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GENERAL AUXILIARY PROBLEM PRINCIPLE AND SOLVABILITY OF A CLASS OF NONLINEAR MIXED VARIATIONAL INEQUALITIES INVOLVING PARTIALLY RELAXED MONOTONE MAPPINGS

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Abstract. The approximation–solvability of the following class of nonlinear variational inequality (NVI) problems based on a new general auxiliary problem principle is presented: Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geqslant 0 \quad \text{for all} \quad x \in K,$$

where $T : K \to H$ is a partially relaxed monotone mapping from a nonempty closed convex subset K of a real Hilbert space H into H, and $f : K \to \mathbf{R}$ is a continuous convex function on K. The general auxiliary problem principle is described as follows: for given iterate $x^k \in K$ and for a constant $\rho > 0$, determine x^{k+1} such that (for $k \geqslant 0$)

$$\langle \rho T(x^k) + \rho L(x^{k+1}) + h'(x^{k+1}) - \rho L(x^k) - h'(x^k), x - x^{k+1} \rangle + \rho[f(x) - f(x^{k+1})] \geqslant 0$$

for all $x \in K$, where $L : K \to H$ is any mapping on K, $h : K \to \mathbf{R}$ is a function on K and h' is the derivative of h.

1. Introduction

Cohen [2] introduced the auxiliary problem principle in the context of the decomposition of the optimization problems, which allows to determine the solution of a problem by solving a sequence of auxiliary problems, that means, the principle of the decomposition of the given problem into subproblems and their coordinations. This principle has been responsible for the development of a variety of abstract algorithms along with their special cases, which have been applied to the convergence of approximate solutions. If we turn our attention mainly to the approximation of the solutions of nonlinear variational inequalities in different space settings, recently Zhu and Marcotte [24] and Verma [21], based on the auxiliary problem principle introduced and studied by Cohen [2-3], investigated the approximation–solvability of a class of variational inequalities involving the co–coercive and partially cocoercive mappings. Just recently, Verma [22] introduced an iterative procedure characterized as an auxiliary variational

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inequality type of algorithm and applied it to the approximation–solvability of a class of nonlinear variational inequalities involving cocoercive as well as partially relaxed monotone mappings [19] in a Hilbert space setting. The partially relaxed monotone mappings are weaker than cocoercive and strongly monotone mappings. Here in this paper, we plan first to introduce a general auxiliary problem principle, and then apply it to the approximation–solvability of a class of nonlinear mixed variational inequalities involving partially relaxed monotone mappings. The results, thus, obtained complement the earlier investigations of Cohen [3], Zhu and Marcotte [24] and Verma [20] on the approximation–solvability of nonlinear variational inequality problems in different space settings. For more details on nonlinear variational inequalities and associated algorithms, we recommend [1-24].

Let H be a real Hilbert space with the inner product $\langle \cdot\ ,\ \cdot \rangle$ and norm $||\cdot||$. Let $T : K \to H$ be any mapping and K a closed convex subset of H. Let $f : K \to \mathbf{R}$ be a proper, convex and lower semi– continuous function. We consider a class of nonlinear variational inequality (abbreviated as NVI) problems: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geqslant 0 \quad \text{for all} \quad x \in K. \tag{1.1}$$

Now we need to recall the following auxiliary result, most commonly used in the context of the approximation–solvability of the nonlinear variational inequality problems based on the iterative procedures.

LEMMA 1.1. *An element $u \in K$ is a solution of the NVI problem (1.1) if*

$$\langle T(u), x - u \rangle + f(x) - f(u) \geqslant 0 \quad \textit{for all} \quad x \in K.$$

A mapping $T : H \to H$ is said to be α–cocoercive [19] if for all $x,\ y \in H$ and for $\alpha > 0$, we have

$$||x-y||^2 \geqslant \alpha^2 ||T(x) - T(y)||^2 + ||\alpha(T(x) - T(y)) - (x-y)||^2,$$

A mapping $T : H \to H$ is called α–cocoercive [12] if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geqslant \alpha ||T(x) - T(y)||^2 \quad \textit{for all} \quad x, y \in H.$$

T is called r–stronglymonotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geqslant r ||x - y||^2 \quad \textit{for a constant} \quad r > 0.$$

This implies that

$$||T(x) - T(y)|| \geqslant r ||x - y||,$$

that is, T is r–expanding, and when $r = 1$, it is expanding. T is called monotone if

$$\langle T(x) - T(y), x - y \rangle \geqslant 0 \quad \textit{for all} \quad x, y \in H.$$

The mapping T is called β–Lipschitz continuous (or β–Lipschitzian) if there exists a constant $\beta \geqslant 0$ such that

$$||T(x) - T(y)|| \leqslant \beta ||x - y|| \quad \textit{for all} \quad x, y \in H.$$

We note that if T is α–cocoercive and expanding, then T is α–strongly monotone. On the top of that, if T is α–strongly monotone and β–Lipschitz continuous, then T is (α/β^2)–cocoercive for $\beta > 0$. Clearly every α–cocoercive mapping T is $(1/\alpha)$–Lipschitz continuous.

PROPOSITION 1.1 [21]. *Let $T : H \to H$ be a mapping from a Hilbert space H into itself. Then the following statements are equivalent:*

(i) For each x, $y \in H$ and for a constant $\alpha > 0$, we have

$$||x-y||^2 \geqslant \alpha^2||T(x)-T(y)||^2 + ||\alpha(T(x)-T(y))-(x-y)||^2.$$

(ii) For each x, $y \in H$, we have

$$\langle T(x)-T(y), x-y\rangle \geqslant \alpha||T(x)-T(y)||^2,$$

where $\alpha > 0$ is a constant.

A mapping $T : H \to H$ is said to be γ*–partially relaxed monotone* [19] if there exists a constant $\gamma > 0$ such that

$$\langle T(x)-T(y), z-y\rangle \geqslant -\gamma||z-x||^2 \quad \text{for all} \quad x,y,z \in H.$$

The γ*–partial relaxed monotonicity introduced by Verma* [19] is a computation–oriented notion–targeted at algorithmic approximations.

When $\gamma = 0$, T is called 0*–partially relaxed monotone (or partially monotone), that is*,

$$\langle T(x)-T(y), z-y\rangle \geqslant 0.$$

We note that the 0–partial relaxed monotonicity implies the monotonicity, that is, for all x, y, $z \in H$,

$$\begin{aligned}\langle T(x)-T(y), x-y\rangle &= \langle T(x)-T(y), z-y+x-z\rangle\\ &= \langle T(x)-T(y), z-y\rangle + \langle T(x)-T(y), x-z\rangle\\ &= \langle T(x)-T(y), z-y\rangle + \langle T(y)-T(x), z-x\rangle \geqslant 0.\end{aligned}$$

On the top of that, every α–cocoercive mapping is $(1/4\alpha)$–partially relaxed monotone. For more details, we refer to [19].

LEMMA 1.2. *For all elements v, $w \in H$, we have*

$$||v||^2 + \langle v, w\rangle \geqslant -(1/4)||w||^2.$$

PROPOSITION 1.2 [19]. *Let $T : H \to H$ be an α–cocoercive mapping on H. Then T is $(1/4\alpha)$–partially relaxed monotone.*

Proof. We include the proof for the sake of the completeness. Since T is α–cocoercive, it implies by Lemma 1.2, for all x, y, $z \in H$, that

$$\begin{aligned}\langle T(x)-T(y), z-y\rangle &= \langle T(x)-T(y), x-y\rangle + \langle T(x)-T(y), z-x\rangle\\ &\geqslant \alpha||T(x)-T(y)||^2 + \langle T(x)-T(y), z-x\rangle\\ &= \alpha\{||T(x)-T(y)||^2 + (1/\alpha)\langle T(x)-T(y), z-x\rangle\}\\ &\geqslant -(1/4\alpha)||z-x||^2,\end{aligned}$$

that is, T is $(1/4\alpha)$–partially relaxed monotone.

PROPOSITION 1.3. *Let $T : H \to H$ be an α–partially relaxed mapping on H. Then T is not $(1/4\alpha)$–cocoercive in general, that is, the converse of Proposition 1.2 may not be true in general.*

Proof. Since T is α–partially relaxed monotone, we can have from Lemma 1.2, for all x, y, $z \in H$, that

$$\begin{aligned}\langle T(x) - T(y), x - y\rangle &= \langle T(x) - T(y), z - y\rangle + \langle T(x) - T(y), x - z\rangle \\ &\geqslant -\alpha||z - x||^2 + \langle T(x) - T(y), x - z\rangle \\ &= -\alpha\{||z - x||^2 + (1/\alpha)\langle T(x) - T(y), z - x\rangle \} \\ &\leqslant (1/4\alpha)||T(x) - T(y)||^2,\end{aligned}$$

that means, T is not $(1/4\alpha)$–cocoercive.

2. General auxiliary problem principle

This section deals first with the introduction of a general auxiliary problem principle — an extension of the auxiliary problem principle first initiated by Cohen [2-3], which is extended and applied by others to optimization problems as well as to variational inequality problems in different problem settings, including Zhu and Marcotte [24] — and then its application to the approximation–solvability of the NVI problem (1.1). Based on the general auxiliary problem principle, the general auxiliary nonlinear variational inequality $(GANVI)$ problem is as follows: for a given iterate x^k, determine an x^{k+1} such that (for $k \geqslant 0$):

$$\begin{aligned}&\langle \rho T(x^k) + \rho L(x^{k+1}) + h'(x^{k+1}) - \rho L(x^k) - h'(x^k), x - x^{k+1}\rangle \\ &\qquad + \rho[f(x) - f(x^{k+1})] \geqslant 0 \quad \text{for all} \quad x \in K, \end{aligned} \tag{2.1}$$

where $L : K \to H$ is 0–partially relaxed monotone, and h', the derivative of h, is b–strongly monotone.

When $L \equiv 0$ in the $GANVI$ problem (2.1), we have the $GANVI$ problem as follows: for a given iterate x^k, determine an x^{k+1} such that (for $k \geqslant 0$):

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1}\rangle + \rho[f(x) - f(x^{k+1})] \geqslant 0 \quad \text{for all} \quad x \in K. \tag{2.2}$$

Next, we recall some auxiliary results crucial to the approximation–solvability of the NVI problem (1.1).

LEMMA 2.1 [24]. *Let $h : K \to \mathbf{R}$ be continuously differentiable on a convex subset K of H. Then we have the following conclusions:*

(i) If h' is b–strongly monotone, then

$$h(x) - h(y) \geqslant \langle h'(y), x - y\rangle + (b/2)||x - y||^2 \quad \textit{for all} \quad x, y \in K.$$

(ii) If the gradient h' is p–Lipschitz continuous, then

$$h(x) - h(y) \leqslant \langle h'(y), x - y\rangle + (p/2)||x - y||^2 \quad \textit{for all} \quad x, y \in K.$$

We are just about ready to present, based on the *GANVI* problem (2.1), the approximation–solvability of the *NVI* problem (1.1) involving γ–partially relaxed monotone mappings in a Hilbert space setting. In what follows, H shall represent a finite–dimensional real Hilbert space.

THEOREM 2.1. *Let H be a real Hilbert space and $T : K \to H$ a γ–partially relaxed monotone and q–Lipschitzian mapping from a nonempty closed convex subset K of H into H. Let $h : K \to \mathbf{R}$ be continuously differentiable, and h' be b–strongly monotone and p–Lipschitz continuous. Suppose that $L : K \to H$ is 0–partially relaxed monotone and s–Lipschitzian, and $f : K \to \mathbf{R}$ is continuous and convex. Then x^{k+1} is a unique solution of (2.1).*

If in addition, $x^* \in K$ is any fixed solution of the NVI problem (1.1), then $\{x^k\}$ is bounded and converges to x^* for $0 < \rho < 2b/\gamma$.

Proof. Before we can show that the sequence $\{x^k\}$ converges to x^*, a solution of the *NVI* problem (1.1), we need to compute the estimates. Since h is b–strongly convex, it ensures the uniqueness of solution x^{k+1} of the *GANVI* problem (2.1). Let us define a function Λ^* by

$$\begin{aligned}\Lambda^*(x) &:= h(x^*) - h(x) - \langle h'(x), x^* - x\rangle \\ &\geqslant (b/2)||x - x^*||^2,\end{aligned}$$

where x^* is any fixed solution of the *NVI* problem (1.1).

Now we can write

$$\begin{aligned}\Lambda^*(x^k) - \Lambda^*(x^{k+1}) &= h(x^{k+1}) - h(x^k) - \langle h'(x^k), x^{k+1} - x^k\rangle \\ &\qquad + \langle h'(x^{k+1}) - h'(x^k), x^* - x^{k+1}\rangle \\ &\geqslant (b/2)||x^{k+1} - x^k||^2 + \rho\langle T(x^k) + L(x^{k+1}) - L(x^k), x^{k+1} - x^*\rangle \\ &\qquad + \rho[f(x^{k+1}) - f(x^*)] \\ &= (b/2)||x^{k+1} - x^k||^2 + \rho\langle T(x^k), x^{k+1} - x^*\rangle \\ &\qquad + \rho\langle L(x^{k+1}) - L(x^k), x^{k+1} - x^*\rangle + \rho[f(x^{k+1}) - f(x^*)] \\ &\geqslant (b/2)||x^{k+1} - x^k||^2 + \rho\langle T(x^k) - T(x^*), x^{k+1} - x^*\rangle \\ &\qquad + \rho\langle L(x^k) - L(x^{k+1}), x^* - x^{k+1}\rangle .\end{aligned}$$

Since T is γ–partially relaxed monotone and L is 0–partially relaxed monotone, we have

$$\begin{aligned}\Lambda^*(x^k) - \Lambda^*(x^{k+1}) &\geqslant (b/2)||x^{k+1} - x^k||^2 - \rho\gamma||x^{k+1} - x^k||^2 \\ &= (1/2)[b - (\rho\gamma/2)]||x^{k+1} - x^k||^2.\end{aligned}$$

Thus, we can write

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \geqslant (1/2)[b - (\rho\gamma/2)]||x^{k+1} - x^k||^2. \tag{2.3}$$

It follows for $x^{k+1} = x^k$ that x^k is a solution of the variational inequality (1.1). If not, the condition $b - (\rho\gamma/2) > 0$ in (2.3) ensures that the $\Lambda^*(x^k) - \Lambda^*(x^{k+1})$ is nonnegative and, as a result, the $\{\Lambda^*(x^k)\}$ is strictly decreasing, which means the difference of the two succeeding terms tends to zero. Therefore, we have

$$\lim_{k\to\infty} ||x^{k+1} - x^k|| = 0.$$

On the top of that, $||x^* - x^k||^2 \leqslant (2/b)\Lambda^*(x^k)$ and the sequence $\{\Lambda^*(x^k)\}$ is decreasing, that means $\{x^k\}$ is a bounded sequence. Hence there exists a strongly convergent subsequence of $\{x^k\}$. Let x' be the limit of a subsequence $\{x^{k'}\}$. Then as $k \to \infty$ in (2.1), x' is a solution of the variational inequality (1.1). There is no loss of generality if x^* is replaced by x'. If we associate x' to a function Λ', defined by

$$\Lambda'(x^k) = h(x') - h(x^k) - \langle h'(x^k), x' - x^k\rangle \leqslant (p/2)||x' - x^k||^2$$

(by Lemma 2.1), then we have

$$\Lambda'(x^k) \leqslant (p/2)||x' - x^k||^2.$$

Since the sequence $\{\Lambda'(x^k)\}$ is strictly decreasing, it follows that $\Lambda'(x^k) \to 0$. On the other hand, we already have

$$\Lambda'(x^k) \geqslant (b/2)||x' - x^k||^2.$$

Thus, we can conclude that the entire sequence $\{x^k\}$ converges to x', and this completes the proof.

THEOREM 2.2. *Let H be a real Hilbert space and T, $L : K \to H$ any mappings such that $T - L$ is γ-partially relaxed monotone and L is monotone. Let $h : K \to$ $\mathbf{R}$ be continuously differentiable, and h' be b–strongly monotone and p–Lipschitz continuous. Then x^{k+1} is a unique solution of (2.1).*

If in addition, $x^* \in K$ is any fixed solution of the NVI problem (1.1), then $\{x^k\}$ is bounded and converges to x^* for $0 < \rho < 2b/\gamma$.

Proof. Since x^* a solution of the NVI problem (1.1), we need to compute the estimates. Since h' is b–strongly monotone, it ensures the uniqueness of solution x^{k+1} of the $GANVI$ problem (2.1). Let us define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x\rangle \geqslant (b/2)||x - x^*||^2,$$

where x^* is any fixed solution of the NVI problem (1.1).

Now we can write

$$\begin{aligned}\Lambda^*(x^k) - \Lambda^*(x^{k+1}) &= h(x^{k+1}) - h(x^k) - \langle h'(x^k), x^{k+1} - x^k\rangle \\ &\qquad + \langle h'(x^{k+1}) - h'(x^k), x^* - x^{k+1}\rangle \\ &\geqslant (b/2)||x^{k+1} - x^k||^2 + \rho\langle T(x^k) + L(x^{k+1}) - L(x^k), x^{k+1} - x^*\rangle \\ &\qquad + \rho[f(x^{k+1}) - f(x^*)]\end{aligned}$$

$$
\begin{aligned}
&= (b/2)\|x^{k+1} - x^k\|^2 + \rho\{\langle T(x^k), x^{k+1} - x^*\rangle \\
&\qquad + \langle L(x^{k+1}) - L(x^k), x^{k+1} - x^*\rangle\} + \rho[f(x^{k+1}) - f(x^*)] \\
&= (b/2)\|x^{k+1} - x^k\|^2 + \rho\{\langle (T - L)(x^k), x^{k+1} - x^*\rangle \\
&\qquad + \langle L(x^{k+1}), x^{k+1} - x^*\rangle\} + \rho[f(x^{k+1}) - f(x^*)] \\
&\geqslant (b/2)\|x^{k+1} - x^k\|^2 + \rho\{\langle (T - L)(x^k) - T(x^*), x^{k+1} - x^*\rangle \\
&\qquad + \langle L(x^{k+1}), x^{k+1} - x^*\rangle\} \\
&= (b/2)\|x^{k+1} - x^k\|^2 + \rho\{\langle (T-L)(x^k) - (T-L)(x^*), x^{k+1}-x^*\rangle \\
&\qquad + \langle L(x^{k+1}) - L(x^*), x^{k+1} - x^*\rangle\}.
\end{aligned}
$$

Since $T-L$ is γ–partially relaxed monotone and L is monotone, it implies that

$$
\begin{aligned}
\Lambda^*(x^k) - \Lambda^*(x^{k+1}) &\geqslant (b/2)\|x^{k+1} - x^k\|^2 - \rho\gamma\|x^{k+1} - x^k\|^2 \\
&= (1/2)[b - (\rho\gamma/2)]\|x^{k+1} - x^k\|^2.
\end{aligned}
$$

The rest of the proof is similar to that of Theorem 2.1.

THEOREM 2.3. *Let H be a real Hilbert space and T, $L : K \to H$ any mappings such that $T-L$ is γ–cocoercive and L is monotone. Let $h : K \to \mathbf{R}$ be continuously differentiable, and h' be b–strongly monotone and p–Lipschitz continuous. Then x^{k+1} is a unique solution of (2.1).*

If, in addition, $x^* \in K$ is any fixed solution of the NVI problem (1.1), then $\{x^k\}$ is bounded and converges to x^* for $0 < \rho < 2b/\gamma$.

THEOREM 2.4. *Let H be a real Hilbert space and $T : K \to H$ any mapping such that T is γ–partially relaxed monotone. Let $h : K \to \mathbf{R}$ be continuously differentiable, and h' be b–strongly monotone and p–Lipschitz continuous. Then x^{k+1} is a unique solution of (2.2).*

If in addition, $x^* \in K$ is any fixed solution of the NVI problem (1.1), then $\{x^k\}$ is bounded and converges to x^* for $0 < \rho < 2b/\gamma$.

THEOREM 2.5. *Let H be a real Hilbert space and $T : K \to H$ any mapping such that T is γ–cocoercive. Let $h : K \to \mathbf{R}$ be continuously differentiable, and h' be b–strongly monotone and p–Lipschitz continuous. Then x^{k+1} is a unique solution of (2.2).*

If, in addition, $x^* \in K$ is any fixed solution of the NVI problem (1.1), then $\{x^k\}$ is bounded and converges to x^* for $0 < \rho < 2b/\gamma$.

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