

THE JENSEN–GRÜSS INEQUALITY

I. BUDIMIR AND J. PEČARIĆ

(communicated by J. Sándor)

Abstract. The Jensen–Grüss inequality is proved, that is a converse of Jensen’s inequality, related to the well known Grüss inequality.

1. Introduction

Let $f : X \rightarrow \mathbf{R}$ be a convex mapping defined on the linear space X and $x_i \in X$, $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m := \sum_{i=1}^m p_i > 0$.

The following inequality is well known in the literature as Jensen’s inequality

$$f \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i \right) \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i). \quad (1.1)$$

Some conversions of Jensen inequality were recently obtained in [1] and [2]. For example, the following result was obtained in [1]:

THEOREM A. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a differentiable convex mapping and $x_i \in \mathbf{R}^n$, $i = 1, \dots, m$. Suppose that there exist vectors $\phi, \Phi \in \mathbf{R}^n$ such that*

$$\phi \leq x_i \leq \Phi \quad (\text{the order is considered coordinatewise}) \quad (1.2)$$

and $m, M \in \mathbf{R}^n$ are such that

$$m \leq \nabla f(x_i) \leq M, \text{ for all } x_i \in \{1, \dots, m\}. \quad (1.3)$$

Then for all $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m > 0$, we have the inequality

$$0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i \right) \leq \frac{1}{4} \|\Phi - \phi\| \|M - m\|, \quad (1.4)$$

where $\|\cdot\|$ is the usual Euclidean norm in \mathbf{R}^n .

It is clear that such results are related to the well known Grüss inequality [3, p. 93], but they are valid for convex functions. Here we shall give some related results for wider class of functions, that is for functions which shouldn’t be convex.

Mathematics subject classification (2000): 26D15, 26B25.

Key words and phrases: Convex functions, Jensen’s inequality, entropy mappings.

2. The Jensen-Grüss inequality in multidimensional case

THEOREM 1. *Let $f : U \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ be a differentiable mapping with continuous partial derivatives, where U is open convex set and $x_i \in U$, $i = 1, \dots, m$. Suppose that there exist vectors $\phi, \Phi \in \mathbf{R}^n$ such that*

$$\phi \leq x_i \leq \Phi \quad (\text{the order is considered coordinatewise}) \quad (2.1)$$

and $m, M \in \mathbf{R}^n$ such that

$$m \leq \nabla f(x) \leq M, \text{ for all } x \in \text{conv}[x_1, \dots, x_m], \quad (2.2)$$

where $\text{conv}[x_1, \dots, x_m]$ is the convex hull. Then for all $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m > 0$, we have the inequality

$$\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \leq \frac{1}{4} \|\Phi - \phi\| \|M - m\|. \quad (2.3)$$

Proof. Using the mean-value theorem in multidimensional case for points $x, y \in \text{conv}[x_1, \dots, x_m]$ we conclude that there exists $\vartheta \in \langle 0, 1 \rangle$ such that

$$f(x) - f(y) = \langle \nabla f(z), x - y \rangle, \quad (2.4)$$

where $z = y + \vartheta(x - y)$.

Using (2.4) for $x = \frac{1}{P_m} \sum_{i=1}^m p_i x_i$, $y = x_j$, $z = z_j$, ($j = 1, \dots, m$), we may simply write that

$$f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - f(x_j) = \left\langle \nabla f(z_j), \frac{1}{P_m} \sum_{i=1}^m p_i x_i - x_j \right\rangle, \quad \forall j \in \{1, \dots, m\}. \quad (2.5)$$

Multiply (2.5) with $p_j \geq 0$ and sum over j , in order to obtain

$$\begin{aligned} p_j f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - p_j f(x_j) &= p_j \left\langle \nabla f(z_j), \frac{1}{P_m} \sum_{i=1}^m p_i x_i - x_j \right\rangle \\ P_m f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \sum_{j=1}^m p_j f(x_j) &= \sum_{j=1}^m p_j \left(\left\langle \nabla f(z_j), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle - \langle \nabla f(z_j), x_j \rangle \right). \end{aligned}$$

Therefore

$$P_m f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \sum_{j=1}^m p_j f(x_j) = \left\langle \sum_{j=1}^m p_j \nabla f(z_j), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle - \sum_{j=1}^m p_j \langle \nabla f(z_j), x_j \rangle.$$

Dividing by $P_m > 0$, we obtain

$$\begin{aligned}
 & f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) \\
 &= \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle - \frac{1}{P_m} \sum_{i=1}^m p_i \langle \nabla f(z_i), x_i \rangle. \tag{2.6}
 \end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
 & \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla f(z_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\rangle \\
 &= \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \langle x_i - x_j, \nabla f(z_i) - \nabla f(z_j) \rangle. \tag{2.7}
 \end{aligned}$$

Taking the modules in both parts of (2.7) we obtain, by Schwartz’s inequality in inner product spaces, i.e. $|\langle a, b \rangle| \leq \|a\| \cdot \|b\|$, $a, b \in \mathbf{R}^n$, that

$$\begin{aligned}
 & \left| \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla f(z_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\rangle \right| \\
 & \leq \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j |\langle x_i - x_j, \nabla f(z_i) - \nabla f(z_j) \rangle| \\
 & \leq \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\| \cdot \|\nabla f(z_i) - \nabla f(z_j)\|. \tag{2.8}
 \end{aligned}$$

Using the Cauchy-Buniakowsky-Schwartz inequality for double sums, we can state that

$$\begin{aligned}
 & \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\| \cdot \|\nabla f(z_i) - \nabla f(z_j)\| \\
 & \leq \left(\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|\nabla f(z_i) - \nabla f(z_j)\|^2 \right)^{\frac{1}{2}}. \tag{2.9}
 \end{aligned}$$

As a simple calculation shows that

$$\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\|^2 = \frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2$$

and

$$\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|\nabla f(z_i) - \nabla f(z_j)\|^2 = \frac{1}{P_m} \sum_{i=1}^m p_i \|\nabla f(z_i)\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\|^2,$$

we can state, by (2.8) and (2.9), that

$$\begin{aligned} & \left| \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla f(z_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\rangle \right| \\ & \leq \left(\frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{P_m} \sum_{i=1}^m p_i \|\nabla f(z_i)\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.10)$$

Now, let us observe that, as in [1], we have

$$\begin{aligned} & \frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \\ & = \left\langle \phi - \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i x_i - \Phi \right\rangle - \frac{1}{P_m} \sum_{i=1}^m p_i \langle \phi - x_i, x_i - \Phi \rangle. \end{aligned} \quad (2.11)$$

As $\phi \leq x_i \leq \Phi$ ($i \in 1, \dots, m$), then $\langle \phi - x_i, x_i - \Phi \rangle \geq 0$ for all $i \in \{1, \dots, m\}$ and then

$$\sum_{i=1}^m p_i \langle \phi - x_i, x_i - \Phi \rangle \geq 0$$

and by (2.11), we obtain

$$\frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \leq \left\langle \phi - \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i x_i - \Phi \right\rangle. \quad (2.12)$$

It is known that if $y, z \in \mathbf{R}^n$, then

$$4\langle z, y \rangle \leq \|z + y\|^2, \quad (2.13)$$

with equality iff $z = y$. Now, if we apply (2.13) for the vectors

$$z = \phi - \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \quad y = \frac{1}{P_m} \sum_{i=1}^m p_i x_i - \Phi,$$

we deduce

$$\left\langle \phi - \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i x_i - \Phi \right\rangle \leq \frac{1}{4} \|\phi - \Phi\|^2$$

and then, by (2.11)–(2.12), we can derive that

$$\frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \leq \frac{1}{4} \|\phi - \Phi\|^2. \quad (2.14)$$

Similarly, one has

$$\frac{1}{P_m} \sum_{i=1}^m p_i \|\nabla f(z_i)\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m \nabla f(z_i) \right\|^2 \leq \frac{1}{4} \|M - m\|^2. \tag{2.15}$$

Finally, by (2.10), (2.14) and (2.15), one can write

$$\begin{aligned} & \left| \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla f(z_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\rangle \right| \\ & \leq \frac{1}{4} \|\Phi - \phi\| \|M - m\|, \end{aligned} \tag{2.16}$$

which, by (2.2), gives the desired inequality (2.1). \square

THEOREM 2. *Let $f : U \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ be a differentiable mapping with continuous partial derivatives, where U is an open convex set, and $x_i \in U$, $i = 1, \dots, m$, $p_i \geq 0$ ($i = 1, \dots, m$) and $P_m > 0$. Suppose that the ∇ -operator satisfies a condition of r -H-Hölder type, i.e.,*

$$\|\nabla f(x) - \nabla f(y)\| \leq M \|x - y\|^r, \quad \forall x, y \in \text{conv}[x_1, \dots, x_m], \tag{2.17}$$

where $M > 0$ and $r \in \langle 0, 1 \rangle$. Let

$$\Delta = \max_{1 \leq i < j \leq m} \|x_i - x_j\|.$$

Then we have the inequality

$$\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \leq \frac{M \Delta^{r+1}}{2P_m^2} \left(P_m^2 - \sum_{i=1}^m p_i^2 \right). \tag{2.18}$$

Proof. As in the proof of Theorem 1, we have (2.8), that is

$$\begin{aligned} & \left| \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla f(z_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\rangle \right| \\ & \leq \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\| \cdot \|\nabla f(z_i) - \nabla f(z_j)\| \end{aligned} \tag{2.19}$$

Because $z_i, z_j \in \text{conv}[x_1, \dots, x_m]$ (2.17) implies that the last expression is

$$\leq \frac{M}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\| \cdot \|z_i - z_j\|^r.$$

As $z_i, z_j \in \text{conv}[x_1, \dots, x_m]$, we can write

$$z_i = \sum_{i=1}^m u_i x_i, \quad z_j = \sum_{j=1}^m v_j x_j.$$

Then

$$\|z_i - z_j\| = \left\| \sum_{i,j=1}^m u_i v_j (x_i - x_j) \right\| \leq \max_{1 \leq i < j \leq m} \|x_i - x_j\| = \Delta.$$

Using (2.19) we can state the inequality

$$\begin{aligned} & \left| \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla f(z_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(z_i) \right\rangle \right| \\ & \leq \frac{M}{2P_m^2} \sum_{i,j=1;i \neq j}^m p_i p_j \|x_i - x_j\| \Delta^r \\ & = \frac{\Delta^r M}{2P_m^2} \sum_{i,j=1;i \neq j}^m p_i p_j \|x_i - x_j\| \leq \frac{\Delta^{r+1} M}{2P_m^2} \sum_{i,j=1;i \neq j}^m p_i p_j. \end{aligned}$$

Since

$$\sum_{1 \leq i < j \leq m} p_i p_j = \frac{1}{2} \left(\sum_{i,j=1}^m p_i p_j - \sum_{i=j}^m p_i p_j \right) = \frac{1}{2} \left(P_m^2 - \sum_{i=1}^m p_i^2 \right),$$

the inequality (2.18) is proved. \square

THEOREM 3. Let $f : U \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ be a differentiable mapping with continuous partial derivatives, where U is an open convex set, $x_i \in U$, $p_i \geq 0$ ($i = 1, \dots, m$) and $P_m > 0$. Suppose that the ∇ -operator is Lipschitzian with the constant $L > 0$, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad (2.20)$$

for all $x, y \in \text{conv}[x_1, \dots, x_m]$. If

$$\phi \leq x \leq \Phi \quad x \in \text{conv}[x_1, \dots, x_m], \quad (2.21)$$

$\phi, \Phi \in \mathbf{R}^n$ ($i = 1, \dots, m$) then we have the inequality

$$\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \leq \frac{L}{4} \|\Phi - \phi\|^2. \quad (2.22)$$

Proof. As in the proof of Theorem 1

$$\begin{aligned} & \left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \\ & \leq \left(\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|\nabla f(z_i) - \nabla f(z_j)\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

(where z_i, z_j are as in Theorem 1)

$$\leq \left(\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j L^2 \|z_i - z_j\|^2 \right)^{\frac{1}{2}}$$

(using the equalities

$$\frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\|^2 = \frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2$$

and

$$\begin{aligned} \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|z_i - z_j\|^2 &= \frac{1}{P_m} \sum_{i=1}^m p_i \|z_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i z_i \right\|^2 \\ &= \left(\frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \right)^{\frac{1}{2}} \times L \left(\frac{1}{P_m} \sum_{i=1}^m p_i \|z_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i z_i \right\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

≤ (we have the following Grüss type inequality

$$\frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \leq \frac{1}{4} \|\Phi - \phi\|^2,$$

and using the fact that $z_i \in \text{conv}[x_1, \dots, x_m]$, so $\phi \leq z_i \leq \Phi$)

$$\leq L \left(\frac{1}{4} \|\Phi - \phi\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{4} \|\Phi - \phi\|^2 \right)^{\frac{1}{2}} = \frac{L}{4} \|\Phi - \phi\|^2. \quad \square$$

REMARK. Some results, related to Theorems 2 and 3, for convex functions were obtained in [2].

3. The Jensen-Grüss inequality in one dimensional case

THEOREM 4. Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping with continuous first derivative and $I = \langle a, b \rangle$. Let $x_i \in I$ ($i = 1, \dots, m$) and suppose that there exist $m, M \in \mathbf{R}$ such that

$$m \leq f'(x) \leq M, \quad \text{for all } x \in I. \tag{3.1}$$

Then for all $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m > 0$ we have the inequality

$$\begin{aligned} &\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i \right) \right| \\ &\leq \frac{\sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i \right)}{P_m^2} (b - a)(M - m), \end{aligned} \tag{3.2}$$

where S is the subset of $\{1, \dots, m\}$ which minimizes the expression $|\sum_{i \in S} p_i - \frac{P_m}{2}|$.

Proof. Using the mean-value theorem for points $x, y \in I$ we can write that there exist $z, x \leq z \leq y$ such that

$$f(x) - f(y) = f'(z)(x - y). \quad (3.3)$$

Using (3.3) for $x = \frac{1}{P_m} \sum_{i=1}^m p_i x_i, y = x_j$ we conclude that there exists $z_j (x \leq z_j \leq x_j)$ such that

$$f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - f(x_j) = f'(z_j) \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i - x_j\right) \quad (3.4)$$

for all $j \in \{1, \dots, m\}$. If we multiply (3.4) by $p_j \geq 0$ and sum over j , we obtain

$$p_j f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - p_j f(x_j) = p_j f'(z_j) \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i - x_j\right).$$

This implies

$$P_m f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \sum_{j=1}^m p_j f(x_j) = \frac{1}{P_m} \sum_{j=1}^m p_j f'(z_j) \left(\sum_{i=1}^m p_i x_i\right) - \sum_{j=1}^m p_j x_j f'(z_j).$$

Dividing by $P_m > 0$, we obtain

$$\begin{aligned} f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \frac{1}{P_m} \sum_{j=1}^m p_j f(x_j) \\ = \frac{1}{P_m} \sum_{j=1}^m p_j f'(z_j) \frac{1}{P_m} \sum_{i=1}^m p_i x_i - \frac{1}{P_m} \sum_{j=1}^m p_j x_j f'(z_j) \end{aligned} \quad (3.5)$$

We note that the following results was proved in [4]:

THEOREM B. If $m_1 \leq a_i \leq M_1, m_2 \leq b_i \leq M_2, i = 1, \dots, m$ and S is the subset of $\{1, \dots, m\}$ which minimizes the expression

$$\left| \sum_{i \in S} p_i - \frac{P_m}{2} \right|$$

then

$$\left| P_m \sum_{i=1}^m p_i a_i b_i - \left(\sum_{i=1}^m p_i a_i\right) \left(\sum_{i=1}^m p_i b_i\right) \right| \leq (M_1 - m_1)(M_2 - m_2) \left(\sum_{i \in S} p_i\right) \left(P_m - \sum_{i \in S} p_i\right)$$

If we choose $a_i = x_i, b_i = f'(z_i)$, then we may state that

$$\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \leq \frac{\sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i\right)}{P_m^2} (M - m)(b - a)$$

and the result follows. \square

THEOREM 5. Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping with continuous first derivative and $I = \langle a, b \rangle$. Let $x_i \in I$, $p_i \geq 0$ ($i = 1, \dots, m$) and $P_m > 0$. Suppose that f' is Lipschitzian with the constant $L > 0$, i.e.,

$$|f'(x) - f'(y)| \leq L|x - y|, \tag{3.6}$$

for all $x, y \in I$. Then we have the inequality

$$\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \leq L \frac{\sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i\right)}{P_m^2} (b - a)^2, \tag{3.7}$$

where S is the subset of $\{1, \dots, m\}$ which minimizes the expression $|\sum_{i \in S} p_i - \frac{P_m}{2}|$.

Proof. As in (3.5) we have the equality

$$\begin{aligned} & \left| f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \frac{1}{P_m} \sum_{j=1}^m p_j f(x_j) \right| \\ &= \left| \frac{1}{P_m} \sum_{j=1}^m p_j f(z_j) - \frac{1}{P_m} \sum_{i=1}^m p_i x_i + \frac{1}{P_m} \sum_{j=1}^m p_j x_j f'(z_j) \right| \\ &\leq \frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |x_i - x_j| \cdot |f'(z_i) - f'(z_j)|. \end{aligned}$$

Using the Cauchy-Buniakowsky-Schwartz inequality, we can state that the last expression is less than

$$\left(\frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |x_i - x_j|^2 \right)^{\frac{1}{2}} \left(\frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |f'(z_i) - f'(z_j)|^2 \right)^{\frac{1}{2}}.$$

Using Theorem B and (3.6) we have that this is

$$\begin{aligned} & \leq \left(\frac{1}{P_m^2} \sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i\right) (b - a)^2 \right)^{\frac{1}{2}} \times L \left(\sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i\right) (b - a)^2 \right)^{\frac{1}{2}} \\ &= L \frac{\sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i\right)}{P_m^2} (b - a)^2 \end{aligned}$$

which proves the Theorem. \square

COROLLARY 1. *With the assumptions of Theorem 5 and assuming bounded second derivative of f , we have the inequality*

$$\left| \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \right| \leq L \frac{\sum_{i \in S} p_i \left(P_m - \sum_{i \in S} p_i \right)}{P_m^2} (b-a)^2, \quad (3.8)$$

where $L = \|f''\|$ and $\|\cdot\|$ is the sup norm.

Proof. Using the mean-value theorem, we conclude that there exist points w_{ij} in the open interval joining z_i to z_j , such that

$$\begin{aligned} |f'(z_i) - f'(z_j)| &= |f''(w_{ij})(z_i - z_j)| \\ &\leq \sup_{x \in I} |f''(x)| |z_i - z_j| = \|f''\| |z_i - z_j|. \end{aligned}$$

As in proof of Theorem 5 we may conclude that

$$\begin{aligned} &\left| f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \frac{1}{P_m} \sum_{j=1}^m p_j f(x_j) \right| \\ &\leq \left(\frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |x_i - x_j|^2 \right)^{\frac{1}{2}} \times \left(\frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |f'(z_i) - f'(z_j)|^2 \right)^{\frac{1}{2}} \\ &\leq \|f''\| \left(\frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |x_i - x_j|^2 \right)^{\frac{1}{2}} \times \left(\frac{1}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j |z_i - z_j|^2 \right)^{\frac{1}{2}} \end{aligned}$$

yielding the desired inequality. \square

REFERENCES

- [1] S. S. DRAGOMIR, *A converse of the Jensen inequality for convex mappings of several variables and applications*, to appear.
- [2] I. BUDIMIR, S. S. DRAGOMIR, AND J. PEČARIĆ, *Further reverse results for Jensen's discrete inequality and application in information theory*, to appear.
- [3] A. M. FINK, *A treatise on Grüss' inequality*, *Analytic and Geometric Inequalities and Applications*, Kluwer Academic Publisher (1999), 93–113.
- [4] D. ANDRICA AND C. BADEA, *Grüss' inequality for positive linear functionals*, *Periodica Mathematica Hungarica* Vol. 19 (2), (1988), 155–167.

(Received May 26, 2000)

Ivan Budimir
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia
e-mail: ivanb@zagreb.tekstil.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia
e-mail: pecaric@mahazu.hazu.hr