Mathematical
Inequalities
& Applications
Volume 3, Number 1 (2000), 63–67

AN EXTENSION OF HLAWKA'S INEQUALITY

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(*communicated by S. Saitoh*)

Abstract. We consider an extension of Hlawka's inequality on a Hilbert space in an integral form and give an extension of Djokovic's inequality as an application of our extension.

1. Introduction and results

Let H be a Hilbert space. The following inequality is well-known as Hlawka's inequality (cf. [3]): for all $x, y, z \in H$,

$$\|x\| + \|y\| + \|z\| + \|x+y+z\| \geqslant \|x+y\| + \|y+z\| + \|z+x\|.$$

Hlawka's inequality has various extensions (cf. [1], [2] and [4]). Here we consider the following formal extension:

$$\text{(GH)} \qquad (\mu(\Omega)-2)\left\|\int_\Omega f(\omega)d\mu\right\| + \int_\Omega \|f(\omega)\|d\mu \geqslant \int_\Omega \left\|f(\omega) - \int_\Omega f\,d\mu\right\|d\mu,$$

where (Ω, μ) is a finite measure space and f is a Bochner integrable Hilbert space-valued function on (Ω, μ).

Of course if Ω consists of the three points and μ is the counting measure on Ω, then (GH) reduces to the original Hlawka inequality. Also if Ω consists of the n-points and μ is the counting measure on Ω, then (GH) reduces to Djokovic's inequality:

$$\sum_{i=1}^n \|x_i\| + (n-2)\left\|\sum_{i=1}^n x_i\right\| \geqslant \sum_{i=1}^n \|x_1 + \cdots + \widehat{x_i} + \cdots + x_n\|,$$

where $x_1, \cdots, x_n$ are in a Hilbert space and the sign $\widehat{\ }$ placed over a vector indicates that this vector is to be deleted from the sum.

Nevertheless, (GH) is not necessarily true. In fact, if μ is a probability measure, H is the complex numbers and f is a non-constant, non-negative function, then (GH)

Mathematics subject classification (1991): 26D15.

Key words and phrases: Hlawka's inequality; Djokovic's inequality; Bochner integralble Hilbert space-valued function.

The content of this paper was presented at the '99 KOTAC International Workshop : Operator Theory and Applications in Keimyung University and Taegu University, June 3-5, 1999.

does not hold. Hence it naturally raises the question under what condition (GH) holds. Here we give a partial solution to this problem. For this, we prepare the following notation :

$$\Omega_f = \{\omega \in \Omega : -f(\omega) \text{ and } \int_\Omega f(t)d\mu(t) \text{ are not codirectional}\}.$$

Here that the vectors x and $y(\neq 0)$ are codirectional means that $\exists\alpha \geqslant 0 : x = \alpha y$. However note that (GH) always holds under the condition $\int_\Omega f(t)d\mu(t) = 0$. So, we can without loss of generality assume that $\int_\Omega f(t)d\mu(t) \neq 0$ and hence the definition of Ω_f is well-defined.

Now our main result is following

THEOREM 1. *Let H be a Hilbert space, (Ω,μ) a finite measure space and f a Bochner integrable H-valued function on (Ω,μ). Suppose that*

$$\text{(H)} \qquad \int_\Omega \|f(t)\|d\mu(t) \geqslant \|f(\omega)\| + \left\|f(\omega) - \int_\Omega f(t)d\mu(t)\right\| \ (a.e.\ \omega \in \Omega_f).$$

Then

$$(\mu(\Omega)-2)\left\|\int_\Omega f(\omega)d\mu\right\| + \int_\Omega \|f(\omega)\|d\mu \geqslant \int_\Omega \left\|f(\omega) - \int_\Omega f\,d\mu\right\|d\mu.$$

The following result is a weighted extension of Djokovic's inequality. In Theorem, if Ω is a finite set and all values of μ are greater than one, then the hypothesis (H) is always true and so the following corollary immediately follows from Theorem 1.

COROLLARY 2. *Let $x_1,\cdots x_n \in H (n \geqslant 3)$ and $\mu_1 \geqslant 1,\cdots,\mu_n \geqslant 1$. Then*

$$\left(\sum_{i=1}^n \mu_i - 2\right)\left\|\sum_{i=1}^n \mu_i x_i\right\| + \sum_{i=1}^n \mu_i\|x_i\| \geqslant \sum_{i=1}^n \mu_i\left\|x_i - \sum_{j=1}^n \mu_j x_j\right\|.$$

2. Proof of Theorem 1

By a simple consideration, we have the following

$$(*) \qquad (\mu(\Omega)-2)\left\|\int_\Omega f(\omega)d\mu\right\|^2 + \int_\Omega \|f(\omega)\|^2 d\mu = \int_\Omega \left\|f(\omega) - \int_\Omega f\,d\mu\right\|^2 d\mu.$$

Set

$$A = (\mu(\Omega)-2)\left\|\int_\Omega f(\omega)d\mu\right\| + \int_\Omega \|f(\omega)\|d\mu - \int_\Omega \left\|f(\omega) - \int_\Omega f\,d\mu\right\|d\mu,$$

$$B = \int_\Omega \|f(\omega)\|d\mu + \left\|\int_\Omega f(\omega)d\mu\right\|,$$

$$C(\omega) = \int_\Omega \|f(\omega)\| d\mu - \|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\, d\mu \right\|,$$

and

$$D(\omega) = \|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\, d\mu \right\| + \left\| \int_\Omega f(\omega) d\mu \right\|.$$

Note that $D(\omega) \geqslant 0$ $(\forall \omega \in \Omega)$ and $\Omega_f = \Omega \backslash \{\omega \in \Omega : D(\omega) = 0\}$. Then we have

$$\begin{aligned}
AB &= (\mu(\Omega) - 2)\left\| \int_\Omega f(\omega) d\mu \right\|^2 + \left(\int_\Omega \|f(\omega)\| d\mu \right)^2 \\
&\quad + (\mu(\Omega) - 2)\left\| \int_\Omega f(\omega) d\mu \right\| \int_\Omega \|f(\omega)\| d\mu + \int_\Omega \|f(\omega)\| d\mu \left\| \int_\Omega f(\omega) d\mu \right\| \\
&\qquad - \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\| d\mu \left(\int_\Omega \|f(\omega)\| d\mu + \left\| \int_\Omega f(\omega) d\mu \right\| d\mu \right) \\[2ex]
&= \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\|^2 d\mu + \left(\int_\Omega \|f(\omega)\| d\mu \right)^2 \\
&\quad - \int_\Omega \|f(\omega)\|^2 d\mu + (\mu(\Omega) - 1)\left\| \int_\Omega f(\omega) d\mu \right\| \int_\Omega \|f(\omega)\| d\mu \\
&\quad - \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\| d\mu \left(\int_\Omega \|f(\omega)\| d\mu + \left\| \int_\Omega f(\omega) d\mu \right\| d\mu \right) \text{ by } (*) \\[2ex]
&= \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\|^2 d\mu + \left(\int_\Omega \|f(\omega)\| d\mu \right)^2 - \int_\Omega \|f(\omega)\|^2 d\mu \\
&\quad + \left\| \int_\Omega f(\omega) d\mu \right\| \left((\mu(\Omega) - 1) \int_\Omega \|f(\omega)\| d\mu - \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\| d\mu \right) \\
&\qquad - \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\| d\mu \int_\Omega \|f(\omega)\| d\mu \\[2ex]
&= \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\|^2 d\mu + \left(\int_\Omega \|f(\omega)\| d\mu \right)^2 - \int_\Omega \|f(\omega)\|^2 d\mu \\
&\quad + \left\| \int_\Omega f(\omega) d\mu \right\| \int_\Omega \left(\int_\Omega \|f(\omega)\| d\mu - \|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\, d\mu \right\| \right) d\mu \\
&\qquad - \int_\Omega \left\| f(\omega) - \int_\Omega f\, d\mu \right\| d\mu \int_\Omega \|f(\omega)\| d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Bigg(\left\| f(\omega) - \int_\Omega f\,d\mu \right\|^2 \\
&\qquad\qquad + \left\| \int_\Omega f(\omega) d\mu \right\| \left(\int_\Omega \|f(\omega)\| d\mu - \|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\,d\mu \right\| \right) \Bigg) d\mu \\
&\qquad + \int_\Omega \|f(\omega)\| d\mu \int_\Omega \left(\|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\,d\mu \right\| \right) d\mu - \int_\Omega \|f(\omega)\|^2 d\mu \\
&= \int_\Omega \Bigg(\left\| f(\omega) - \int_\Omega f\,d\mu \right\|^2 \\
&\qquad\qquad + \left\| \int_\Omega f(\omega) d\mu \right\| \left(\int_\Omega \|f(\omega)\| d\mu - \|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\,d\mu \right\| \right) \Bigg) d\mu \\
&\qquad + \int_\Omega \left(\int_\Omega \|f(\omega)\| d\mu - \|f(\omega)\| - \left\| f(\omega) - \int_\Omega f\,d\mu \right\| \right) \|f(\omega)\| d\mu \\
&\qquad - \int_\Omega \left(\int_\Omega \|f(\omega)\| d\mu \left\| f(\omega) - \int_\Omega f\,d\mu \right\| - \left\| f(\omega) - \int_\Omega f\,d\mu \right\| \|f(\omega)\| \right) d\mu \\
&= \int_\Omega C(\omega) D(\omega) d\mu = \int_{\Omega_f} C(\omega) D(\omega) d\mu \\
&\geqslant 0 \text{ (by hypothesis H)} .
\end{aligned}
$$

Since $B \geqslant 0$, it follows that $A \geqslant 0$ and hence (GH) holds. $\square$

Acknowledgment. The authors would like to thank the referee for his invaluable suggestions. The first*-and-second** named authors are partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan (*10640150, **10440048).

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(Received July 15, 1999)

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