

INEQUALITIES FOR VOLUMES AND MEDIANS OF TWO SIMPLICES

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Abstract. In this paper, some inequalities for volumes and medians of two n -dimensional simplices are established.

1. Introduction

Let $a_i (i = 0, 1, 2)$ denote the sides of the triangle $A_0A_1A_2$ with area Δ , and $a'_i (i = 0, 1, 2)$ denote the sides of the triangle $A'_0A'_1A'_2$ with area Δ' . D. Pedoe [1] proved an important inequality as follows

$$\sum_{i=0}^2 a_i^2 \left(\sum_{j=0}^2 a_j'^2 - 2a_i'^2 \right) \geq 16\Delta\Delta', \quad (1)$$

with equality if and only if the triangle $A_0A_1A_2$ is similar to the triangle $A'_0A'_1A'_2$.

Inequality (1) is the well-known Neuberger – Pedoe inequality. In 1984, P. Chiakuei [2] proved the following sharpening of the Neuberger – Pedoe inequality:

$$\sum_{i=0}^2 a_i^2 \left(\sum_{j=0}^2 a_j'^2 - 2a_i'^2 \right) \geq 8 \left(\frac{a_0'^2 + a_1'^2 + a_2'^2}{a_0^2 + a_1^2 + a_2^2} \Delta^2 + \frac{a_0^2 + a_1^2 + a_2^2}{a_0'^2 + a_1'^2 + a_2'^2} \Delta'^2 \right), \quad (2)$$

with equality if and only if the triangle $A_0A_1A_2$ is similar to the triangle $A'_0A'_1A'_2$.

Let Ω_n be an n -dimensional simplex in n -dimensional Euclidean space E^n with vertex set $\tau_n = \{A_0, A_1, \dots, A_n\}$ and V its volume, $a_{ij} = |A_iA_j|$. Let Ω'_n be another n -dimensional simplex in E^n with vertex set $\tau'_n = \{A'_0, A'_1, \dots, A'_n\}$ and V' its volume, $a'_{ij} = |A'_iA'_j|$. Let m_i be the median of the simplex Ω_n from vertex A_i and m'_i the median of the simplex Ω'_n from vertex A'_i . In 1983, Yang Lu and Zhang Jingzhong [3] extended the Neuberger – Pedoe inequality (1) to n -dimensional simplices. Recently,

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Leng Gangsong [4] has extended the Neuberg – Pedoe inequality (1) to the edge lengths and volumes of two n -dimensional simplices, and obtained an inequality as follows:

$$\sum_{0 \leq i < j \leq n} a_{ij}^{2\theta} \left(\sum_{0 \leq k < l \leq n} a'_{kl}{}^{2\theta} - 2a'_{ij}{}^{2\theta} \right) \geq 2^{2\theta-3} n^2 (n^2 - 1) \left[\frac{(n!)^2}{n+1} \right]^{2\theta/n} \left(\frac{\sum_{0 \leq i < j \leq n} a'_{ij}{}^{2\theta}}{\sum_{0 \leq i < j \leq n} a_{ij}^{2\theta}} V^{4\theta/n} + \frac{\sum_{0 \leq i < j \leq n} a_{ij}^{2\theta}}{\sum_{0 \leq i < j \leq n} a'_{ij}{}^{2\theta}} V^{4\theta/n} \right), \tag{3}$$

where $\theta \in (0, 1]$.

In this paper, we extend inequality (2) to the volumes and medians of two n -dimensional simplices. As corollaries, we get some inequalities for medians and volumes of two n -dimensional simplices. Our main result is the following theorem.

THEOREM 1. *For any two n -dimensional simplices Ω_n and $\Omega'_n (n \geq 3)$ in E^n and two arbitrary real numbers $\alpha, \theta \in (0, 1]$, we have*

$$\sum_{i=0}^n m_i^{2\alpha} \left(\sum_{j=0}^n m_j'^{2\theta} - 2m_i'^{2\theta} \right) \geq \frac{1}{2} (n^2 - 1) \left(\rho_n^\alpha \frac{\sum_{i=0}^n m_i'^{2\theta}}{\sum_{i=0}^n m_i^{2\alpha}} V^{4\alpha/n} + \rho_n^\theta \frac{\sum_{i=0}^n m_i^{2\alpha}}{\sum_{i=0}^n m_i'^{2\theta}} V^{4\theta/n} \right), \tag{4}$$

with equality if Ω_n and Ω'_n are regular simplices. Where

$$\rho_n = \left[\frac{(n!)^2 (n+1)^{n-1}}{n^n} \right]^{2/n}. \tag{5}$$

Using inequality (4) and the arithmetic–geometric mean inequality, we get the following corollary.

COROLLARY 1. *For any two n -dimensional simplices Ω_n and $\Omega'_n (n \geq 3)$ and two arbitrary real numbers $\alpha, \theta \in (0, 1]$, we have*

$$\sum_{i=0}^n m_i^{2\alpha} \left(\sum_{j=0}^n m_j'^{2\theta} - 2m_i'^{2\theta} \right) \geq (n^2 - 1) \rho_n^{(a+\theta)/2} V^{2\alpha/n} V'^{2\theta/n}, \tag{6}$$

with equality if Ω_n and Ω'_n are regular simplices.

If taking $\alpha = \theta = 1$ in inequality (4), we get a corollary as follows.

COROLLARY 2. *For any two n -dimensional simplices Ω_n and $\Omega'_n (n \geq 3)$ we have*

$$\sum_{i=0}^n m_i^2 \left(\sum_{j=0}^n m_j'^2 - 2m_i'^2 \right) \geq \frac{1}{2} (n^2 - 1) \rho_n^2 \left(\frac{\sum_{i=0}^n m_i'^2}{\sum_{i=0}^n m_i^2} V^{4/n} + \frac{\sum_{i=0}^n m_i^2}{\sum_{i=0}^n m_i'^2} V'^{4/n} \right), \tag{7}$$

with equality if Ω_n and Ω'_n are regular simplices.

2. Some Lemmas and Proof of Theorem 1

To prove Theorem 1, we need some lemmas as follows.

LEMMA 1. *Given real numbers $x_i > 0 (i = 0, 1, \dots, n)$ and $\sum_{i=0}^n x_i = 1 (n \geq 3)$, then*

$$1 - 2 \sum_{i=0}^n x_i^2 \geq (n + 1)^{(n+2)/n} (n - 1) \prod_{i=0}^n x_i^{2/n}, \tag{8}$$

with equality if and only if $x_0 = x_1 = \dots = x_n = (n + 1)^{-1}$.

For the proof of inequality (8), the reader is referred to [5]. Here we state the idea of the proof of inequality (8). Put

$$f(x_0, x_1, \dots, x_n) = 1 - 2 \sum_{i=0}^n x_i^2 - (n + 1)^{(n+1)/n} (n - 1) \prod_{i=0}^n x_i^{2/n},$$

$$\varphi(x_0, x_1, \dots, x_n) = \sum_{i=0}^n x_i - 1.$$

By Lagrange’s method of multipliers, we can prove that function $f(x_0, x_1, \dots, x_n)$ has an absolute minimum at the point $(\frac{1}{n + 1}, \frac{1}{n + 1}, \dots, \frac{1}{n + 1})$ under the condition $\varphi(x_0, x_1, \dots, x_n) = 0$. So $f(x_0, x_1, \dots, x_n) \geq f(\frac{1}{n + 1}, \frac{1}{n + 1}, \dots, \frac{1}{n + 1}) = 0$ under the condition $x_0 + x_1 + \dots + x_n = 1$, thus inequality (8) holds.

LEMMA 2. *For an n -dimensional simplex $\Omega_n (n \geq 3)$ and an arbitrary real number $\alpha \in (0, 1]$, we have*

$$\frac{\prod_{i=0}^n m_i^{2\alpha}}{\sum_{i=0}^n m_i^{2\alpha}} \geq \frac{1}{n + 1} \left[\frac{(n!)^2 (n + 1)^{n-1}}{n^n} \right]^\alpha V^{2\alpha}, \tag{9}$$

with equality if Ω_n is regular simplex.

By inequality (9) and the arithmetic–geometric mean inequality, we have

$$V \leq \frac{n^{n/2}}{n! \cdot (n + 1)^{(n-1)/2}} \left(\prod_{i=0}^n m_i \right)^{n/(n+1)}, \tag{10}$$

with equality if Ω_n is regular simplex.

Proof. Let point G denote the barycenter of simplex Ω_n , V_i denote the volume of n -dimensional simplex $A_0 \dots A_{i-1} G A_{i+1} \dots A_n$, then $V_i = (n + 1)^{-1} V$, $|GA_i| = n(n + 1)^{-1} m_i (i = 0, 1, \dots, n)$. Let $\vec{GA}_i = n(n + 1)^{-1} m_i \alpha_i$, where α_i is the unit vector of \vec{GA}_i . Using inequality (15) in [6], we have

$$\sum_{i=0}^n \det(\alpha_l \cdot \alpha_k)_{l,k \neq i} \leq \left(\frac{n + 1}{n} \right)^n. \tag{11}$$

Equality in (11) holds if and only if the nonzero eigenvalues of matrix $(\alpha_i \cdot \alpha_j)_{i,j=0}^n = 0$ are all the same.

By the formula for the volume of a simplex, we have

$$\begin{aligned} \frac{1}{n+1}V &= V_i = \frac{1}{n!}[\det(|GA_l| \cdot |GA_k|\alpha_l \cdot \alpha_k)_{l,k \neq i}]^{1/2} \\ &= \frac{1}{n!}[\det(n^2(n+1)^{-2}m_l m_k \alpha_l \cdot \alpha_k)_{l,k \neq i}]^{1/2} \\ &= \frac{n^n}{n! \cdot (n+1)^n} \left(\prod_{\substack{j=0 \\ j \neq i}}^n m_j \right) [\det(\alpha_l \cdot \alpha_k)_{l,k \neq i}]^{1/2}, \end{aligned}$$

i.e.

$$\det(\alpha_l \cdot \alpha_k)_{l,k \neq i} = \frac{(n!)^2(n+1)^{2(n-1)}}{n^{2n}} V^2 \frac{m_i^2}{\prod_{j=0}^n m_j^2} \quad (i = 0, 1, \dots, n). \tag{12}$$

Using equality (12) and inequality (11) we get that

$$\frac{\prod_{i=0}^n m_i^2}{\sum_{i=0}^n m_i^2} \geq \frac{(n!)^2(n+1)^{n-2}}{n^n} V^2. \tag{13}$$

It is easy to know that equality in (13) holds if simplex Ω_n is regular.

From inequality (13) we show that inequality (9) holds for $\alpha = 1$. Now we prove that inequality (9) holds for $\alpha \in (0, 1)$. Using the well-known inequality

$$\frac{1}{n+1} \sum_{i=0}^n m_i^2 \geq \left(\frac{1}{n+1} \sum_{i=0}^n m_i^{2\alpha} \right)^{1/\alpha}, \tag{14}$$

with equality if and only if $m_0 = m_1 = \dots = m_n$.

By inequalities (13) and (14), we have

$$\begin{aligned} \prod_{i=0}^n m_i^{2\alpha} &\geq \left[\frac{(n!)^2(n+1)^{n-1}}{n^n} \right]^\alpha V^{2\alpha} \left(\frac{1}{n+1} \sum_{i=0}^n m_i^2 \right)^\alpha \\ &\geq \frac{1}{n+1} \left[\frac{(n!)^2(n+1)^{n-1}}{n^n} \right]^\alpha V^{2\alpha} \sum_{i=0}^n m_i^{2\alpha}. \end{aligned}$$

Thus inequality (9) holds for $\alpha \in (0, 1)$.

LEMMA 3. Let numbers $x_i > 0, y_i > 0 (i = 0, 1, \dots, n), \lambda_n = \sum_{i=0}^n x_i, \lambda'_n = \sum_{i=0}^n y_i,$

then

$$\lambda_n \lambda'_n - 2 \sum_{i=0}^n x_i y_i \geq \frac{1}{2} \left[\frac{\lambda'_n}{\lambda_n} (\lambda_n^2 - 2 \sum_{i=0}^n x_i^2) + \frac{\lambda_n}{\lambda'_n} (\lambda_n'^2 - 2 \sum_{i=0}^n y_i^2) \right], \tag{15}$$

with equality if and only if $\frac{y_0}{x_0} = \frac{y_1}{x_1} = \dots = \frac{y_n}{x_n}$.

Proof. Inequality (15) is

$$\frac{\lambda'_n}{\lambda_n} \sum_{i=0}^n x_i^2 + \frac{\lambda_n}{\lambda'_n} \sum_{i=0}^n y_i^2 \geq 2 \sum_{i=0}^n x_i y_i. \tag{16}$$

Now we prove that inequality (16) holds. Using the arithmetic–geometric mean inequality, we have

$$\frac{\lambda'_n}{\lambda_n} x_i^2 + \frac{\lambda_n}{\lambda'_n} y_i^2 \geq 2x_i y_i \quad (i = 0, 1, \dots, n).$$

Adding up these $n + 1$ inequalities, we get inequality (16). Equality in (16) holds if and only if $\frac{\lambda'_n}{\lambda_n} x_i^2 = \frac{\lambda_n}{\lambda'_n} y_i^2 (i = 0, 1, \dots, n)$, i.e.

$$\frac{y_0}{x_0} = \frac{y_1}{x_1} = \dots = \frac{y_n}{x_n} = \frac{\lambda'_n}{\lambda_n}.$$

LEMMA 4. For simplex $\Omega_n (n \geq 3)$ and an arbitrary real number $\alpha \in (0, 1]$, we have

$$\left(\sum_{i=0}^n m_i^{2\alpha} \right)^2 - 2 \sum_{i=0}^n m_i^{4\alpha} \geq (n^2 - 1) \rho_n^\alpha V^{4\alpha/n}, \tag{17}$$

with equality if $m_0 = m_1 = \dots = m_n$.

Proof. Taking $x_i = m_i^{2\alpha} / \sum_{j=0}^n m_j^{2\alpha} (i = 0, 1, \dots, n)$ in inequality (8), we get

$$\left(\sum_{i=0}^n m_i^{2\alpha} \right)^2 - 2 \sum_{i=0}^n m_i^{4\alpha} \geq (n + 1)^{(n+2)/n} (n - 1) \left(\frac{\prod_{i=0}^n m_i^{2\alpha}}{\sum_{i=0}^n m_i^{2\alpha}} \right)^{2/n}. \tag{18}$$

Using inequalities (18) and (9), we get inequality (17). It is easy to see that equality in (17) holds if Ω_n is regular simplex.

PROOF OF THEOREM 1. Let $x_i = m_i^{2\alpha}$, $y_i = m_i'^{2\theta} (i = 0, 1, \dots, n)$. By Lemma 3 and Lemma 4, we have

$$\begin{aligned} & \sum_{i=0}^n m_i^{2\alpha} \left(\sum_{j=0}^n m_j'^{2\theta} - 2m_i'^{2\theta} \right) = \left(\sum_{i=0}^n m_i^{2\alpha} \right) \left(\sum_{i=0}^n m_i'^{2\theta} \right) - 2 \sum_{i=0}^n m_i^{2\alpha} m_i'^{2\theta} \\ & \geq \frac{1}{2} \left\{ \frac{\sum_{i=0}^n m_i'^{2\theta}}{\sum_{i=0}^n m_i^{2\alpha}} \left[\left(\sum_{i=0}^n m_i^{2\alpha} \right)^2 - 2 \sum_{i=0}^n m_i^{4\alpha} \right] + \frac{\sum_{i=0}^n m_i^{2\alpha}}{\sum_{i=0}^n m_i'^{2\theta}} \left[\left(\sum_{i=0}^n m_i'^{2\theta} \right)^2 - 2 \sum_{i=0}^n m_i'^{4\theta} \right] \right\} \\ & \geq \frac{1}{2} (n^2 - 1) \left(\rho_n^\alpha \frac{\sum_{i=0}^n m_i'^{2\theta}}{\sum_{i=0}^n m_i^{2\alpha}} V^{4\alpha/n} + \rho_n^\theta \frac{\sum_{i=0}^n m_i^{2\alpha}}{\sum_{i=0}^n m_i'^{2\theta}} V'^{4\theta/n} \right). \end{aligned}$$

Thus inequality (4) holds. It is easy to show that equality in (4) holds if simplices Ω_n and Ω'_n are regular.

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