Mathematical
Inequalities
& Applications
Volume 1, Number 4 (1998), 517–526

HARDY–BENNETT–TYPE THEOREMS

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(*communicated by J. Pečarić*)

Abstract. The factorization of inequalities, introduced and treated systematically by G. Bennett, is a new and very effective method providing the best possible version of several classical and recent inequalities. Here we moderately improve two factorization-theorems proved by us.

1. Introduction

The problem of factorization of inequalities was raised and treated systematically by G. Bennett [2]. For precise definition and explanation of the great advantage of the factorization we refer to this basic monograph of G. Bennett.

Recently in the papers [3], [4] we also studied some problems of factorization.

In the present paper we shall generalize two theorems of [4] which themselves, disregarding two tolerable assumptions and the best possible constants, are certain factorized extensions of two theorems of G. Bennett [1].

In order to recall these results and to formulate our new results we have to present some notions and notations.

Let $\mathbf{x} := \{x_n\}$ denote an arbitrary sequence of real (or complex) numbers. Let $\lambda := \{\lambda_n\}$ be a sequence of positive numbers. We shall use the following notations:

$$H_n := \sum_{k=1}^{n} \lambda_k \qquad \text{and} \qquad \Lambda_n := \sum_{k=n}^{\infty} \lambda_k, \quad (\Lambda_1 < \infty);$$

furthermore, for $c \geqslant 0$,

$$\Lambda_n^{(c)} := \sum_{k=n}^{\infty} \lambda_k \Lambda_k^{-c} \quad \text{and} \quad R_n^{(c)} := \sum_{k=n}^{\infty} \lambda_k H_k^{-c}.$$

We also define, for $p > 0$ and $c \geqslant 0$, the following sets:

$$\lambda(p,c) := \left\{ \mathbf{x} : \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

Mathematics subject classification (1991): 26D15, 40A05, 40A99.
Key words and phrases: Inequalities for sums, factorization, l^p -spaces, classical and recent inequalities.

Paper MIA-01-49

and

$$\Lambda(p,c) := \left\{ \mathbf{x} : \sum_{k=1}^{n} |x_k|^p = O(\Lambda_n^{(1-p)(1-c)}) \right\};$$

furthermore if $p > 1,\ 0 \leqslant c < 1$ and $\Lambda_1 < \infty$ the norms:

$$||\mathbf{x}||_{\lambda(p,c)} := \left\{ \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{n} |x_k| \right)^p \right\}^{\frac{1}{p}}$$

and

$$||\mathbf{x}||_{\Lambda(p,c)} := \sup_n \left(\Lambda_n^{(p-1)(1-c)} \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$

If $c = 0$ we shall simply write $\lambda(p) := \lambda(p,0)$ and $\Lambda(p) := \Lambda(p,0)$.

Moreover denote

$$\lambda(p,c,H) := \left\{ \mathbf{x} : \sum \lambda_n H_n^{-c} \left(\sum_{k=1}^{n} |x_k| \right)^p < \infty \right\},$$

$$\Lambda(p,c,H) := \left\{ \mathbf{x} : \sum_{k=1}^{n} |x_k|^p = O(H_n^{(1-p)(1-c)}) \right\},$$

$$||\mathbf{x}||_{\lambda(p,c,H)} := \left\{ \sum \lambda_n H_n^{-c} \left(\sum_{k=1}^{n} |x_k| \right)^p \right\}^{\frac{1}{p}}$$

and

$$||\mathbf{x}||_{\Lambda(p,c,H)} := \sup_n \left(H_n^{(p-1)(1-c)} \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called β*-power-monotone decreasing* if

$$n^{\beta}\gamma_n \leqslant m^{\beta}\gamma_m$$

holds for any $n \geqslant m$ and for all m.

It will be convenient to lay down here that henceforth a summation sign $\sum$ in which the limits of the summation are omitted will denote summation from 1 to ∞. We shall use K to denote a positive constant. Not necessarily the same on any two occurrences. If we wish to express the dependence explicitly, we write K in the form $K(\alpha,\ldots)$.

Now we establish two interesting theorems of G. Bennett [1], and their fractional extansions proved in [4].

THEOREM A. *If $p > 1$ and $0 \leqslant c < 1$, then*

$$\sum \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{n} a_k \right)^p \leqslant \left(\frac{p}{1-c} \right)^p \sum \lambda_n^{1-p} \Lambda_n^{p-c} a_n^p.$$

The constant is best possible.

THEOREM B. *If* $1 < c \leqslant p$, *then*

$$\sum \lambda_n H_n^{-c} \left(\sum_{k=1}^{n} a_k \right)^p \leqslant \left(\frac{p}{c-1} \right)^p \sum \lambda_n^{1-p} H_n^{p-c} a_n^p. \tag{1.1}$$

The constant is best possible.

THEOREM C. *Let* $p > 1$, $0 \leqslant c < 1$ *and let* $\lambda := \{\lambda_n\}$ *be a given sequence of positive terms.*

(i) *If a sequence* $\mathbf{x}$ *belongs to* $\lambda(p,c)$ *then it admits a factorization*

$$\mathbf{x} = \mathbf{y} \cdot \mathbf{z} \qquad (x_n = y_n z_n) \tag{1.2}$$

with

$$\mathbf{y} \in \ell^p \quad \text{and} \quad \mathbf{z} \in \Lambda(p^*, c), \quad p^* := \frac{p}{p-1}; \tag{1.3}$$

moreover

$$!\mathbf{x}!_{p,\Lambda,c} \leqslant ||\mathbf{x}||_{\lambda(p,c)}, \tag{1.4}$$

where

$$!\mathbf{x}!_{p,\Lambda,c} := \inf ||\mathbf{y}||_p ||\mathbf{z}||_{\Lambda(p^*,c)},$$

the infimum being extended over all factorization (1.2) *with* (1.3).

(ii) *Conversely, if the sequence* λ *satisfies the additional condition*

$$n\lambda_n \leqslant K\Lambda_n, \quad n = 1, 2, \ldots, \tag{1.5}$$

furthermore the sequence $\{\Lambda_n^{(c)}\}$ *is* β*-power-monotone decreasing with some positive* β *and the sequence* $\mathbf{x}$ *admits a factorization* (1.2) *with* (1.3), *then* $\mathbf{x}$ *belongs to* $\lambda(p,c)$*; moreover*

$$||\mathbf{x}||_{\lambda(p,c)} \leqslant K(\lambda, p, c) !\mathbf{x}!_{p,\Lambda,c}. \tag{1.6}$$

THEOREM D. *Let* $1 < c \leqslant p$ *and let* λ *be a given sequence of positive terms.*

(i) *If a sequence* $\mathbf{x}$ *belongs to* $\lambda(p,c,H)$ *then it admits a factorization* (1.2) *with*

$$\mathbf{y} \in \ell^p \qquad \text{and} \qquad \mathbf{z} \in \Lambda(p^*, c, H); \tag{1.7}$$

moreover

$$!\mathbf{x}!_{p,H,c} \leqslant ||\mathbf{x}||_{\lambda(p,c,H)}, \tag{1.8}$$

where

$$!\mathbf{x}!_{p,H,c} := \inf ||\mathbf{y}||_p ||\mathbf{z}||_{\Lambda(p^*,c,H)}$$

the infimum is being extended over all factorization (1.2) *with* (1.7).

(ii) *Conversely, if the sequence* λ *satisfies the additional conditions:* $\sum \lambda_n = \infty$ *and*

$$n\lambda_n \leqslant KH_n, \quad n = 1, 2, \ldots, \tag{1.9}$$

furthermore the sequence $\{R_n^{(c)}\}$ *is* β*-power-monotone decreasing with some positive* β *and the sequence* $\mathbf{x}$ *admits a factorization* (1.2) *with* (1.7), *then* $\mathbf{x}$ *belongs to* $\lambda(p,c,H)$*; moreover*

$$||\mathbf{x}||_{\lambda(p,c,H)} \leqslant K(\lambda, p, c) !\mathbf{x}!_{p,H,c}. \tag{1.10}$$

Perhaps it is not superfluous to show that Theorem C contains Theorem A, and Theorem D includes Theorem B, under the assumptions on λ and disregarding the best constants. E.g. we demonstrate that Theorem D contains Theorem B. Let us suppose that the right-hand side of (1.1) is finite. Then we can factorize $\mathbf{x} := \mathbf{a}$ as follows:

$$y_n := \lambda_n^{\frac{1-p}{p}} H_n^{1-\frac{c}{p}} a_n \quad \text{and} \quad z_n := \lambda_n^{\frac{p-1}{p}} H_n^{\frac{c}{p}-1}.$$

The finiteness of the right-hand side of (1.1) implies that $\mathbf{y} \in \ell^p$ and by Lemma 4 (see below) $\mathbf{z} \in \Lambda(p^*, c, H)$ also holds, i.e. $\mathbf{a}$ is factorizable as in (1.2) with (1.7), so the inequality (1.10) of Theorem D implies the inequality (1.1) disregarding the best constant $\left(\frac{p}{c-1}\right)^p$, as stated.

We also emphasize, that the finiteness of the right-hand side of (1.1) always, without any additional assumption on λ, implies that $\mathbf{a}$ has a factorization (1.2) with (1.7). Namely then, by Theorem B, $\mathbf{a}$ belongs to $\lambda(p, c, H)$ and by the part (i) of Theorem D this fact implies the required factorization of the sequence $\mathbf{a}$.

We also note that in [4] our Theorems C and D were formulated not in the present form. By reformulating we intend to emphasize that the additional assumptions on λ are required only to the proof of the assertions $\mathbf{x} \in \lambda(p, c)$ and $\mathbf{x} \in \lambda(p, c, H)$.

According to my conjecture these assertions also follow from the factorization of $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ without any additional condition on λ, but, unfortunately, I am not able to prove this. It is proposed to the reader.

Thus our modest aim is to weaken the conditions (1.5) and (1.9) moderately.

2. Theorems

Now we formulate our results.

THEOREM 1. *Let $p > 1, 0 \leqslant c < 1$ and let $\lambda := \{\lambda_n\}$ be a given sequence of positive terms.*

(i) If a sequence $\mathbf{a}$ belongs to $\lambda(p, c)$ then it admits a factorization (1.2) with (1.3); furthermore (1.4) also holds.

(ii) Conversely, if the sequence λ satisfies the additional conditions:

$$\sum_{k=2}^{n} \frac{\lambda_{k-1}}{\Lambda_k} k \leqslant Kn, \quad n = 1, 2 \ldots, \tag{2.1}$$

furthermore the sequence $\{\Lambda_n^{(c)}\}$ is β-power-monotone decreasing with some positive β, and the sequence $\mathbf{x}$ admits a factorization (1.2) with (1.3), then $\mathbf{x}$ belongs to $\lambda(p, c)$ and (1.6) also holds.

THEOREM 2. *Let $1 < c \leqslant p$ and let λ be a given sequence as in the part (i) of Theorem 1.*

(i) If a sequence $\mathbf{x}$ belongs to $\lambda(p, c, H)$ then it admits a factorization (1.2) with (1.7); furthermore (1.8) also holds.

(ii) *Conversely, if the sequence* λ *satisfies the additional conditions*: $\sum \lambda_n = \infty$,

$$\sum_{k=1}^{n} \frac{\lambda_{k+1}}{H_k} k \leqslant Kn, \quad n = 1, 2, \ldots, \tag{2.2}$$

and the sequence $\{R_n^{(c)}\}$ *is* β*-power-monotone decreasing with some positive* β*, and the sequence* $\mathbf{x}$ *admits a factorization* (1.2) *with* (1.7)*, then* $\mathbf{x}$ *belongs to* $\lambda(p, c, H)$ *and* (1.10) *also holds.*

It is easy to show that the assumptions (1.5) and (1.9) in Theorems C and D imply the conditions (2.1) and (2.2) in Theorems 1 and 2, respectively, but the converse assertions are not true. See e.g. the case

$$\lambda_n = \begin{cases} \frac{1}{n}, & \text{if} \quad n = 2^m, \\ \frac{1}{n^2}, & \text{if} \quad n \neq 2^m. \end{cases}$$

Then (2.1) holds, but (1.5) is not satisfied for $n = 2^m$. Similarly (2.2) holds and (1.9) does not stay if

$$\lambda_n := \begin{cases} n, & \text{if} \quad n = 2^m, \\ 1, & \text{if} \quad n \neq 2^m. \end{cases}$$

3. Lemmas

To prove our theorems we need the following lemmas.

LEMMA 1. *Let* **u, v, w** *be sequences with non-negative terms and suppose that* w_k *decreases with* k*. If*

$$\sum_{k=1}^{n} u_k \leqslant \sum_{k=1}^{n} v_k \quad (n = 1, 2, \ldots),$$

then

$$\sum_{k=1}^{n} u_k w_k \leqslant \sum_{k=1}^{n} v_k w_k \quad (n = 1, 2, \ldots).$$

This lemma is known, see e.g. Lemma 3.6 in [2].

The proof of the following three lemmas can be found in [4].

LEMMA 2. *If* $0 \leqslant c < 1$*,* λ *is a sequence of positive terms* λ_n *and* $\sum \lambda_n < \infty$*, then*

$$\Lambda_n^{1-c} \leqslant \sum_{k=n}^{\infty} \lambda_k \Lambda_k^{-c} \leqslant K(c) \Lambda_n^{1-c}$$

hold for all n*.*

LEMMA 3. *If* $1 < c$, λ *is a sequence of positive terms such that* $\sum \lambda_n = \infty$, *then*

$$\sum_{k=n}^{\infty} \lambda_k H_k^{-c} \leqslant K(c) H_n^{1-c}$$

holds for all n.

LEMMA 4. *If* $1 < c \leqslant p$, λ *is a sequence of positive terms such that* $\sum \lambda_n = \infty$, *then*

$$\sum_{k=1}^{n} \lambda_k H_k^{\frac{c-p}{p-1}} \leqslant K(c,p) H_n^{\frac{c-1}{p-1}}$$

holds for all n.

Finally we prove one more lemma.

LEMMA 5. *If* $1 < c$, λ *is a sequence of positive terms such that* $\sum \lambda_n = \infty$ *and (2.2) stays, then*

$$H_n^{1-c} \leqslant K(c) \sum_{k=n}^{\infty} \lambda_k H_k^{-c} \tag{3.1}$$

holds for all n.

Proof. For any n let $N(n)$ denote the only natural number satisfying the inequalities

$$\sum_{k=n}^{N(n)} \lambda_k \leqslant H_n < \sum_{k=n}^{N(n)+1} \lambda_k. \tag{3.2}$$

Hence we get that

$$H_{N(n)} \leqslant 2H_n \tag{3.3}$$

also holds.

It is also clear that (2.2) implies the inequality

$$\lambda_{n+1} \leqslant KH_n,$$

whence

$$H_{n+1} \leqslant K_1 H_n,$$

and thus

$$H_n^{-1} \leqslant K_1 H_{n+1}^{-1} \tag{3.4}$$

follows.

Combining the inequalities (3.2), (3.3) and (3.4) yields the following

$$\sum_{k=n}^{\infty} \lambda_k H_k^{-c} \geqslant \sum_{k=n}^{N(n)+1} \lambda_k H_k^{-c} \geqslant H_{N(n)+1}^{-c} \sum_{k=n}^{N(n)+1} \lambda_k > K H_n^{1-c}$$

and this plainly proves (3.1).

4. Proofs

Proof of Theorem 1. First we prove the special case $c = 0$.

The part (i) of Theorem 1 was proved in [3], see the necessity part of the proof of Theorem, namely the given proof does not use that all λ_n are positive.

Thus we can turn to the proof of (ii) assuming $c = 0$. Then $\mathbf{x}$ admits a factorization (1.2) with (1.3). For such a sequence $\mathbf{z}$ we have, by definition, that

$$\sum_{i=1}^{n} |z_i|^{p^*} \leqslant ||\mathbf{z}||_{\Lambda(p^*)}^{p^*} \Lambda_n^{\frac{1}{1-p}} = ||\mathbf{z}||_{\Lambda(p^*)}^{p^*} \sum_{i=1}^{n} \left(\Lambda_i^{\frac{1}{1-p}} - \Lambda_{i-1}^{\frac{1}{1-p}} \right),$$

where $\Lambda_0 := 0$. This and Lemma 1 with the monotone decreasing sequence $w_i := \Lambda_i^{\frac{1}{p-1}} i^{\varepsilon}$, where $0 < \varepsilon < \frac{\beta}{p-1}$, imply

$$\begin{aligned}\sum_{i=1}^{n} |z_i|^{p^*} \Lambda_i^{\frac{1}{p-1}} i^{\varepsilon} &\leqslant ||\mathbf{z}||_{\Lambda(p^*)}^{p^*} \sum_{i=1}^{n} \left(\Lambda_i^{\frac{1}{1-p}} - \Lambda_{i-1}^{\frac{1}{1-p}} \right) \Lambda_i^{\frac{1}{p-1}} i^{\varepsilon} \\ &\leqslant ||\mathbf{z}||_{\Lambda(p^*)}^{p^*} \left\{ 1 + K \sum_{i=2}^{n} \frac{\lambda_{i-1}}{\Lambda_i} i^{\varepsilon} \right\}. \qquad (4.1)\end{aligned}$$

Next we show that (2.1) implies that

$$\sigma_n(\varepsilon) := \sum_{i=2}^{n} \frac{\lambda_{i-1}}{\Lambda_i} i^{\varepsilon} \leqslant K n^{\varepsilon}. \qquad (4.2)$$

If $\varepsilon \geqslant 1$ then the assertion is obvious.

If $(0 <)\varepsilon < 1$, then let $\delta = 1 - \varepsilon$. Assuming that $2^{N-1} < n \leqslant 2^N$, then by (2.1)

$$\sigma_n(\varepsilon) \leqslant \sum_{m=1}^{N} \sum_{i=2^{m-1}+1}^{2^m} i^{1-\delta} \frac{\lambda_{i-1}}{\Lambda_i} \leqslant K \sum_{m=1}^{N} 2^{-m\delta} \cdot 2^m \leqslant K_1 2^{N\varepsilon},$$

whence (4.2) follows plainly.

Thus, (4.1) and (4.2) imply that

$$\sum_{i=1}^{n} |z_i|^{p^*} \Lambda_i^{\frac{1}{p-1}} i^{\varepsilon} \leqslant K ||\mathbf{z}||_{\Lambda(p^*)}^{p^*} n^{\varepsilon}.$$

Using this and Hölder's inequality, we have

$$\begin{aligned}\left(\sum_{k=1}^{n} |x_k| \right)^p &= \left(\sum_{k=1}^{n} |y_k| w_k^{\frac{-1}{p^*}} w_k^{\frac{1}{p^*}} |z_k| \right)^p \\ &\leqslant \left(\sum_{k=1}^{n} |y_k|^p w_k^{-\frac{p}{p^*}} \right) K_1 ||\mathbf{z}||_{\Lambda(p^*)}^{p} n^{\varepsilon \frac{p}{p^*}}. \qquad (4.3)\end{aligned}$$

Since $\frac{p}{p^*} = p-1$ and $w_k = \Lambda_k^{\frac{1}{p-1}} k^{\varepsilon}$, thus (4.3) conveys that

$$\sum \lambda_n \left(\sum_{k=1}^{n} |x_k| \right)^p \leqslant K||\mathbf{z}||^p_{\Lambda(p^*)} \sum \lambda_n \sum_{k=1}^{n} |y_k|^p \Lambda_k^{-1} \left(\frac{n}{k}\right)^{\varepsilon(p-1)}$$
$$= K||\mathbf{z}||^p_{\Lambda(p^*)} \sum_{k=1}^{\infty} |y_k|^p \Lambda_k^{-1} k^{\varepsilon(1-p)} \sum_{n=k}^{\infty} \lambda_n n^{\varepsilon(p-1)}. \tag{4.4}$$

Now utilizing the assumptions $\Lambda_n n^{\beta} \downarrow$ and $\varepsilon(p-1) < \beta$, we get that

$$\sum_{n=k}^{\infty} \lambda_n n^{\varepsilon(p-1)} = \sum_{n=k}^{\infty} (\Lambda_n - \Lambda_{n+1}) n^{\varepsilon(p-1)}$$
$$\leqslant \Lambda_k k^{\varepsilon(p-1)} + K \sum_{n=k}^{\infty} \Lambda_n n^{\varepsilon(p-1)-1}$$
$$\leqslant \Lambda_k k^{\varepsilon(p-1)} + K\Lambda_k k^{\beta} \sum_{n=k}^{\infty} n^{\varepsilon(p-1)-1-\beta} \leqslant K_1 \Lambda_k k^{\varepsilon(p-1)}.$$

This and (4.4) yield

$$\sum \lambda_n \left(\sum_{k=1}^{n} |x_k| \right)^p \leqslant K||\mathbf{z}||^p_{\Lambda(p^*)} ||\mathbf{y}||^p_p.$$

This inequality plainly proves that $\mathbf{x}$ belongs to $\lambda(p)$ and that (1.6) also holds. Herewith we have proved Theorem 1 in the special case $c=0$.
The case $c>0$ of Theorem 1 can be proved by its special case $c=0$.
Let us define a new sequence $\overline{\lambda} := \{\overline{\lambda}_n\}$ as follows:

$$\overline{\lambda}_n := \lambda_n \Lambda_n^{-c}.$$

First we show that this new sequence $\overline{\lambda}$ also satisfies the same conditions as λ does in the special case $c=0$. It is clear that $\overline{\lambda}_n > 0$. Since

$$\overline{\Lambda}_n := \sum_{k=n}^{\infty} \overline{\lambda}_k = \sum_{k=n}^{\infty} \lambda_k \Lambda_k^{-c} = \Lambda_n^{(c)},$$

thus, by the assumption on $\{\Lambda_n^{(c)}\}$, the sequence $\{\overline{\Lambda}_n\}$ is also β-power-monotone decreasing.

Using Lemma 2, the monotonicity of $\{\Lambda_i\}$ and (2.1) we get that

$$\sum_{i=2}^{n} i \frac{\overline{\lambda}_{i-1}}{\overline{\Lambda}_i} \leqslant K \sum_{i=2}^{n} i \frac{\lambda_{i-1}\Lambda_{i-1}^{-c}}{\Lambda_i^{1-c}} \leqslant K \sum_{i=2}^{n} i \frac{\lambda_{i-1}}{\Lambda_i} \leqslant K_1 n,$$

that is, the sequence $\overline{\lambda}$ also satisfies the condition (2.1).

Summing up we see that $\overline{\lambda}$ satisfies all of the assumptions of Theorem 1 with $c=0$.

Moreover let $\overline{\lambda}(p)$ and $\overline{\Lambda}(p)$ denote the spaces defined with this sequence $\overline{\lambda}$. By Lemma 2 it is clear that

$$\overline{\lambda}(p) \equiv \lambda(p,c) \qquad \text{and} \qquad \overline{\Lambda}(p) \equiv \Lambda(p,c).$$

Thus Theorem 1 with $\overline{\lambda}$ and $c=0$ conveys the statements of Theorem 1 with the given positive c, herewith Theorem 1 is verified.

Proof of Theorem 2. Conceptually the proof of Theorem 2 follows the line of Theorem 1 with $c>0$, but now we define the sequence $\tilde{\lambda} := \{\tilde{\lambda}_n\}$ as follows

$$\tilde{\lambda}_n := \lambda_n H_n^{-c}.$$

Then

$$\tilde{\Lambda}_n := \sum_{k=n}^{\infty} \tilde{\lambda}_k = \sum_{k=n}^{\infty} \lambda_k H^{-c} = R_n^{(c)},$$

thus, by the assumption on $\{R_n^{(c)}\}$, the sequence $\{\tilde{\Lambda}_n\}$ is also β-power-monotone decreasing.

Since $\sum \lambda_n = \infty$ we can apply both Lemma 3 and Lemma 5, thus there exist two positive constants K_1 and K_2 such that

$$K_1 H_n^{1-c} \leqslant \tilde{\Lambda}_n \leqslant K_2 H_n^{1-c}. \tag{4.5}$$

Hereupon we can easily verify that the sequence $\tilde{\lambda}$ satisfies the condition (2.1), too.
Namely (2.2) implies that

$$H_{n-1}^{-1} \leqslant K H_n^{-1}, \tag{4.6}$$

see (3.4). Thus, by (2.2), (4.5) and (4.6), an elementary consideration gives that

$$\begin{aligned}\sum_{k=2}^{n} k\frac{\tilde{\lambda}_{k-1}}{\tilde{\Lambda}_k} &\leqslant K\sum_{k=2}^{n} k\frac{\lambda_{k-1}H_{k-1}^{-c}}{H_k^{1-c}} \leqslant K_1\sum_{k=2}^{n} k\frac{\lambda_{k-1}}{H_k}\\ &\leqslant K_2\left(1+\sum_{k=1}^{n}\frac{\lambda_{k+1}}{H_k}k\right) \leqslant K_3 n,\end{aligned}$$

i.e. the sequence $\tilde{\lambda}$ satisfies (2.1) indeed.

Now let $\tilde{\lambda}(p)$ and $\tilde{\Lambda}(p)$ denote the spaces defined with the sequence $\tilde{\lambda}$. Then, by (4.5), it is plain that

$$\tilde{\lambda}(p) \equiv \lambda(p,c,H) \qquad \textit{and} \qquad \tilde{\Lambda}(p) \equiv \Lambda(p,c,H).$$

The final conclusion is self-evident: Theorem 1 with the sequence $\tilde{\lambda}$ and $c=0$ proves Theorem 2 for the sequence λ.

REFERENCES

[1] G. BENNETT, *Some elementary inequalities*, Quart. J. Math. Oxford **38** (1987), 401–425.
[2] G. BENNETT, *Factorizing the Classical Inequalities*, Memoirs of Amer. Math. Soc. **120** (1996), Number 576, 1–130.
[3] L. LEINDLER, *A theorem of Hardy-Bennett-type*, Acta Math. Hung. **78(4)** (1998), 315-325.
[4] L. LEINDLER, *Two theorems of Hardy-Bennett-type*, Acta Math. Hung. **79(4)** (1998), 341–350.

(Received February 3, 1998)

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