{cases}$$

Analoguously to the previous point, $\{d_1 = d_2\} \Longleftrightarrow \dfrac{z_1 a+m}{z_1} = \dfrac{z_2 a+m}{z_2} := s>0$, $z_1 = \dfrac{m}{s-a}$, $z_2 = \dfrac{m}{s-b}$, and

$$z_1 a + z_2 b = 1 \Longleftrightarrow G(s) := \frac{am}{s-a} + \frac{bm}{s-b} - 1 = 0 \tag{70}$$

The equation (70) has (unique) positive root s_0 on $(\max\{a;b\}, +\infty)$ which is defined by

$$s_0 = \frac{m+1}{2}\left[a + b + \sqrt{(a+b)^2 - \frac{4(2m+1)}{(m+1)^2}ab}\,\right]. \tag{71}$$

Further,

$$\{d_1 = d_2 > 1\} \Longleftrightarrow \frac{(m+z_1 a)^{2m+1} \cdot (m+z_2 b)^{2m+1}}{z_1 z_2 \cdot m^{4m}} > 1 \Longleftrightarrow s_0^{\frac{1}{m}} > \frac{(s_0-a)(s_0-b)}{s_0^2} \tag{72}$$

One can check that

$$\frac{(s_0-a)(s_0-b)}{s_0^2} = \frac{m}{m+1}\left[1 + \frac{ab}{(m+1)(a+b)s_0 - (2m+1)ab}\right].$$

Thus the condition (71) is equivalent to

$$B(s_0) := s_0^{\frac{1}{m}} - \frac{m}{m+1}\left[1 + \frac{ab}{(m+1)(a+b)s_0 - (2m+1)ab}\right] > 0 \tag{73}$$

and therefore the following statement holds:

COROLLARY 5.12. *Let $k = 2m$, $m \geqslant 1$ and s_0 be defined by (71). Then the following statements are true:*

1° *If $B(s_0) > 0$ then all solutions of Eq. (66) are oscillatory.*

2° *If $B(s_0) \leqslant 0$ then there exists at least one non-oscillatory solution of Eq. (66) and this equation has no regularly oscillatory solution.*

REMARK. The statement of Cor. 5.11 for k is *odd* (and not Cor. 5.12 for k is *even*!) was proved in [6].

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