

STRONG DOUBLING CONDITIONS

STEPHEN M. BUCKLEY

(communicated by L.-E. Persson)

Abstract. We show that the class of strong doubling measures depends essentially on the parameter t , and that the measure of the boundary layer of a QHBC domain decays geometrically, if the measure is suitably strong doubling.

0. Introduction

Various areas of analysis utilize doubling measures, i.e. positive Borel measures on \mathbb{R}^n satisfying (some variation of) the condition:

$$\mu(B(x, r)) \leq C\mu(B(x, r/2)), \quad \text{for all } x \in \mathbb{R}^n, r > 0. \quad (0.1)$$

For instance, Chapter I of [S2] investigates many questions in harmonic analysis within a general framework involving a measure that satisfies a doubling condition relative to a set of generalized balls in \mathbb{R}^n , and [HKM] develops the potential theory of a certain class of degenerate elliptic partial differential equations that involve admissible weights, where a weight w is admissible if the measure $w dx$ satisfies certain conditions including (0.1).

Much of this analysis takes place on an open subset Ω of \mathbb{R}^n , rather than on all of \mathbb{R}^n (for instance, this is often the case for PDE-related analysis). Some such results require only a local doubling condition for balls $B(x, 2r) \subset \Omega$, for instance, but often a stronger form of doubling is required. It is then quite common to assume that the measure is defined on all of \mathbb{R}^n and satisfies (0.1); this, for example, is the approach adopted in [HKM] for the definition of an admissible weight. However, there exist rather nice measures defined on an open set Ω which are not restrictions of global doubling measures, e.g. *power-weight measures* $d\mu = \delta_\Omega^a dx$ for certain domains Ω , where $\delta_\Omega(x)$ is the distance from x to $\partial\Omega$, and $a > 0$. The author wishes to thank Paul MacManus for kindly providing an explicit example of this type (given at the end of Section 1).

Mathematics subject classification (1991): 28A75, 42B25.

Key words and phrases: Doubling condition, QHBC domain, maximal functions, covering lemmas.

The author was partially supported by Forbairt.

One doubling condition applicable to measures on Ω is the *boundary doubling condition*:

$$\mu(B(x, r) \cap \Omega) \leq C\mu(B(x, r/2) \cap \Omega), \quad \text{for all } x \in \Omega, r > 0. \quad (0.2)$$

This condition, however, places restrictions on Ω as well as on μ , since even Lebesgue measure does not always satisfy (0.2) (see the proof of Theorem 1.1). The concept of a *strong doubling measure*, employed in [BKL] and [BO] to prove inequalities of Poincaré and Trudinger type, is an attractive intermediate option; there are actually a family of such strong doubling conditions indexed by a parameter $1 < t < \infty$ (see Section 1). These conditions are strong enough to do some non-local analysis, but weaker than boundary doubling. Additionally, they are all satisfied by the measure $\delta_\Omega^a dx$, $a \geq 0$, no matter how bad the geometry of the domain Ω .

In Section 1, we determine how strong doubling conditions relate to each other and to other doubling conditions; in particular, we show that all strong doubling conditions are different, since there exist measures which are strong doubling for all parameters less than t , but not for parameter t . In Section 2, we prove that if a measure is appropriately strong doubling on a QHBC domain Ω , then the measure of the part of Ω lying within a distance ε of $\partial\Omega$ is dominated by a power of ε . This result, which generalizes a result of Smith and Stegenga on the Minkowski dimension of $\partial\Omega$, has been used in [BO, Theorem 3.10] to prove a theorem on Trudinger-type inequalities.

1. Various doubling conditions

Throughout this paper, Ω is a proper open subset of \mathbb{R}^n , which we may further restrict as necessary. If $B = B(x, r)$ is a ball, and $t > 0$, we write $tB = B(x, tr)$ (and so $t^{-1}B = B(x, r/t)$). We also write $\delta_\Omega(x) \equiv \text{dist}(x, \partial\Omega)$, $x \in \Omega$, and define the *quasihyperbolic length* of a rectifiable path $\gamma \subset \Omega$ to be

$$k_\Omega(\gamma) = \int_\gamma \delta_\Omega(x)^{-1} ds.$$

The *quasihyperbolic distance* between $x, y \in \Omega$, $k_\Omega(x, y)$, is then defined to be the infimum of $k_\Omega(\gamma)$, as γ ranges over all paths linking x and y . There exists a *quasihyperbolic geodesic* between any pair of points $x, y \in \Omega$, i.e. a path $\gamma_{x,y}$ such that $k_\Omega(x, y) = k_\Omega(\gamma_{x,y})$; see [GO].

A (necessarily bounded) domain Ω satisfies a *quasihyperbolic boundary condition* (more briefly, Ω is *QHBC*) with respect to its *QHBC center* $x_0 \in \Omega$ if there exists a constant $C \geq 1$ such that for all $x \in \Omega$,

$$k_\Omega(x, x_0) \leq C \log \left(\frac{C}{\delta(x)} \right).$$

The *QHBC path* for x is the quasihyperbolic geodesic for x, x_0 , and the *QHBC constant* of Ω , denoted C_Ω , is the smallest value of C for which the above inequality is valid.

We say that a bounded domain Ω is a *John domain* with respect to its *John center* $x_0 \in \Omega$ if there exists a constant $K \geq 1$ such that for all $x \in \Omega$, there is a path

$\gamma = \gamma_x : [0, l] \rightarrow \Omega$ parametrized by arclength satisfying $\gamma(0) = x$, $\gamma(l) = x_0$, and $\delta(\gamma(t)) \geq t/K$. We call γ_x the *John path* for x , and we define K_Ω , the *John constant* of Ω , to be the smallest value of K for which the above inequality is valid.

Clearly every John domain is a QHBC domain, but it is not difficult to construct examples of non-John QHBC domains (e.g. see [BO, Section 5]). Note that the choice of center point $x_0 \in \Omega$ in the definitions of John and QHBC domains is unimportant, in the sense that if Ω is John (or QHBC) with respect to one point, it is John (or QHBC) with respect to all of its points (of course, the John/QHBC constant tends to infinity as we let x_0 approach $\partial\Omega$).

Suppose that $0 < t \leq \infty$ and that μ is a positive Borel measure on Ω . We say that μ is *t-doubling* on Ω , denoted $\mu \in D_t(\Omega)$, if there exists a constant C such that

$$\mu(B \cap \Omega) \leq C\mu(2^{-1}B \cap \Omega) < \infty$$

whenever B is a ball for which $t^{-1}B \subset \Omega$ (in the case $t = \infty$, we merely require the center of B to lie in Ω , or equivalently in $\overline{\Omega}$). We denote by $C_{\mu,t}$ the smallest such constant C for which this doubling condition is true ($0 < t \leq \infty$).

Note that the t -doubling condition imposes restrictions on the boundary behaviour of the measure precisely when $t \geq 1$. We say that a t -doubling measure μ is a *locally doubling* if $t < 1$, *strong doubling* if $t > 1$, and *boundary doubling* if $t = \infty$. Obviously, strong doubling is logically stronger than local doubling but weaker than boundary doubling. In fact, it is not difficult to construct examples of a measure that is local doubling but not strong doubling, or strong doubling but not boundary doubling. Whether or not strong doubling depends on the parameter $t \in (1, \infty)$ is a more difficult question which we now answer.

THEOREM 1.1. *Suppose $0 < t < t' \leq \infty$. If $t' \geq 1$, $D_{t'}(\Omega) \setminus D_t(\Omega)$ is non-empty for some QHBC domain $\Omega \subset \mathbb{R}^n$. If $t' < 1$, then $D_t(\Omega) = D_{t'}(\Omega)$ for every proper open set Ω .*

Before proving this theorem, we first state a simple but useful lemma.

LEMMA 1.2. *A sphere $S \subset \mathbb{R}^n$ of radius $a > 0$ can be covered by balls $\{B_i\}_{i=1}^m$, centered on S and of radius $b \in (0, a)$, for some m dependent only on n and b/a .*

Proof. We choose a sequence of disjoint balls B'_1, B'_2, \dots , centered on S and of radius $b/3$ as long as we can continue to do so; this process must halt in a bounded number of steps since each ball covers a fixed fraction (dependent only on $b/a, n$) of the surface measure of S . If the resulting balls are B'_1, \dots, B'_m , then the required balls are $B_i = 3B'_i$, $1 \leq i \leq m$. \square

Proof of Theorem 1.1. The equivalence of all local doubling conditions is intuitively rather obvious, but we prove it for completeness. Assume that $\mu \in D_t(\Omega)$ for some $t < 1$, and so $\mu(B) \leq C\mu(2^{-1}B)$ whenever $B = B(x, r)$, $0 < r \leq t\delta_\Omega(x)$. We fix such a ball $B(x, r)$ with $r = t\delta_\Omega$, and write $c = (2 - 2t)/(2 - t)$. Applying Lemma 1.2 with $a = (1 - c/2)r$, $b = cr/4$, to the sphere $S = \{y : |x - y| = a\}$, we get balls B_1, \dots, B_m covering S , where m is bounded by some number dependent

only on n and t . Our choice of parameters ensures that

$$\begin{aligned} 2B_i &\subset B(x, r), \\ 4B_i &\supset B(x, (1 + (c/4))r) \setminus B(x, r), \\ 4t^{-1}B_i &\subset \Omega. \end{aligned}$$

We deduce that $\mu \in D_{f(t)}$, where $f(t) = (5t - 3t^2)/(4 - 2t) = (1 + c/4)t$. Defining $t_0 = t$ and $t_k = f(t_{k+1})$ for all $k > 0$, it follows iteratively that $\mu \in D_{t_k}$ for every $k > 0$. Note that $ct/4 < 1 - t$ and so the sequence (t_k) is increasing and bounded by 1. Since f is continuous on $(0, 2)$ and 1 is the only fixed point there, we deduce that t_k tends to 1 as $k \rightarrow \infty$. Thus $\mu \in D_{t'}$ for all $t' < 1$, as required.

Letting Ω be the unit ball in \mathbb{R}^n , it is easy to find $\mu \in D_t(\Omega) \setminus D_1(\Omega)$ whenever $t < 1$. For example, $d\mu = (1 - |x|)^{-1} dx$ is such a measure. Alternatively, we could take $d\mu = [\log(2/(1 - |x|))]^{-2} (1 - |x|)^{-1} dx$; in this latter case, $\mu(\Omega) < \infty$.

In the remaining cases, we give only planar counterexamples to equality; these are easily modified to give counterexamples in any larger dimension. The domains we use will consist of a central square with small narrow pieces attached. It is convenient for us to take as our central square

$$Q_0 = \{(x, y) \in \mathbb{R}^2 : -1 < x < 0, 0 < y < 1\}.$$

We first consider the case $t' = \infty$ (even though it follows from the case $t' < \infty$), because we can produce a counterexample here with μ equal to Lebesgue measure. Note first that Lebesgue measure lies in $D_t(\Omega)$ for all $t < \infty$, regardless of the domain Ω . We define Ω to be the union of Q_0 and the rectangles

$$R_k = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 2^{-k}, 1 - 2^{-k}(1 + 1/k) < y < 1 - 2^{-k}\}.$$

Then $|\cdot|$ is not boundary doubling because $|2^{-1}B_k \cap \Omega| \approx |B_k \cap \Omega|/k$, where B_k is the ball whose center is the same as the center of R_k and whose radius is 2^{-k} . The only possible obstacle to Ω being QHBC is the narrowness of the rectangles R_k . Since the length-to-width ratio of R_k , i.e. k , is dominated by the logarithm of the reciprocal of R_k 's diameter, this is not a genuine obstacle, and it is easy to check that Ω is QHBC.

To prove the remaining cases, it suffices to find, for all $1 < t < t' < \infty$, a QHBC domain Ω and a measure μ such that $\mu \in D_t(\Omega) \setminus D_{t'}(\Omega)$. By elementary geometry, we note that if B is a ball inscribed in the cone $K_a = \{(x, y) : |y| < ax\}$, $a > 0$, then the dilate sB , $s > 1$, contains the vertex of K_a if and only if $a > f(s) = [s^2 - 1]^{-1/2}$. We shall define Ω to be a union of Q_0 and a sequence of diamond-shaped sets S_k . First, we define the preliminary diamond-shaped sets

$$S'_k = \{(x, y) \in \mathbb{R}^2 : 0 \leq |x| < x_k, |y| < f(t')|x - x_k|\},$$

where $x_k = 2^{-k}c$, and $c = \min\{1, [4f(t')]^{-1}\}$. We then write $y_k = 1 - 2^{-k}$ and define S_k to be the translate of S'_k by the vector $(x_k - x_k^2, y_k)$. Note that the sets S_k have a small overlap with Q_0 but are disjoint from each other. The sets S_k are of a fixed length-to-width ratio, but there is a new potential obstacle to Ω being a QHBC domain: each S_k is attached to Q_0 by a narrow neck whose width is proportional to

x_k^2 . However, it is a routine exercise to check that if “satellite pieces” (such as S_k) are adjoined to the main part of the domain via bottlenecks of width proportional to a fixed power of the length of the satellite, then this does not destroy the QHBC condition. Consequently, Ω is QHBC.

Let us denote by U_k and V_k the vertices $(-x_k^2, y_k)$ and $(2x_k - x_k^2, y_k)$, respectively, of S_k . Defining

$$\begin{aligned} g_k(x) &= x_k - |x - x_k + x_k^2|, \quad x \in \mathbb{R}, \\ w_s(x, y) &= [x_k^2 / g_k(x)]^s, \quad (x, y) \in S_k, \\ d\mu_s &= w_s(x, y) dx dy, \quad (x, y) \in S_k, \end{aligned}$$

we see that $\mu_s(S_k) < \infty$ for $0 < s < 2$, but not for $s = 2$. Furthermore, as $s \rightarrow 2^-$, more and more of the μ_s -mass of S_k is concentrated closer and closer to U_k and V_k . More precisely,

$$\lim_{s \rightarrow 2^-} \frac{\mu_s(\{X \in S_k : \min(|X - U_k|, |X - V_k|) < (2 - s)x_k\})}{\mu_s(S_k)} = 1.$$

By a routine calculation, this last limit reduces to the fact that $\lim_{t \rightarrow 0^+} t^t = 1$.

We are now ready to define a measure $\mu \in D_t(\Omega) \setminus D_{t'}(\Omega)$. Specifically, we take $d\mu \equiv w(x, y) dx dy$, where

$$w(x, y) = \begin{cases} 1, & (x, y) \in Q_0, \\ [x_k^2 / g_k(x)]^{2-2/k}, & (x, y) \in S_k \setminus Q_0. \end{cases}$$

Note that w is continuous across the necks of the sets S_k (i.e. at $x = 0$) and, by the above considerations, most of the μ -measure of S_k is concentrated very near V_k if k is large. Considering balls inscribed in S_k near this vertex, we deduce that any t' -doubling condition is violated for sufficiently large k . By contrast, μ is t -doubling for all $0 < t < t'$. To see this, note that balls centered in S_k satisfy a t -doubling condition (because their t^{-1} -dilates stay away from V_k), and that balls centered in Q_0 actually satisfy an ∞ -doubling condition (because the average value of w on S_k is bounded, as can easily be checked). \square

In the above proof, we chose Ω to be the unit ball when defining a locally doubling measure on Ω which is not 1-doubling. By contrast, the fact that, for $1 < t \leq \infty$, the D_t -conditions are all distinct, made use of a domain which, although QHBC, was nevertheless rather nasty. We now show that such nastiness is in fact unavoidable.

PROPOSITION 1.3. *If Ω is a John domain, then $D_t(\Omega) = D_\infty(\Omega)$ for all $t \geq t_0$, where t_0 depends only on K_Ω , the John constant of Ω .*

Proof. Let x_0 be the John center of Ω . We fix a ball $B = B(x, r)$, $x \in \Omega$. If $x_0 \in B$, then either $B \subset \Omega$, or B contains a ball of radius $\delta_\Omega(x_0)/2$. In both cases, the required estimate

$$\mu(B \cap \Omega) \leq C\mu(2^{-1}B \cap \Omega) < \infty$$

follows easily from the assumption that $\mu \in D_t(\Omega)$ for sufficiently large $t = t(K_\Omega)$. Thus we may assume that $x_0 \notin B$. We choose any point y on the John path for x with

respect to x_0 which lies in the annulus $B(x, r/3) \setminus B(x, r/6)$. The John condition ensures that $B' = B(y, r') \subset \Omega$ where $r' = r/6K_\Omega$. Since $B' \subset 2^{-1}B$ and $8K_\Omega B' \supset B$, it follows that $\mu(B \cap \Omega) \leq C\mu(2^{-1}B \cap \Omega)$ if $\mu \in D_{8K_\Omega}(\Omega)$. \square

We end this section by giving an example, essentially due to Paul MacManus, of a strong doubling measure which is not the restriction of a global doubling measure. Let us fix $s > 0$ and define $d\mu_\Omega = \delta_\Omega^s dx$ for any proper non-empty open subset Ω of \mathbb{R}^n . Note that $\mu_\Omega \in D_t(\Omega)$ for every proper open subset Ω of \mathbb{R}^n , with doubling constant dependent only on s , n , and t . We define Ω_k to consist of the interval $(0, 2)$ with the points i/k removed, $1 \leq i \leq k$. Suppose that μ_{Ω_k} is a restriction of a global doubling measure μ_k . Since the measure of a countable set is zero (for any global doubling measure), $\mu_k(1, 2)/\mu_k(0, 1) \rightarrow \infty$ as $k \rightarrow \infty$. By piecing together sets like Ω_k , it is thus easy to define a set Ω such that μ_Ω is not the restriction of a global doubling measure. We could for instance take Ω to be the bounded open set given by

$$\Omega = \{2^{-k-1}x + 1 - 2^{-k+1} : x \in \Omega_k, k \in \mathbb{N}\}.$$

One can even define a domain $D \subset \mathbb{R}^n$, $n > 1$, such that $\delta_D^s dx$ is strong doubling but not the restriction of a global doubling measure. For instance, if Ω is as above, then $D = \Omega \times (0, 1) \cup (-1, 0] \times (0, 1)$ is one such domain. Note that here we need the rather well-known fact that line segments are null sets for all doubling measures on \mathbb{R}^2 ; this fact is, for example, an easy corollary of Theorem 2.4).

2. Geometric decay of the measure of a QHBC boundary layer

In this section, we shall prove that the measure of the boundary layer of a QHBC domain decays like a power of its thickness if the measure is appropriately strong doubling. We begin, though, with some preliminary definitions and lemmas. If p is an exponent and S is a set, we write $p' = p/(p-1)$, and χ_S for the characteristic function of S . If Ω is a bounded domain, we denote by $\text{diam}(\Omega)$ and $\text{inrad}(\Omega)$ its diameter and inradius (the latter being the radius of the largest ball that fits inside Ω). If $t > 0$ and $f \in L^1_{\text{loc}}(\Omega)$, we define the maximal function

$$M_t f(x) \equiv M_{t;\Omega,\mu} f(x) = \sup_{x \in B \subset \Omega} \frac{1}{\mu(tB)} \int_{tB \cap \Omega} |f| d\mu,$$

where the supremum is taken over all balls B satisfying the indicated conditions.

Our first lemma is both a generalization of the well-known Besicovitch Covering Theorem, and a special case of a theorem of Morse [M] (also stated in [G]), and consequently needs no proof.

LEMMA 2.1. *Suppose that $0 < s < 1$, that $A \subset \mathbb{R}^n$, and that \mathcal{F} is a family of balls of bounded radius. If for every $x \in A$, \mathcal{F} contains a ball B_x of radius at most R such that $x \in sB_x$, then there exist subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_k \subset \mathcal{F}$ such that each \mathcal{F}_i is a pairwise disjoint collection of balls, $\bigcup_{i=1}^k \mathcal{F}_i$ covers A , and $k \leq N$ for some N dependent only on n and s .*

The following lemma belongs to the large family of results that state that various maximal operators are bounded on L^p , $1 < p \leq \infty$.

LEMMA 2.2. *If Ω is a bounded domain in \mathbb{R}^n , and μ is a positive Borel measure on \mathbb{R}^n , then $M_{t;\Omega,\mu}$ is bounded on $L^p(\Omega, \mu)$ for all $1 < p \leq \infty$, $1 < t$. Furthermore, its operator norm is bounded by Cp' , for some constant C dependent only on n and t .*

Note that we do not assume that μ satisfies any doubling assumption. If we assumed that $\mu \in D_{5t}(\Omega)$, then the alternative “5-covering lemma” (see e.g. [S1, Section 1.1]) could be used in place of Lemma 2.1 in the following proof sketch; additionally, the lemma would be true for all $t > 0$, and not just $t > 1$.

Sketch of proof of Lemma 2.2. As usual for results of this type, the proof consists of an interpolation between the (obvious) boundedness of M_t on $L^\infty(\Omega, \mu)$, and its boundedness from $L^1(\Omega, \mu)$ to the Lorentz (or “weak-type”) space $L^{1,\infty}(\Omega, \mu)$. Such weak-type boundedness results are always proved by means of a covering theorem (see, for example, [S2, Section I.3.1]). Here, we take $f \in L^1(\Omega, \mu)$, fix a cut-off value $\alpha > 0$ and, for each x such that $A = \{x : M_t f(x) > \alpha\}$, we associate a ball B'_x such that $x \in B'_x \subset \Omega$, and such that the μ -average of $|f|$ on $B_x \equiv tB'_x$ exceeds α . By applying Lemma 2.1 with $s = 1/t$ to the family $\{B_x : x \in A\}$, weak boundedness follows in the usual manner. \square

The next lemma is also a variant of a rather well-known lemma (e.g. see [Bo]); we include a proof for completeness. In its proof and later, we use $A \lesssim B$ if $A \leq CB$ for some constant C dependent only on allowed parameters. In particular, we stress that C is not allowed to depend on p in this lemma.

LEMMA 2.3. *Suppose that $1 \leq p < \infty$, $1 < t$, $\Omega \subset \mathbb{R}^n$, and $\mu \in D_t(\Omega)$. Let \mathcal{F} be a family of balls contained in Ω , and let a_B be a non-negative number for each $B \in \mathcal{F}$. Then*

$$\left\| \sum_{B \in \mathcal{F}} a_B \chi_{tB} \right\|_{L^p(\Omega, \mu)} \leq Cp \left\| \sum_{B \in \mathcal{F}} a_B \chi_B \right\|_{L^p(\Omega, \mu)}.$$

where C depends only on n , t , and $C_{\mu,t}$.

Proof. Let g be a non-negative function in $L^{p'}(\Omega, \mu)$. Since μ is t -doubling,

$$A \equiv \int_{\Omega} \left(\sum_{B \in \mathcal{F}} a_B \chi_{tB} \right) g \, d\mu \lesssim \sum_{B \in \mathcal{F}} a_B \left[\frac{1}{\mu(tB)} \int_{tB \cap \Omega} g \, d\mu \right] \cdot \mu(B).$$

We now use the fact that the bracketed quantity is dominated by $M_t g(x)$ for every $x \in B$, together with Hölder's inequality and Lemma 2.2, to get

$$\begin{aligned} A &\lesssim \sum_{B \in \mathcal{F}} a_B \int_B M_t g \, d\mu = \int_{\Omega} M_t g \cdot \sum_{B \in \mathcal{F}} a_B \chi_B \, d\mu \\ &\leq \|M_t g\|_{L^{p'}(\Omega, \mu)} \cdot \left\| \sum_{B \in \mathcal{F}} a_B \chi_B \right\|_{L^p(\Omega, \mu)} \\ &\lesssim \|g\|_{L^{p'}(\Omega, \mu)} \cdot \left\| \sum_{B \in \mathcal{F}} a_B \chi_B \right\|_{L^p(\Omega, \mu)} \end{aligned}$$

Taking a supremum over all $g \geq 0$ in the unit ball of $L^{p'}(\Omega, \mu)$, the required result follows by duality. \square

In [SS], Smith and Stegenga prove that if $\Omega \subset \mathbb{R}^n$ is a QHBC domain, then the Minkowski dimension d of $\partial\Omega$ is bounded away from n , i.e. the Lebesgue measure of the “boundary layer” decays geometrically; for more on Minkowski content and the decay of the Lebesgue measure of boundary layers of sets, we refer the reader to [MV]. In the planar simply-connected case, Smith and Stegenga’s result follows, with a sharp estimate of d , from the results in [JM]. Koskela and Rohde [KR], reproved Smith and Stegenga’s result, in the process getting the sharp estimate of d in all dimensions. The next theorem generalizes this boundary layer decay to the setting of strong doubling measures; our proof is based on the method of [KR].

THEOREM 2.4. *Suppose that Ω is QHBC and that $\mu \in D_t(\Omega)$, for some $t > t_0$, where $t_0 \in (1, \infty)$ is dependent only on n and C_Ω . Then there exist $C, \alpha > 0$ dependent only on n , C_Ω , and $C_{\mu,t}$, such that*

$$\mu(\Omega_r) \leq C(r/\text{diam}(\Omega))^\alpha \mu(\Omega) < \infty, \quad \text{for all } r > 0.$$

In the above statement, recall that C_Ω is the QHBC constant of Ω and $C_{\mu,t}$ is the t -doubling constant of μ . The QHBC condition is necessary in the above theorem—just take μ to be Lebesgue measure, and $\Omega \subset \mathbb{R}^n$ to be any domain whose boundary has Minkowski dimension n . It is also necessary to assume a $D_t(\Omega)$ condition for sufficiently large t . For instance, if $\Omega \subset \mathbb{R}^2$ consists of all points in the unit disk whose argument is at most $\theta \in (0, \pi)$, the measure $d\mu(x) = (|x| \log^2(2/x))^{-1} dx$ does not satisfy the conclusion of the theorem even though $\mu \in D_t(\Omega)$ for $t < \sec^{-1} \theta$ (note that both $\sec^{-1} \theta$ and C_Ω tend to infinity as $\theta \rightarrow 0$).

Proof of Theorem 2.4. Assuming $t \geq t_1 \equiv \text{diam}(\Omega)/\text{inrad}(\Omega)$, the doubling condition ensures that $\mu(\Omega) < \infty$; note also that t_1 is bounded above by a constant dependent only on C_Ω . Without loss of generality, we normalize Ω so that $\text{diam}(\Omega) = 1$, and μ so that $\mu(\Omega) = 1$. Let $\varepsilon = 1/C_\Omega$ and $c = 1/10$. For each $x \in \partial\Omega$, and $n > 0$, we define

$$\begin{aligned} A_n(x) &= \{y \in \mathbb{R}^n : (1 + \varepsilon)^{-n} < |x - y| < (1 + \varepsilon)^{-n+1}\}, \\ \chi_n(x) &= \begin{cases} 1, & \text{if } \exists y \in \Omega \cap A_n(x) : d(y, \partial\Omega) > c\varepsilon|x - y|, \\ 0, & \text{otherwise,} \end{cases} \\ \sigma_n(x) &= \sum_{k=1}^n \chi_k(x). \end{aligned}$$

Koskela and Rohde [KR] prove that the boundary of a QHBC domain is what they term an ε -mean porous set (with auxiliary constant $c = 1/10$, as here). This means that there exists a number n_0 , depending only on C_Ω , such that $\sigma_n(x) > n/2$ for all $n \geq n_0$ (actually, the mean porosity of a set only implies the existence of certain holes in its complement, but an examination of the proof of Theorem 5.1 in [KR] reveals that one can assume that these holes are contained in the domain itself, as we do here).

It follows, as in Theorem 2.1 of [KR], that we can find a collection \mathcal{F} of pairwise disjoint open balls and constants $t_2 > 1$, $j_0 \geq 1$, $c' > 0$, all dependent only on n and

C_Ω , such that

$$\sum_{B \in \mathcal{F}} \chi_{t_2 B}(x) > c'j, \quad x \in \Omega_{2^{-j}}, j \geq j_0.$$

Note that in [KR], an initial reduction argument (which we do not use here) gives $n_0 = 1$, and hence $j_0 = 1$. We define $t_0 = \max\{t_1, t_2\}$.

Writing $u(x) = \sum_{B \in \mathcal{F}} \chi_{t_2 B}(x)$ for all $x \in \Omega$, we have $\exp(au(x)) > \exp(ac'j)$ for all $x \in \Omega_{2^{-j}}$, $j > j_0$, and $a > 0$. It therefore suffices to find a constant $a = a(n, C_\Omega, C_{\mu,t}) > 0$ such that

$$\int_{\Omega_{2^{-j}}} e^{au(x)} d\mu(x) \lesssim \mu(\Omega_1).$$

Now,

$$\int_{\Omega_{2^{-j}}} e^{au} d\mu \leq \sum_{k \geq 0} \int_{\Omega_1} \frac{(au)^k}{k!} d\mu \leq \mu(\Omega_1) + \sum_{k > 0} \frac{a^k}{k!} \int_{\Omega_1} \left(\sum_{B \in \mathcal{F}} \chi_{t_2 B} \right)^k d\mu.$$

Since $\mu \in D_t(\Omega) \subset D_{t_2}(\Omega)$, we may use Lemma 2.3 to get

$$\begin{aligned} \int_{\Omega_{2^{-j}}} e^{au(x)} d\mu(x) &\leq \mu(\Omega_1) + \sum_{k > 0} \frac{(aCk)^k}{k!} \int_{\Omega_1} \left(\sum_{B \in \mathcal{F}} \chi_B \right)^k d\mu \\ &\lesssim \mu(\Omega_1) \left(1 + \sum_{k > 0} \frac{(aCk)^k}{k!} \right). \end{aligned}$$

This last series converges for all $a < 1/Ce$, and so we are done. \square

REFERENCES

- [Bo] B. BOJARSKI, *Remarks on Sobolev imbedding inequalities*, Proc. of the conference on Complex Analysis, Joensuu 1987, Lecture Notes in Math. 1351, Springer-Verlag, Berlin, 1989, pp. 52–68.
- [BKL] S.M. BUCKLEY, P. KOSKELA, AND G. LU, *Subelliptic Poincaré inequalities: the case $p < 1$* , Publ. Mat. **39** (1995), 313–334.
- [BO] S.M. BUCKLEY AND J. O'SHEA, *Weighted Trudinger type inequalities*, preprint.
- [GO] F.W. GEHRING AND B. OSGOOD, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.
- [G] M. DE GUZMÁN, *Differentiation of integrals in \mathbb{R}^n* , Lecture Notes in Math. 481, Springer-Verlag, Berlin, 1975.
- [HKM] J. HEINONEN, T. KILPELÄINEN, AND O. MARTIO, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Univ. Press, Oxford, 1993.
- [JM] P.W. JONES AND N.G. MAKAROV, *Density properties of harmonic measure*, Ann. of Math. (2) **142** (1995), 427–455.
- [KR] P. KOSKELA AND S. ROHDE, *Hausdorff dimension and mean porosity*, Math. Ann. **309** (1997), 593–609.
- [MV] O. MARTIO AND M. VUORINEN, *Whitney cubes, p -capacity, and Minkowski content*, Expo. Math. **5** (1987), 17–40.
- [M] A.P. MORSE, *Perfect blankets*, Trans. Amer. Math. Soc. **6** (1947), 418–442.
- [SS] W. SMITH AND D.A. STEGENGA, *Exponential integrability of the quasihyperbolic metric in Hölder domains*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **16** (1991), 345–360.

- [S1] E.M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.
- [S2] E.M. STEIN, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, 1993.

Department of Mathematics
National University of Ireland
Maynooth
Co. Kildare
Ireland
e-mail: sbuckley@maths.may.ie