

ON THE ROOTS OF LACUNARY POLYNOMIALS

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(communicated by J. Borwein)

Abstract. We prove estimates for the roots of lacunary polynomials. They are deduced from the study of the equation $x^n - x^{n-1} = a$, where $n \in \mathbf{N}$, $n \geq 2$, $a > 0$.

Let $P(X) = \sum_{i=0}^m a_i X^i$ be a nonzero complex polynomial. We are interested to find bounds for the absolute values of the roots of P in function of the coefficients a_i . Such estimates were obtained by P. Montel [4] and more recently by M. Mignotte [3]. Other bounds are given by evaluations valid for arbitrary polynomials (good references are to be found, for example, in the monographs of P. Henrici [2] and P. Borwein–T. Erdélyi [1]).

In this paper we obtain such bounds in the following way:

To the polynomial P we associate convenient $a > 0$ and $n \in \mathbf{N} \setminus \{0, 1\}$, with $a = a(a_0, a_1, \dots, a_m) > 0$. Therefore we obtain bounds for the unique root $\xi > 1$ of the equation $x^n - x^{n-1} = a$. This allows us to describe bounds for the roots of the original polynomial P . In particular, this method gives good estimates for the case of lacunary polynomials, i.e. for polynomials with a certain number of consecutive zero coefficients.

1. Bounds for the Solutions of $x^n - x^{n-1} \leq a$

We shall study separately the cases $0 < a < 1$ and $a \geq 1$.

1.1. The equation $x^n - x^{n-1} - a = 0$ for $0 < a < 1$.

Our evaluation follows from the study of the condition $(z - c)(z + 1)^m \geq 1$, where $c, z > 0$, $m \in \mathbf{N}$, $m \geq 1$.

LEMMA 1.1. *If $z > 0$, $c > 0$, $m \in \mathbf{N}^* := \mathbf{N} \setminus \{0\}$ and $(z - c)(z + 1)^m = 1$, then*

$$c < z < \min\{d_1, d_2\},$$

where

$$d_1 = d_1(c, m) = \frac{mc - 1 + \sqrt{(1 - mc)^2 + 4(1 + c)m}}{2m}, \quad d_2 = d_2(c, m) = c + \sqrt{t_2}$$

and

$$t_2 = \frac{2}{(1 + c) \left(m + \sqrt{m^2 + \frac{2}{3}m(m - 1)(m - 2)(m - 3)(1 + c)} \right)}.$$

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Proof. First observe that we must take $z > c$ and that the function $z \mapsto (z-c)(z+1)^m$ is increasing for $z > c$.

The inequality $(z-c)(z+1)^m \geq 1$ is fulfilled if the following condition is satisfied:

$$(z-c)(1+mz) \geq 1, \quad (1)$$

Now look to the previous condition. It is equivalent to $mz^2 + (1-mc)z - c - 1 \geq 0$ and it holds when

$$z \geq d_1 := \frac{mc - 1 + \sqrt{(1-mc)^2 + 4(1+c)m}}{2m}. \quad (2)$$

To obtain another evaluation for z , we put $t = z - c$. Then $(z+c)(z+1)^m \geq 1$ becomes

$$t(1+c+t)^m \geq 1. \quad (3)$$

Note that (3) is satisfied if one has:

$$t \left((1+c)mt + \frac{(1+c)^3 \cdot m(m-1)(m-2)(m-3)}{6} t^3 \right) \geq 1. \quad (4)$$

The condition (4) is fulfilled if

$$\begin{aligned} t^2 \geq t_2 &:= \frac{-m(1+c) + \sqrt{m^2(1+c)^2 + \frac{2}{3}m(m-1)(m-2)(m-3)(1+c)^3}}{\frac{m(m-1)(m-2)(m-3)(1+c)^3}{3}} \\ &= \frac{2}{(1+c) \left(m + \sqrt{m^2 + \frac{2}{3}m(m-1)(m-2)(m-3)(1+c)} \right)}. \end{aligned}$$

The corresponding upper bound for z is

$$d_2 := c + \sqrt{t_2}. \quad (5)$$

Therefore $z < \min\{d_1, d_2\}$. \square

THEOREM 1.2. *If $0 < a < 1$, then the unique root $\xi > 1$ of the equation $x^n - x^{n-1} - a = 0$ satisfies*

$$1 < \xi < \min\{b_1, b_2\},$$

where

$$b_1 = a^{\frac{1}{n}}(1 + d_1(c, n-1)), \quad b_2 = a^{\frac{1}{n}}(1 + d_2(c, n-1)), \quad c = a^{-1/n} - 1.$$

Proof. Suppose $0 < a < 1$. Let $b = a^{-1/n}$ and $x = a^{\frac{1}{n}}y$. The equation becomes

$$y^n - by^{n-1} - 1 = 0. \quad (6)$$

Let $b = c + 1$. Note that $c > 0$ because $a^{-1/n} > 1$. Put $y = 1 + z$. It follows

$$(1 + z)^n - (1 + c)(1 + z)^{n-1} = 1,$$

i.e.

$$(z - c)(z + 1)^{n-1} = 1. \tag{7}$$

The conclusion follows from Lemma 1.1. \square

EXAMPLES.

Let $L(c, m) = \min\{d_1, d_2\}$ and $S(a, n) = \min\{b_1, b_2\}$. Then $L(c, m)$, respectively $S(a, n)$, is attained in both cases, as follows from the following table.

| c | m | $L(c, m)$ | Attained for |
|------|-----|-----------|--------------|
| 0.12 | 4 | 0.468 | d_1 |
| 1.02 | 7 | 1.132 | d_1 |
| 0.2 | 21 | 0.267 | d_2 |

| a | n | $S(a, n)$ | Attained for |
|-------|-----|-----------|--------------|
| 0.57 | 5 | 1.311 | b_1 |
| 0.003 | 8 | 1.052 | b_1 |
| 0.02 | 22 | 1.057 | b_2 |

1.2. The equation $x^n - x^{n-1} - a = 0$ for $a \geq 1$.

We first study the condition $(z + c)(z + 1)^m \geq 1$, where $z, c > 0, m \in \mathbb{N}^*$.

LEMMA 1.3. *Let $m \in \mathbb{N}^*, c > 0$. If $z \geq 0$ and $(c + z)(1 + z)^m = 1$, then $z \leq K(c, m)$, where*

$$K(c, m) = \min\{m_1, m_2, m_3\}$$

and

$$m_1 = \begin{cases} c^{-1/m} - 1, & \text{if } 0 < c < 1, \\ 1 - c^{-1/m}, & \text{if } c \geq 1, \end{cases}$$

$$m_2 = \frac{\log((1.514065m) / \log(m + 1))}{m},$$

$$m_3 = \frac{-c + \sqrt{c^2 + 2cm}}{4cm}.$$

Proof. Observe first that the function $z \mapsto (z+c)(z+1)^m$ is increasing for $z > 0$. Note that if $0 < c < 1$, then for $z = c^{-1/m} - 1 \geq 0$ one has $(z+c)(1+z)^m = (z+c)/c \geq 1$.

On the other hand the inequality $(z+c)(1+z)^m \geq 1$ holds if one of the following two conditions is satisfied:

$$z(1+z)^m = 1, \quad (8)$$

$$2\sqrt{zc}(1+z)^m \geq 1. \quad (9)$$

Suppose that (8) is satisfied. Put $y = z + 1$, then (8) becomes

$$y^{m+1} - y^m - 1 = 0. \quad (10)$$

Let $\xi = \xi_{m+1}$ be the real root > 1 of the equation (10). We define λ_{m+1} by the condition

$$\frac{\log(\lambda_{m+1}m/\log(m+1))}{m} = \xi_{m+1} - 1.$$

Therefore

$$\lambda_{m+1} = \frac{\log(m+1)}{m} \exp\{m(\xi_{m+1} - 1)\}.$$

We may estimate ξ_{m+1} with arbitrary precision and deduce an approximate value of λ_{m+1} , for example for $m < 100$. A computation (done in Reduce) gives $\lambda_{m+1} \leq 1.514065$ (the maximum is reached for $m = 11$).

Let $g(z) = (c+z)(1+z)^m$.

Suppose now $m \geq 100$. Because $z > 0$ we have

$$g(z) \geq z \exp\left(m\left(z - \frac{1}{2}z^2\right)\right).$$

It follows that, for $1.514065(m+1)/m \leq \lambda \leq \log(m+1)$, one has

$$\begin{aligned} g\left(\frac{\log(\lambda m/\log(m+1))}{m}\right) &\geq \frac{\log(\lambda m/\log(m+1))}{m} \cdot \frac{\lambda m}{\log(m+1)} \cdot \exp\left(-\frac{\log^2 m}{2m}\right) \\ &= \frac{\lambda}{\log(m+1)} \cdot \log\left(\frac{\lambda m}{\log(m+1)}\right) \cdot \exp\left(-\frac{\log^2 m}{2m}\right) \\ &> 0.899 \lambda \cdot \frac{\log(m+1) - \log \log(m+1)}{\log(m+1)} > 0.6734 \lambda. \end{aligned}$$

Therefore $g\left(\frac{\log(\lambda m/\log(m+1))}{m}\right) \geq 1$ if $m \geq 100$ and $\lambda \geq 1.514065$.

It follows that the real solution $\xi > 1$ of the equation (10) satisfies

$$\xi < 1 + \frac{\log(1.514065m/\log(m+1))}{m}.$$

Therefore the corresponding solution $z > 0$ of (8) satisfies

$$z < \frac{\log(1.514065m / \log(m + 1))}{m}.$$

The condition (9) is equivalent to

$$4cz(1 + z)^{2m} \geq 1. \tag{11}$$

But (11) is satisfied if the following inequality is fulfilled:

$$4cz\left(1 + \binom{2m}{1}z\right) \geq 1, \tag{12}$$

The relation (12) is satisfied if $z \geq \frac{-c + \sqrt{c^2 + 2cm}}{4cm}$.
 Thus we obtained the stated estimations. \square

EXAMPLES.

| c | m | $K(c, m)$ | Attained for |
|------|-----|-----------|--------------|
| 0.98 | 8 | 0.002 | m_1 |
| 1.09 | 5 | 0.017 | m_1 |
| 0.2 | 18 | 0.062 | m_2 |
| 0.3 | 4 | 0.266 | m_3 |

THEOREM 1.4. *Let $n \in \mathbb{N}$, $n \geq 2$ and $a > 1$. Then the unique root $\xi > 1$ of the trinomial $x^n - x^{n-1} - a$ satisfies*

$$a^{1/n} < \left(\frac{1 + \sqrt{4a + 1}}{2}\right)^{2/n} \leq \xi \leq a^{\frac{1}{n}} \left(1 + K(1 - a^{-\frac{1}{n}}, n - 1)\right) := T(a, n).$$

Proof. We observe that, by Lemma 1 from [1], the polynomial has a unique root $\xi > 1$.

Note that $a > 1$. Let $b = a^{-1/n}$ and $x = a^{1/n}y$. Consider $y = 1 + z$. The equation $x^n - x^{n-1} - a = 0$ becomes

$$y^n - by^{n-1} - 1 = 0.$$

Put $b = 1 - c$. Note that $0 < b < 1$, so $0 < c = 1 - b < 1$. It follows

$$(1 + z)^n - (1 - c)(1 + z)^{n-1} = 1,$$

i.e.

$$(z + c)(1 + z)^{n-1} = 1. \tag{14}$$

Therefore we study the inequality

$$(c+z)(1+z)^m \geq 1,$$

where c and z are real numbers with $0 < c < 1$, $z > 0$, $m \in \mathbb{N}^*$.

We establish now a lower bound for the root ξ . Note that 1 is such a bound. On the other hand, because $\xi > 1$, and $n \geq 2$

$$a = \xi^n - \xi^{n-1} \leq \xi^n - \xi^{n/2},$$

therefore

$$\xi \geq \left(\frac{1 + \sqrt{4a+1}}{2} \right)^{2/n},$$

which is another lower bound for the root ξ . \square

EXAMPLES. Denote by $T(a, n)$ the upper bound of ξ given by Theorem 1.4. The last column refers to the corresponding formulas given in Lemma 1.3.

| a | n | $T(a, n)$ | Attained for |
|-------------------|-----|-----------|--------------|
| 898×10^9 | 9 | 21.416 | m_1 |
| 6.2 | 123 | 1.023 | m_2 |
| 5.9 | 4 | 1.975 | m_3 |

COROLLARY 1.5. Let $a > 0$, $n \in \mathbb{N}$, $n \geq 2$. Then the unique root $\xi > 1$ of $x^n - x^{n-1} - a = 0$ fulfills the conditions

$$\max \left\{ 1, a^{\frac{1}{n}} \right\} \leq \xi \leq M(a, n) := \begin{cases} S(a, n), & \text{if } 0 < a < 1, \\ T(a, n), & \text{if } a > 1. \end{cases}$$

2. A Bound for the Roots of Complex Polynomials

PROPOSITION 2.1. Let $P = a_n X^n + a_{n-k} X^{n-k} + \dots + a_0 \in \mathbb{C}[X]$, $k \geq 1$, $K = K(P) = \max_{0 \leq i \leq n-k} |a_i / a_n|$. If $z \in \mathbb{C}$ is a root of P , then

$$|z| \leq M(K, k),$$

where $M(a, n)$ is defined in Corollary 1.5 for $n \geq 2$ and $M(a, 1) = a + 1$.

Proof. Let

$$H_1 = \max_{0 \leq i \leq n-k} |a_i|.$$

If $z \in \mathbf{C}$, $|z| > 1$, then

$$\begin{aligned}
 |P(z)| &\geq |a_n| \cdot |z|^n - (|a_0| + |a_1| \cdot |z| + \cdots + |a_{n-1}| \cdot |z|^{n-k}) \\
 &\geq |a_n| \cdot |z|^n - H_1 \cdot (1 + |z| + \cdots + |z|^{n-k}) \\
 &> |a_n| \cdot |z|^n - \frac{H_1 |z|^{n-k+1}}{|z| - 1} \\
 &= |a_n| \frac{|z|^{n+1} - |z|^n - \frac{H_1 |z|^{n-k+1}}{|a_n|}}{|z| - 1} \\
 &= |a_n| \frac{|z|^{n+1} - |z|^n - K |z|^{n-k+1}}{|z| - 1}.
 \end{aligned}$$

Consider the upper bound $M(K, k)$ of the root > 1 of the equation $x^k - x^{k-1} = K$ given by Corollary 1.5 if $k \geq 2$ and which is obvious if $k = 1$:

$$\text{If } |z| \geq M(K, k), \text{ then } P(z) \neq 0.$$

Therefore $M(K, k)$ is an upper bound for the moduli of the roots of P . \square

REMARK. If $|z| > 1 + K$, then

$$|z|^{n+1} - |z|^n - K = |z|^n (|z| - 1) - K > (1 + K)^n \cdot K - K > 0.$$

If P is a monic integer polynomial, then $K = H(P) = \max_{0 \leq i \leq n} |a_i|$, so $1 + K = 1 + H(P)$ which is the upper bound for the roots of P , obtained by Cauchy (1829).

REMARK. Following M. Mignotte [3], instead of K one may consider a convenient Hölder norm L_p^q .

3. Applications to Lacunary Polynomials

We now use the estimate of the unique root > 1 of a polynomial $x^n - x^{n-1} - a$ to derive bounds for the absolute values of the roots of lacunary polynomials. A polynomial $P(X) = \sum_{i=0}^n a_i X^i \in \mathbf{C}[X]$ is called *lacunary* if there exists $k \geq 2$ such that $a_{n-1} = \cdots = a_{n-k+1} = 0$ and $a_{n-k} \neq 0$.

3.1. Complex lacunary polynomials.

THEOREM 3.1. *Let $P(X) = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbf{C}[X]$, with $a_n, a_{n-k} \neq 0$ and $k \geq 2$. Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and denote*

$$L_p = \sqrt[p]{\sum_{j=0}^{n-k} |a_j|^p}.$$

If $w \in \mathbf{C}$ is a root of P , then

$$|\omega| \leq \left(M \left(\left(\frac{L_p^q}{|a_n|} \right) \right), k \right)^{\frac{1}{q}},$$

where $M(a, n)$ is given by Corollary 1.5.

Proof. By M. Mignotte [3], if $\omega \in \mathbf{C}$ is a root of P , then

$$|\omega|^k - |\omega|^{k-1} - \frac{L_p^q}{|a_n|} < 0.$$

We now take $a = L_p^q/|a_n|$ and $n = k$ in Corollary 1.5. \square

3.2. Lacunary polynomials with nonnegative coefficients.

THEOREM 3.2. *Let $P(X) = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbf{R}[X]$. If*

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-k}, \quad a_n > 0,$$

then all roots of P lie in the domain

$$\{z \in \mathbf{C}; |z| \leq 1\} \cup \{z \in \mathbf{C}; a_n |z|^k - a_{n-k} \leq |a_n z^{k-1} - a_{n-k}|\}.$$

Proof.

Let $\omega \in \mathbf{C}$ be a root of P . We put $\sigma = |\omega|$.

We observe that

$$\begin{aligned} (1 - \omega)P(\omega) &= a_0 + a_1 \omega + \cdots + a_{n-k} \omega^{n-k} + a_n \omega^n - a_0 \omega - a_1 \omega^2 - \cdots \\ &\quad - a_{n-k-1} \omega^{n-k} - a_{n-k} \omega^{n-k+1} - a_n \omega^{n+1} \\ &= a_0 + (a_1 - a_0) \omega + (a_2 - a_1) \omega^2 + \cdots \\ &\quad + (a_{n-k} - a_{n-k-1}) \omega^{n-k} + (a_n \omega^{k-1} - a_{n-k}) \omega^{n-k+1} - a_n \omega^{n+1}. \end{aligned}$$

Therefore, for $\sigma = |\omega| > 1$,

$$\begin{aligned} |(1 - \omega)P(\omega)| &\geq a_n |\omega|^{n+1} - |a_0 + (a_1 - a_0) \omega + (a_2 - a_1) \omega^2 \\ &\quad + \cdots + (a_{n-k} - a_{n-k-1}) \omega^{n-k} + (a_n \omega^{k-1} - a_{n-k}) \omega^{n-k+1}| \\ &\geq a_n \sigma^{n+1} - (|a_0| + |a_1 - a_0| \sigma + |a_2 - a_1| \sigma^2 + \cdots \\ &\quad + |a_{n-k} - a_{n-k-1}| \sigma^{n-k} + |a_n \omega^{k-1} - a_{n-k}| \sigma^{n-k+1}) \\ &\geq a_n \sigma^{n+1} - (|a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_{n-k} - a_{n-k-1}| \\ &\quad + |a_n \omega^{k-1} - a_{n-k}|) \sigma^{n-k+1} \\ &= a_n \sigma^{n+1} - (a_{n-k} + |a_n \omega^{k-1} - a_{n-k}|) \sigma^{n-k+1} \\ &= (a_n \sigma^k - a_{n-k} - |a_n \omega^{k-1} - a_{n-k}|) \sigma^{n-k+1}. \end{aligned}$$

If

$$a_n |\omega|^k - a_{n-k} > |a_n \omega^{k-1} - a_{n-k}|,$$

then $|(1 - \omega)P(\omega)| > 0$, so ω cannot be a root of P .

Therefore

$$|\omega| \leq 1 \quad \text{or} \quad a_n |\omega|^k - a_{n-k} \leq |a_n \omega^{k-1} - a_{n-k}|.$$

□

REMARK. Theorem 3.2 is a lacunary version of Eneström–Kakeya criterion, cf. [1], 1.2.

COROLLARY 3.3. *If $z \in \mathbb{C}$ is a root of the polynomial P , then*

$$|z| \leq M \left(2 \frac{a_{n-k}}{a_n}, k \right).$$

Proof. First note that $|a_n z^{k-1} - a_{n-k}| \leq a_n |z|^{k-1} + a_{n-k}$. Hence $a_n |z|^k - a_n |z|^{k-1} - 2a_{n-k} \leq 0$, i.e.

$$|z| \leq 1 \quad \text{or} \quad |z|^k - |z|^{k-1} - 2 \frac{a_{n-k}}{a_n} \leq 0.$$

□

REMARK. *Let $z \in \mathbb{C}$ be a root of the polynomial*

$$P(X) = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbb{R}[X].$$

a) *If*

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-k} \leq \frac{a_n}{2},$$

then

$$|z| < S \left(\frac{2a_{n-k}}{a_n}, k \right).$$

b) *If*

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-k} \quad \text{and} \quad a_{n-k} > \frac{a_n}{2},$$

then

$$|z| < T \left(\frac{2a_{n-k}}{a_n}, k \right).$$

PROPOSITION 3.4. *Let $P(X) = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbb{R}[X]$, where $a_n, a_{n-k}, \dots, a_1, a_0 > 0$. If $z \in \mathbb{C}$ is a root of P , then*

$$|z| \leq \alpha M \left(\frac{2a_{n-k}}{\alpha^k a_n}, k \right),$$

where $\alpha = \max_{1 \leq i \leq n-k} \left(\frac{a_{i-1}}{a_i} \right)$.

Proof. For $\alpha > 0$ consider the associated polynomial

$$P_\alpha(Y) = P(\alpha Y) = a_n \alpha^n Y^n + \sum_{i=0}^{n-k} a_i \alpha^i Y^i.$$

If

$$0 < a_0 \leq a_1 \alpha \leq \cdots \leq a_{i-1} \alpha^{i-1} \leq a_i \alpha^i \leq \cdots \leq a_{n-k} \alpha^{n-k} \quad (15)$$

then, by Corollary 3.3, the roots of P_α are bounded by $M \left(\frac{2a_{n-k}}{\alpha^k a_n}, n \right)$.

But (15) is fulfilled if and only if

$$\alpha \geq \frac{a_{i-1}}{a_i}, \quad \forall i \in \{1, 2, \dots, n-k\}.$$

We take

$$\alpha = \max_{1 \leq i \leq n-k} \left(\frac{a_{i-1}}{a_i} \right)$$

and the roots of P are bounded by $\alpha M \left(\frac{2a_{n-k}}{\alpha^k a_n}, k \right)$. \square

4. Another Estimate for the Roots of Lacunary Polynomials

PROPOSITION 4.1. *Let $P(X) = X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbb{C}[X] \setminus \mathbb{C}$. Then the moduli of the roots of P are bounded by*

$$B := \max \left\{ 1, \left(|a_0| + |a_1| + \cdots + |a_{n-k}| \right)^{\frac{1}{k}} \right\}.$$

Proof. Let $z \in \mathbb{C}$ be a root of P . If $|z| > 1$ and if $|z| > \left(|a_0| + |a_1| + \cdots + |a_{n-k}| \right)^{\frac{1}{k}}$, then

$$|z|^n - \left(|a_0| + |a_1| + \cdots + |a_{n-k}| \right) |z|^{n-k} > 0.$$

On the other hand

$$\left| \sum_{j=0}^{n-k} a_j z^j \right| \leq \sum_{j=0}^{n-k} |a_j| \cdot |z|^j \leq \left(\sum_{j=0}^{n-k} |a_j| \right) \cdot |z|^{n-k}.$$

Therefore

$$|P(z)| \geq |z^n| - \left| \sum_{j=0}^{n-k} a_j z^j \right| \geq |z|^n - \left(\sum_{j=0}^{n-k} |a_j| \right) \cdot |z|^{n-k} > 0,$$

so $P(z) \neq 0$.

Therefore the absolute values of the roots of P are $\leq \max\{1, (|a_0| + |a_1| + \cdots + |a_{n-k}|)^{\frac{1}{k}}\}$. \square

REMARK. If in Proposition 4.1 we consider P an integer polynomial which is not a monomial, then we may take $B := (|a_0| + |a_1| + \cdots + |a_{n-k}|)^{\frac{1}{k}}$.

5. Examples

We consider the bounds for the moduli of the roots of lacunary polynomials obtained throughout this paper. In addition we refer also to the estimates from [3] and [4].

The following notations will be used:

$$A = A(\infty) = M(H(P), k),$$

$$A(p) = M(L_p^q, k)^{\frac{1}{q}},$$

$$B = (|a_0| + |a_1| + \cdots + |a_{n-k}|)^{\frac{1}{k}}, \quad (\text{Prop. 4.1})$$

$$C(p) = \left(1 + \frac{\log k}{k-1} \right)^{\frac{1}{q}} \left(\left(\sum_{j=0}^{n-k} |a_j|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{k}}, \quad (\text{M. Mignotte [3]})$$

$$C = C(\infty) = \left(1 + \frac{\log k}{k-1} \right) H^{\frac{1}{k}},$$

$$D(p) = \left(1 + \left(\sum_{j=0}^{n-k} |a_j|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, \quad (\text{P. Montel [4]})$$

$$E = M\left(\frac{2a_{n-k}}{\alpha^k a_n}, n\right), \quad (\text{Prop. 3.4})$$

Note that the bound E will be used only for polynomials with positive coefficients.

Consider the polynomials:

$$P_1(X) = X^{10} + 29X^6 + 27X^5 + 30X^4 - 28X^3 + 29X^2 + 28X - 9,$$

$$P_2(X) = X^{10} + 13X^6 + 15X^5 + 14X^4 - 9X^3 + 14X^2 - 13X + 14,$$

$$P_3(X) = X^{11} + 6X^6 + 9X^5 + 8X^4 - 7X^3 + 4X^2 + 7X + 6,$$

$$P_4(X) = X^{12} - 32X^4 - 17X^3 + 18X^2 + 18X + 19,$$

$$P_5(X) = X^{17} - 8X^4 - 3X^3 + 5X^2 + 7X - 6,$$

$$P_6(X) = X^4 + 3X + 2,$$

$$P_7(X) = X^5 + 3X^2 + 2X + 12,$$

$$P_8(X) = X^7 + 17X^4 + 18X^3 + 2X^2 + 3X + 12,$$

$$P_9(X) = X^9 + 321X^4 + 24X^3 + 3X^2 + 17X + 5.$$

$$P_{10}(X) = X^{11} + 2X^3 + 4X^2 + 5X + 2,$$

$$P_{11}(X) = X^{12} + 29X^6 + 27X^5 + 26X^4 + 26X^3 + 25X^2 + 25X + 24,$$

$$P_{12}(X) = X^{14} + 12X^3 + 13X^2 + 10X + 9.$$

The following estimates include the cases $p = 1.1$, $p = 2.9$ and $p = \infty$.

| P | A | $A(2.9)$ | $A(1.1)$ | B | C | $C(2.9)$ | $C(1.1)$ | $D(2.9)$ | $D(1.1)$ | E |
|----------|-------|----------|----------|--------|--------|----------|----------|----------|----------|------|
| P_1 | 2.81 | 4224.65 | 4.56 | 3.66 | 3.42 | 3.94 | 5.12 | 53.11 | 151.48 | — |
| P_2 | 2.49 | 954.42 | 3.72 | 3.09 | 2.87 | 3.30 | 4.33 | 26.25 | 77.16 | — |
| P_3 | 1.98 | 81.74 | 2.43 | 2.15 | 2.17 | 2.36 | 2.92 | 13.87 | 39.47 | — |
| P_4 | 1.79 | 46.89 | 1.97 | 1.78 | 2.0003 | 2.04 | 2.27 | 38.96 | 90.14 | — |
| P_5 | 1.28 | 4.704 | 1.35 | 1.29 | 1.42 | 1.45 | 1.55 | 11.04 | 25.17 | — |
| P_6 | 2.07 | 28.26 | 1.86 | 1.709 | 2.23 | 2.304 | 2.59 | 3.63 | 4.704 | 2.28 |
| P_7 | 2.93 | 1084.06 | 3.04 | 2.57 | 3.54 | 3.55 | 3.89 | 12.27 | 15.83 | 6.30 |
| P_8 | 3.31 | 6963.79 | 4.75 | 3.73 | 4.06 | 4.43 | 5.55 | 23.61 | 46.01 | 5.39 |
| P_9 | 3.48 | 16448.63 | 4.13 | 3.26 | 4.447 | 4.448 | 4.53 | 321.11 | 353.89 | 6.53 |
| P_{10} | 1.42 | 6.56 | 1.44 | 1.37 | 1.58 | 1.62 | 1.76 | 6.21 | 11.55 | 2.04 |
| P_{11} | 2.08 | 247.61 | 2.75 | 2.3805 | 2.3809 | 2.61 | 3.13 | 51.10 | 152.51 | 2.26 |
| P_{12} | 1.405 | 9.16 | 1.49 | 1.41 | 1.56 | 1.61 | 1.72 | 18.22 | 38.83 | 1.48 |

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