

EXTENSIONS OF THE GEOMETRIC-ARITHMETIC MEANS INEQUALITY TO A DISC OF THE COMPLEX PLANE

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Abstract. For complex numbers z_j with $|z_j - 1| \leq r$, $r < 1$, we consider the arithmetic mean $A_n := (1/n) \sum_{j=1}^n z_j$ and the geometric mean $G_n := \prod_{j=1}^n z_j^{1/n}$ and prove, amongst other results, that

$$\frac{1-r^2}{|2-A_n|} \leq |G_n| \leq \frac{|A_n|}{\sqrt{1-r^2}}.$$

Introduction and Statement of the Results

The inequality

$$\left(\prod_{j=1}^n z_j \right)^{1/n} \leq \frac{1}{n} \sum_{j=1}^n z_j, \quad 0 < z_j, \quad n = 1, 2, \dots,$$

is a most important result and in any case one of the keystones of the general theory of inequalities. Numerous proofs and generalizations (see [1, 3, 4]) are known; for example,

$$\prod_{j=1}^n z_j^{\lambda_j} \leq \sum_{j=1}^n \lambda_j z_j, \quad 0 \leq \lambda_j, \quad \sum_{j=1}^n \lambda_j = 1. \quad (1)$$

In this note we shall be concerned by extensions of (1) to the case where the quantities z_j are assumed to represent complex numbers. Our point of view is the following: let $G := \prod_{j=1}^n z_j^{\lambda_j}$, $G_n := \left(\prod_{j=1}^n z_j \right)^{1/n}$, $A := \sum_{j=1}^n \lambda_j z_j$ and $A_n := 1/n \sum_{j=1}^n z_j$, where the complex numbers z_j are assumed to satisfy $|z_j - 1| \leq r < 1$; we wish to obtain bounds for $|G|$ (resp. $|G_n|$) in terms of A (resp. A_n) and r . Our main results are

THEOREM 1. $\frac{1-r^2}{|2-A|} \leq |G| \leq \frac{|A|}{\sqrt{1-r^2}}.$

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THEOREM 2. $(1-r)^{(r+|1-A|)/2r}(1+r)^{(r-|1-A|)/2r} \leq |G| \leq \sqrt{1+2|1-A|+r^2}$.

THEOREM 3. $|A_n|^{1/n}(1-r)^{1-1/n} \leq |G_n| \leq |A_n|^{1/n}(1+r)^{1-1/n}$.

All our results are sharp and it should be noted that no upper (resp. lower) bound given for $|G|$ in our Theorems is universally better (that is, for all eligible complex numbers z_j and all reals r and λ_j) than any other one. All our inequalities but one also sharpen the trivial estimates $1-r \leq |G| \leq 1+r$. Our method is based on well-known results concerning starlike univalent functions (we refer to the book of Duren [2] for basic facts concerning geometric function theory).

Proof of Theorem 1

Let \mathbb{D} denote the open unit disc $\{z \mid |z| < 1\}$ of the complex plane. We consider the class S_α of functions $f(z) := z + \sum_{n=2}^{\infty} a_n(f)z^n$ analytic in \mathbb{D} and starlike of order α , i.e., $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ if $z \in \mathbb{D}$. It is a result of Sheil-Small [6] that $f \in S_{1/2}$ if and only if, for each $x \in \mathbb{D}$, the function F defined by

$$F(z) := \frac{x}{f(x)} \frac{f(z) - f(x)}{z - x} = 1 + \frac{f(x)/x - 1}{f(x)}z + \frac{f(x)/x - 1 - a_2(f)x}{xf(x)}z^2 + \dots,$$

satisfies $\operatorname{Re}(F(z)) > 1/2$ if $z \in \mathbb{D}$. It then follows from Schwarz Lemma that for $x \in \mathbb{D}$,

$$\left| \frac{f(x)/x - 1 - a_2(f)x}{xf(x)} - \left(\frac{f(x)/x - 1}{f(x)} \right)^2 \right| \leq 1 - \left| \frac{f(x)/x - 1}{f(x)} \right|^2$$

i.e., $\left| (1 + a_2(f)x)f(x)/x + (1 - 2f(x)/x) \right| \leq -(1 - |x|^2)|f(x)/x|^2 + 2\operatorname{Re}(f(x)/x) - 1$ and

$$\operatorname{Re} \left((1 + a_2(f)x) \frac{f(x)}{x} \right) \geq (1 - |x|^2) \left| \frac{f(x)}{x} \right|^2 > 0, \quad x \in \mathbb{D}. \quad (2)$$

In particular

$$\left| \frac{f(x)}{x} \right| \leq \frac{|1 + a_2(f)x|}{1 - |x|^2}, \quad x \in \mathbb{D}. \quad (3)$$

The class S_α^0 of functions f defined as

$$f(z) = z \prod_{j=1}^n (1 + x_j z)^{-2(1-\alpha)\lambda_j}, \quad |x_j| \leq 1$$

is known to be a dense subset of S_α (endowed with the topology of uniform convergence over compact subsets of \mathbb{D}). For such functions f , with $\alpha = 1/2$, we have $a_2(f) = -\sum_{j=1}^n \lambda_j x_j$ and an application of (3) with $x = r \in (0, 1)$ yields

$$\frac{1-r^2}{|2-A|} \leq |G|. \quad (4)$$

Clearly equality shall hold in (4) if we choose $z_j = 1 + r$ for all $1 \leq j \leq n$ or if $z_j = 1 - r$ for all $1 \leq j \leq n$.

The proof of the right-hand side inequality in Theorem 1 is rather elementary. Indeed

$$\begin{aligned}
 |A| &= \left| \sum_{j=1}^n \lambda_j (1 + x_j r) \right| \\
 &\geq \sum_{j=1}^n \lambda_j |1 + x_j r| \cos(\arg(1 + x_j r)) \\
 &\geq \sqrt{1 - r^2} \sum_{j=1}^n \lambda_j |1 + x_j r| \\
 &\geq \sqrt{1 - r^2} \prod_{j=1}^n |1 + x_j r|^{\lambda_j},
 \end{aligned}$$

where we have made use of the “real” geometric-arithmetic means inequality. We obtain

$$|G| \leq \frac{|A|}{\sqrt{1 - r^2}}. \quad (5)$$

For values of $|A|$ larger than $(1 + r)\sqrt{1 - r^2}$, the estimate (5) is rather crude. Nevertheless equality may hold in (5) for each fixed $r \in (0, 1)$; indeed if $n = 2$, $z_1 = 1 + ie^{i\theta}r$, $z_2 = \bar{z}_1$, $\lambda_1 = \lambda_2 = 1/2$ and finally θ is an acute angle with $\cos(\theta) = \sqrt{1 - r^2}$, we obtain $|G| = \sqrt{1 - r^2}$ and $|A| = 1 - r^2$.

The following corollary can be obtained by a density argument and should be compared with (3):

COROLLARY 1. *The sharp inequality*

$$\frac{\sqrt{1 - |z|^2}}{|1 - a_2(f)z|} \leq \left| \frac{f(z)}{z} \right|, \quad z \in \mathbb{D},$$

holds for functions $f \in S_{1/2}$.

We shall conclude this section by a remark. By applying (2) to functions $f(z) = z \prod_{j=1}^n (1 + x_j z)^{-\lambda_j} \in S_{1/2}^0$, we obtain that $F(z) := \sum_{j=1}^n \lambda_j (1 - x_j z) / \prod_{j=1}^n (1 + x_j z)^{\lambda_j}$ is an analytic function with positive real part in \mathbb{D} ; as in the proof of Theorem 2 (see below), we obtain

$$\text{COROLLARY 2. } |G| \leq \frac{|2 - A|}{1 - r^2} (1 + 2|1 - A| + r^2).$$

This last inequality is stronger than (5) when A is close enough to $1 + r$. It can also seen to be sharp if $A = G = 1 + r$.

Proof of Theorem 2

We consider functions $f(z) = z + a_2(f)z^2 + \cdots \in S_0$. We have the representation

$$z \frac{f'(z)}{f(z)} = \frac{1 + m(z)}{1 - m(z)}, \quad z \in \mathbb{D}, \quad m'(0) = \frac{a_2(f)}{2}, \quad (6)$$

where m is analytic in \mathbb{D} and, by Schwarz Lemma,

$$|m(z)| \leq |z| \frac{|z| + |m'(0)|}{1 + |m'(0)||z|} < 1 \quad \text{if } z \in \mathbb{D}. \quad (7)$$

By (6) and (7),

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) \leq \frac{1 + |m(z)|}{1 - |m(z)|} \leq \frac{1 + |a_2(f)||z| + |z|^2}{1 - |z|^2} \quad (8)$$

and

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) \geq \frac{1 - |m(z)|}{1 + |m(z)|} \geq \frac{1 - |z|^2}{1 + |a_2(f)||z| + |z|^2}. \quad (9)$$

Since

$$\frac{d}{dr} \ln \left(\frac{|f(re^{i\theta})|}{r} \right) = \frac{1}{r} \operatorname{Re} \left(re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} - 1 \right), \quad \theta \in \mathbb{R},$$

we obtain upon integration of the last identity, for $z \in \mathbb{D}$,

$$\frac{|z|}{1 + |a_2(f)||z| + |z|^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^{1+|a_2(f)|/2} (1 + |z|)^{1-|a_2(f)|/2}}, \quad (10)$$

We now choose $f(z) := z \prod_{j=1}^n (1 + x_j z)^{-2\lambda_j} \in S_0^0$; clearly $a_2(f) = -2 \sum_{j=1}^n \lambda_j x_j$ and an application of (10) with $z = r \in (0, 1)$ yields

$$(1 - r)^{1/2+|1-A|/2r} (1 + r)^{1/2-|1-A|/2r} \leq |G| \quad (11)$$

and

$$|G| \leq \sqrt{1 + 2|1 - A| + r^2}. \quad (12)$$

Equality shall hold in (11) if $n = 2$, $z_1 = 1 + r$, $z_2 = 1 - r$, $\lambda_1 = \lambda \in [0, 1/2]$ and $\lambda_2 = 1 - \lambda$. Equality shall hold in (12) if $n = 2$, $z_1 = 1 + e^{i\theta}r$, $\lambda_1 = 1/2 = \lambda_2$ and $z_2 = 1 + e^{-i\theta}r$, where $\cos(\theta) \geq 0$. We finally mention

COROLLARY 3. $\sqrt{1 - r^2} \leq |G| \leq \sqrt{1 + r^2}$ if $A = 1$.

Proof of Theorem 3

Let us consider $H(\mathbb{D})$, the set of analytic functions in \mathbb{D} , endowed with the topology of uniform convergence over compact subset of \mathbb{D} . Let $n \geq 1$ and

$$\mathcal{P}_n = \{p \mid p \text{ is a polynomial of degree } \leq n, p(0) = 1, \text{ with no zeroes in } \mathbb{D}\}.$$

\mathcal{P}_n is a compact and rotationally invariant subset of $H(\mathbb{D})$. Let us also consider ℓ_1 and ℓ_2 , two continuous linear functionals over $H(\mathbb{D})$ with $0 \notin \ell_2(\mathcal{P}_n)$. It follows from Ruscheweyh's Duality Theory [5] that

$$\left\{ \frac{\ell_1(p)}{\ell_2(p)} \mid p \in \mathcal{P}_n \right\} = \left\{ \frac{\ell_1(p)}{\ell_2(p)} \mid p \in \mathcal{P}'_n \right\} \quad (13)$$

where \mathcal{P}'_n denotes the subset of \mathcal{P}_n consisting of the polynomials $p(z) = (1+xz)^n$, $|x| \leq 1$.

We choose, for fixed $r \in (0, 1)$,

$$\ell_1(f) := f(r) \quad \text{and} \quad \ell_2(f) := f(0) + \frac{f'(0)}{n}r.$$

Any $p \in \mathcal{P}_n$ can be represented as

$$p(z) = \prod_{j=1}^n (1 + x_j z), \quad |x_j| \leq 1.$$

For any p as above,

$$\ell_2(p) = 1 + \frac{1}{n} \sum_{j=1}^n x_j r = \frac{1}{n} \sum_{j=1}^n (1 + x_j r) \neq 0$$

and by (13), for $n \geq 1$,

$$\left\{ \frac{\prod_{j=1}^n (1 + x_j r)}{1/n \sum_{j=1}^n (1 + x_j r)} \mid |x_j| \leq 1 \right\} = \{(1 + xr)^{n-1} \mid |x| \leq 1\}.$$

It follows easily that

$$|A_n|(1-r)^{n-1} \leq |G_n|^n \leq |A_n|(1+r)^{n-1},$$

with obvious cases of equality.

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