Mathematical
Inequalities
& Applications
Volume 1, Number 3 (1998), 437–444

SOME DIFFERENCE INEQUALITIES WITH WEIGHTS AND INTERPOLATION

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Abstract. The well-known Grisvard-Jakovlev inequality (see Theorems 1 and $1'$) can be interpreted as a fractional order Hardy inequality or as a weighted difference inequality. Some inequalities of this type have been recently proved and discussed by the authors and H. Heinig, and this paper coincides mostly with a lecture held by the first author at the International workshop on difference and differential inequalities (July 3 – 7, 1996, Marmara Research Center, Turkey) where some historical remarks, ideas and results from the papers of the authors and H. Heinig have been presented. Additionally we present and prove some new difference inequalities with weights. Mostly, we omit the proofs which can be found in the papers mentioned and in the references there, and for simplicity, we consider functions on the interval $(0,\infty)$.

1. A remarkable inequality and its extensions

THEOREM 1. *(Grisvard 1963) If $1<p<\infty$, $0<\lambda<1, \lambda\neq 1/p$ and $u\in C_0^\infty(0,\infty)$, then*

$$\int_0^\infty |u(x)|^p x^{-\lambda p}dx \leqslant C^p \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}}dxdy \tag{*}$$

with $C>0$ independent of u.

THEOREM 1'. *(Jakovlev 1961) If $1<p<\infty$, $0<\lambda<1$, $\lambda\neq 1/p$ and $u\in W^{\lambda,p}(0,\infty)$, then*

$$\int_0^\infty |u(x)-u(0)|^p x^{-\lambda p}dx \leqslant C^p \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}}dxdy \tag{**}$$

with $C>0$ independent of u.

REMARK 1. The additional term $u(0)$ at the left hand side of $(**)$ is essential for the case $\lambda>1/p$.

Inequalities $(*)$ and $(**)$ can be treated as *difference inequalities* and we are concerned with the following problems:

PROBLEM 1. Find *weighted* difference inequalities of the form

$$\int_0^\infty |u(x)|^p w_0(x)dx \leqslant C^p \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}}w(x,y)dxdy \tag{1}$$

Mathematics subject classification (1991): 26D15, 46E30.

Key words and phrases: Inequalities, difference inequalities, fractional order derivatives, Hardy inequalities, weight functions, L^p-spaces, mixed norm spaces, real interpolation.

with $w_0(x)$ and $w(x,y)$ weight functions, i.e., functions measurable and positive a.e. in $(0,\infty)$ and in $(0,\infty)\times(0,\infty)$, respectively.

PROBLEM 2. Find weighted *mixed norm* difference inequalities of the following form (with $p \neq q$):

$$\Big(\int_0^\infty |u(x)|^q w_0(x)dx\Big)^{1/q} \leqslant C\Big(\int_0^\infty \Big(\int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}} w(x,y)dx\Big)^{q/p} dy\Big)^{1/p}. \quad (2)$$

The inequalities (1) and (2) can be also treated as *fractional order Hardy inequalities* since the double integrals at the right hand sides are in fact a part of a weighted norm of the „derivative of order λ", $0<\lambda<1$, i.e., a part of the norm in the fractional order weighted Sobolev space $W^{\lambda,p}(w)$ and $W^{\lambda,(p,q)}(w)$, respectively.

If we denote by $\|u\|_{p,w_0}^p$ the left hand side in (1) (i.e., the p-th power of the norm in the weighted Lebesgue space $L^p(w_0))$ and by $J_{\lambda,p,w}(u)$ the right hand side in (1), then we can rewrite (1) as

$$\|u\|_{p,w_0} \leqslant C[J_{\lambda,p,w}(u)]^{1/p}. \quad (1')$$

Besides this inequality, we will deal in Section 5 with the inequality

$$[J_{\lambda,p,w}(u)]^{1/p} \leqslant C\|u'\|_{r,w_1} \quad (3)$$

where u' is the *derivative* of u. Let us mention that from $(1')$ and (3) we obtain the *Hardy inequality*

$$\|u\|_{p,w_0} \leqslant C\|u'\|_{r,w_1} \quad (4)$$

which is dealt with in detail in [4]. Thus, the *simultaneous* validity of $(1')$ and (3) can be treated as a certain *refinement* of the Hardy inequality (4).

2. Inequalities and interpolation

One way how to derive (weighted) difference inequalities is offered by the theory of interpolation of Banach spaces. Let us shortly describe the idea.

Let (A_0,A_1) and (B_0,B_1) be compatible Banach couples. The real (Lions-Peetre) interpolation method states that a bounded linear operator T mapping A_i into B_i with norm $M_i (i=0,1)$ maps also the interpolation space $(A_0,A_1)_{\theta,q}$ $(0<\theta<1,\ 0<q\leqslant\infty)$ into the interpolation space $(B_0,B_1)_{\theta,q}$ with norm $M_\theta \leqslant M_0^{1-\theta}M_1^\theta$.

In terms of inequalities, this result can be formulated as follows: If

$$\|Tf\|_{B_0} \leqslant M_0\|f\|_{A_0} \quad \text{and} \quad \|Tf\|_{B_1} \leqslant M_1\|f\|_{A_1}, \quad (5)$$

then

$$\|Tf\|_{(B_0,B_1)_{\theta,q}} \leqslant M_\theta\|f\|_{(A_0,A_1)_{\theta,q}}. \quad (6)$$

Let us describe how this general result can be used. For this purpose, we need the following assertion:

THEOREM 2. *(Hardy's inequality) Let* $1 < r \leqslant p < \infty$ *and let* u *be a differentiable function on* $(0,\infty)$ *such that* $u(0) = 0$. *Then inequality* (4) *holds if and only if*

$$C_1 := \sup_{x\in(0,\infty)} \Big(\int_x^\infty w_0(t)dt\Big)^{1/p}\Big(\int_0^x w_1^{1-r'}(t)dt\Big)^{1/r'} < \infty \tag{7}$$

with $r' = \frac{r}{r-1}$.

If we denote by $\dot{W}^{1,r}(w_1)$ the (homogeneous) weighted Sobolev space of differentiable functions u satisfying $u(0) = 0$ and such that the norm $\|u'\|_{r,w_1}$ is finite, then inequality (4) is in fact the second inequality in (5) with $A_1 = \dot{W}^{1,r}(w_1)$, $B_1 = L^p(w_0)$ and T the identity operator. Using the trivial continuous imbedding $L^r(w_1) \hookrightarrow L^r(w_1)$ as the first inequality in (5) [with $A_0 = B_0 = L^r(w_1)$], we obtain as the result of (6) the following *fractional order Hardy inequality*:

$$\|u\|_{(L^r(w_1),L^p(w_0))_{\theta,q}} \leqslant M_\theta \|u\|_{(L^r(w_1),\dot{W}^1_p(w_1))_{\theta,q}}. \tag{8}$$

The *problem* which remains is to *describe appropriately the interpolation spaces* appearing in (8). This problem is in general very complicated but for some particular cases, these interpolation spaces can be described even in terms of classical function spaces. E.g., the following theorem holds:

THEOREM 3. *Let* $1 < r \leqslant p < \infty$, $0 < \lambda < 1$, *and denote* $\frac{1}{p(\lambda)} = \frac{1-\lambda}{r} + \frac{\lambda}{p}$. *Let* u *be a differentiable function on* $(0,\infty)$ *such that* $u(0) = 0$. *If*

$$C_1 := \sup_{x\in(0,\infty)} \Big(\int_x^\infty w_0(t)dt\Big)^{1/p} x^{1/r'} < \infty, \tag{9}$$

then, for any $\delta > 0$,

$$\int_0^\infty |u(x)|^{p(\lambda)} w_0^{\lambda p(\lambda)/p}(x)dx \leqslant C\int_0^\delta t^{-\lambda p(\lambda)-1}\Big(\int_{-t}^\infty |u(x+t)-u(x)|^r dx\Big)^{p(\lambda)/r} dt. \tag{10}$$

REMARK 2. (i) Inequality (10) is in fact inequality (8) for the special choice $\theta = \lambda$, $q = r$ and $w_1(x) \equiv 1$. Notice that condition (9) is nothing else than condition (7).

(ii) The proof of Theorem 3 as well as its analogue for more general cases can be found in [3].

(iii) Inequality (10) contains on its right hand side a *mixed norm* (for $p(\lambda) \neq r$), and consequently we have an inequality of the form (2) (of course, with $w(x,y) \equiv 1$). For $p = r$, it is $p(\lambda) = p$, and thus, we obtain an inequality of the type (1), i.e., $(1')$. Let us mention that in [3], also inequalities of the type (3) are dealt with via interpolation theory.

EXAMPLE 1. If $r = p$, $\delta = \infty$ and $w_0(x) = x^{-p}$ then inequality (10) essentially coincides with the Jakovlev-Grisvard inequality $(*)$.

3. Some further results connected with inequality (1)

The approach via interpolation theory is rather complicated and still not satisfactory. Therefore, let us mention some results where inequalities of the type (1) have been derived *directly*. The first result is due to V. Burenkov and W. D. Evans (see [1]), the other one can be found in [2] - [3].

THEOREM 4. *Let* $0 < p < \infty$, *let* v *be a weight function on* $(0,\infty)$ *and define*

$$w_0(x) := \int_x^\infty v(t)dt. \tag{11}$$

Moreover, suppose that there is a constant $c \in (1,2)$ *such that* $w_0(t) \leqslant cw_0(2t)$ *for every* $t > 0$. *Then for all* $u \in L^p$,

$$\int_0^\infty |u(x)|^p w_0(x)dx \leqslant C^p \int_0^\infty \int_0^\infty |u(x)-u(y)|^p v(|x-y|)dxdy \tag{12}$$

with $C > 0$ *independent of* u.

THEOREM 5. *Let* $1 < p < \infty$ *and* $\lambda \geqslant -1/p$. *Let* $w_0(x)$, $w_1(x)$ *be weight functions on* $(0,\infty)$ *and let*

$$B := \sup_{x\in(0,\infty)} \left(\int_0^x w_0(t)dt\right)\left(\int_x^\infty w_1^{1-p'}(t)dt\right)^{p-1} < \infty.$$

Moreover, assume that $\lim_{x\to\infty} \frac{1}{x}\int_0^x u(t)dt = 0$. *Then, for every* $\beta \geqslant 0$,

$$\int_0^\infty |u(x)|^p w_0(x)dx \leqslant C^p \int_0^\infty \int_0^x \frac{|u(x)-u(y)|^p}{|x-y|^\beta} W(x)dydx \tag{13}$$

where $W(x) = x^{\beta-1}(w_0(x) + x^{-p}w_1(x))$ *and* $C^p = 2^{p-1}\max(1, C_p)$ *with* $C_p \leqslant Bp^p(p-1)^{1-p}$.

Taking in Theorem 5 $w_0(x) = x^{\alpha-\lambda p}$ and $w_1(x) = x^{\alpha-\lambda p+p}$ we obtain the following

COROLLARY 1. *Let* $1 < p < \infty$, $\beta \geqslant 0$, $\lambda \geqslant -1/p$ *and* $\alpha > \lambda p - 1$. *If* $\lim_{x\to\infty} \frac{1}{x}\int_0^x u(t)dt = 0$, *then*

$$\int_0^\infty |u(x)|^p x^{\alpha-\lambda p}dx \leqslant C^p \int_0^\infty \int_0^x \frac{|u(x)-u(y)|^p}{|x-y|^\beta} x^{\beta-1-\lambda p+\alpha}dydx. \tag{14}$$

EXAMPLE 2. For $\beta = 1 + \lambda p$, (14) reads

$$\int_0^\infty |u(x)|^p x^{\alpha-\lambda p}dx \leqslant C^p \int_0^\infty \int_0^x \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}} x^\alpha dydx,$$

and for $\alpha = 0$, again inequality $(*)$ follows.

EXAMPLE 3. Since $x^{-1-\lambda p+\alpha} \leqslant |x-y|^{-1-\lambda p+\alpha}$ provided $0 < y \leqslant x$ and $\alpha < 1+\lambda p$, inequality (14) implies for $\beta = 0$ that

$$\int_0^\infty |u(x)|^p x^{\alpha-\lambda p} dx \leqslant C^p \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}} |x-y|^\alpha dy dx.$$

Notice that the same result can be obtained also by applying Theorem 4 with $v(t) = t^{\alpha-\lambda p-1}$, *but only for* $\alpha < \lambda p$, since for $\alpha \geqslant \lambda p$, we have $w_0(x) \equiv \infty$ (see formula (11)!).

REMARK 3. In the foregoing results concerning inequality (1), we dealt with special weight functions $w(x,y)$, namely, of the form $w(x,y) = v(|x-y|)$ or $w(x,y)$ depending only on x. Results concerning more general weights can be found in [2].

4. The mixed norm case

First, let us introduce some notation: For weight functions $v(x)$ and $w(x)$ on $(0,\infty)$, denote

$$V(y) := \int_0^y v(x) dx,$$

$$w_0(y) := \Big(\frac{1}{y}\int_0^y v^{p'}(x) w^{1-p'}(x) dx\Big)^{-q/p'} \Big(\frac{1}{y}\int_0^y v(x) dx\Big)^q v(y) y^{-\lambda q}.$$

Some results concerning the general mixed norm case (see (2)) have been proved and discussed in [2]. Here, we only present the following result:

THEOREM 6. *Let* $1 < p \leqslant q < \infty,\ \lambda \geqslant -1/p$ *and*

$$C_{p,q} := \sup_{r\in(0,\infty)} \Big(\int_r^\infty w_0(x) V^{-q}(x) dx\Big)^{1/q} \Big(\int_0^r (w_0(x) v^{-q}(x))^{1-q'} dx\Big)^{1/q'} < \infty.$$

Then

$$\Big(\int_0^\infty |u(x)|^q w_0(x) dx\Big)^{1/q} \leqslant \frac{1}{1-K}\Big(\int_0^\infty \Big(\int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}} w(x) dx\Big)^{q/p} v(y) dy\Big)^{1/q}$$

provided

$$K = \frac{C_{p,q}q}{(q-1)^{1/q'}} < 1.$$

EXAMPLE 4. By applying Theorem 6 with $w(x) \equiv 1,\ v(y) \equiv 1,\ p = q$ and with $u(x)$ replaced by $u(x) - u(0)$, we obtain, for $\lambda > 1/p$, the Jakovlev inequality $(**)$ with $C = (\lambda p + p - 1)/(\lambda p - 1)$.

REMARK 4. (i) The assumption $p \leqslant q$ was essential; thus, the problem of mixed norm inequalities of the form (2) for $p > q$ remains still open.

(ii) The conditions of the validity of the inequalities derived in the foregoing sections have been only sufficient. Consequently, we have the following *open problem: Find necessary and sufficient conditions* for the validity of each of the weighted difference inequalities derived in Sections 1 – 4.

5. Concerning inequality (3)

In this section we will prove the following new result:

THEOREM 7. *Let* $1 < r \leqslant p < \infty$. *Let* $w(x,y)$ *and* $W(x,y)$ *be weight functions on* $(0,\infty)\times(0,\infty)$ *and denote* $\widetilde{w}(x,y) = w(x,y) + w(y,x)$. *Suppose that for a.e.* $x \in (0,\infty)$,

$$B(x) := \sup_{t\in(0,x)} \Big(\int_0^t \widetilde{w}(x,y)dy\Big)^{1/p} \Big(\int_t^x W^{1-r'}(x,y)dy\Big)^{1/r'} < \infty \tag{15}$$

and denote

$$w_1(y) = \Big(\int_y^\infty B^p(x) W^{p/r}(x,y)dx\Big)^{r/p}. \tag{16}$$

Then the following inequality holds:

$$\Big(\int_0^\infty \int_0^\infty |u(x)-u(y)|^p w(x,y)dydx\Big)^{1/p} \leqslant C\Big(\int_0^\infty |u'(y)|^r w_1(y)dy\Big)^{1/r} \tag{17}$$

with $C > 0$ *independent of* u.

REMARK 5. Replacing $w(x,y)$ by $w(x,y)|x-y|^{-1-\lambda p}$ in (17) we obtain inequality (3), of course, for $r \leqslant p$.

Choosing in Theorem 7 $r = p$, $w(x,y) = |x-y|^{-1-\lambda p}$ with $0 < \lambda < 1$, and $W(x,y) = |x-y|^{-1-\lambda p+p}$, we can verify that (15) is satisfied (with $B(x)$ a constant) and that $w_1(y) = \text{const}\, y^{(1-\lambda)p}$. Thus we have the following

COROLLARY 2. *If* $1 < p < \infty$ *and* $0 < \lambda < 1$, *then*

$$\int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+\lambda p}} dydx \leqslant C^p \int_0^\infty |u'(x)|^p x^{(1-\lambda)p} dx \tag{18}$$

with $C^p = \frac{2\lambda^{-p}}{(1-\lambda)p}$.

Inequality (15) can be considered as a counterpart of inequality $(*)$. It is inequality (3) for $r = p$, $w(x,y) \equiv 1$ and $w_1(x) = x^{(1-\lambda)p}$.

Proof of Theorem 7. Using Fubini's theorem, we have that

$$\begin{aligned} J &= \int_0^\infty \int_0^\infty |u(x)-u(y)|^p w(x,y)dydx \\ &= \int_0^\infty \int_0^x \ldots dydx + \int_0^\infty \int_x^\infty \ldots dydx \\ &= \int_0^\infty \int_0^x \ldots dydx + \int_0^\infty \int_0^y \ldots dxdy \\ &= \int_0^\infty \int_0^x |u(x)-u(y)|^p \widetilde{w}(x,y)dydx \\ &= \int_0^\infty \Big[\int_0^x \Big|\int_y^x u'(t)dt\Big|^p \widetilde{w}(x,y)dy\Big]dx. \end{aligned} \tag{19}$$

For an arbitrary $x \in (0,\infty)$, the Hardy inequality, used for the interval $[0,x]$, yields in view of (16) that

$$\Big(\int_0^x \Big|\int_y^x u'(t)dt\Big|^p \widetilde{w}(x,y)dy\Big)^{1/p} \leqslant \text{const}\, B(x)\Big(\int_0^x |u'(y)|^r W(x,y)dy\Big)^{1/r}. \tag{20}$$

Using this estimate in (19), the Minkowski integral inequality together with (16) yields

$$\begin{aligned} J &\leqslant \text{const} \int_0^\infty B^p(x)\Big(\int_0^x |u'(y)|^r W(x,y)dy\Big)^{p/r}dx \\ &\leqslant \text{const}\,\Big(\int_0^\infty |u'(y)|^r\Big(\int_y^\infty B^p(x)W^{p/r}(x,y)dx\Big)^{r/p}dy\Big)^{p/r} \\ &= \text{const}\,\Big(\int_0^\infty |u'(y)|^r w_1(y)dy\Big)^{p/r}. \end{aligned}$$

Thus, the theorem is proved.

EXAMPLE 5. In Theorem 7, taking the weight functions $w(x,y)$ and $W(x,y)$ in the special form $w(|x-y|)$ and $W(|x-y|)$, we obtain for $r=p$ the inequality

$$\int_0^\infty\int_0^\infty |u(x)-u(y)|^p w(|x-y|)dydx \leqslant C^p \int_0^\infty |u'(y)|^p w_1(y)dy \tag{21}$$

where $w_1(y) = \int_y^\infty B^p(x)W(|x-y|)dx$ provided

$$B^p(x) := \sup_{t\in(0,x)}\Big(\int_{x-t}^x w(s)ds\Big)\Big(\int_0^{x-t} W^{1-p'}(s)ds\Big)^{p-1} < \infty.$$

Inequality (21) is a counterpart of the Burenkov-Evans inequality (12).

REMARK 6. Inequality (17) holds also for $r>p$, but in this case, we have to replace condition (15) by

$$B'(x) := \Big(\int_0^x\Big(\int_0^t \widetilde{w}(x,y)dy\Big)^{s/r}\Big(\int_t^x W^{1-r'}(x,y)dy\Big)^{s/r'} W^{1-r'}(x,t)dt\Big)^{1/s} < \infty \tag{15'}$$

for a.e. $x\in(0,\infty)$ with $\frac{1}{s}=\frac{1}{p}-\frac{1}{r}$ [this is the necessary and sufficient condition for the validity of the Hardy inequality (20) if $r>p$] and we need the additional condition

$$C_0 := \Big(\int_0^\infty B'^s(x)dx\Big)^{1/s} < \infty.$$

In this case, we have, instead of (16),

$$w_1(y) = \int_y^\infty W(x,y)dx. \tag{16'}$$

Finally, we will derive another *sufficient* condition for the validity of inequality (18), which does not depend on the mutual position of the parameters p and r.

THEOREM 8. *Let* $1 < p,\ r < \infty$. *Let* $w(x,y)$ *and* $w_1(x)$ *be weight functions on* $(0,\infty)\times(0,\infty)$ *and* $(0,\infty)$, *respectively. Denote*

$$V(x) = \int_0^x w_1^{1-r'}\,dt$$

and suppose that

$$B := \int_0^\infty \int_0^\infty |V(x)-V(y)|^{p/r'} w(x,y)\,dxdy < \infty.$$

Then the inequality (17) *holds with* $C = B^{1/p}$.

Proof. The Hölder inequality yields

$$\begin{aligned}
\int_0^\infty \int_0^\infty |u(x)-u(y)|^p w(x,y)\,dxdy &= \int_0^\infty \int_0^\infty \Big| \int_x^y u'(t)\,dt \Big|^p w(x,y)\,dxdy \\
&= \int_0^\infty \int_0^\infty \Big| \int_x^y u'(t) w_1^{1/r}(t) w_1^{-1/r}(t)\,dt \Big|^p w(x,y)\,dxdy \\
&\leqslant \int_0^\infty \int_0^\infty \Big| \int_x^y |u'(t)|^r w_1(t)\,dt \Big|^{p/r} \Big| \int_x^y w_1^{1-r'}(t)\,dt \Big|^{p/r'} w(x,y)\,dxdy \\
&\leqslant \int_0^\infty \int_0^\infty |V(x)-V(y)|^{p/r'} w(x,y) \Big| \int_0^\infty |u'(t)|^r w_1(t)\,dt \Big|^{p/r} dxdy \\
&= B\Big(\int_0^\infty |u'(t)|^r w_1(t)\,dt \Big)^{p/r}.
\end{aligned}$$

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