Mathematical
Inequalities
& Applications
Volume 5, Number 4 (2002), 697–705

ON THE FEKETE–SZEGÖ AND ARGUMENT INEQUALITIES FOR STRONGLY CLOSE–TO–STAR FUNCTIONS

NAK EUN CHO AND SHIGEYOSHI OWA

(*communicated by H. M. Srivastava*)

Abstract. Let $\mathcal{CS}(\beta)$ be the class of normalized strongly close-to-star functions of order β in the open unit disk. We obtain sharp Fekete-Szegö inequalities for functions belonging to the class $\mathcal{CS}(\beta)$. Some sufficient conditions for close-to-star functions also are investigated in a sector. Furthermore, we consider the integral preserving properties for functions in $\mathcal{CS}(\beta)$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions. We also denote by $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{U}$ (see, e.g., Srivastava and Owa [18]).

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g(z) \prec h(z)$.

Let

$$\mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \ \ (z \in \mathcal{U}\,;\ -1 \leqslant B < A \leqslant 1) \right\}$$

and

$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \ \ (z \in \mathcal{U}\,;\ -1 \leqslant B < A \leqslant 1) \right\}.$$

Mathematics subject classification (2000): 30C45.

Key words and phrases: univalent, starlike, convex, close-to-convex, subordinate, strongly close-to-star, Fekete-Szegö inequality, argument, integral operator.

This work was supported by Korea Research Foundation Grant (KRF-99-015-DP0019).

© ELEMENT, Zagreb
Paper MIA-05-70

The class $\mathcal{S}^*[A,B]$ and related classes were studied by Janowski [5] and Silverman and Silvia [17]. Applying the Briot-Bouquet differential subordination [10, p. 81], we can easily see that $\mathcal{K}[A,B] \subset \mathcal{S}^*[A,B]$. We also note that $\mathcal{S}^*[1,-1] = \mathcal{S}^*$ and $\mathcal{K}[1,-1] = \mathcal{K}$. Furthermore, Silverman and Silvia [17] proved that a function $f(z)$ is in $\mathcal{S}^*[A,B]$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (z \in \mathcal{U}\ ;\ B \neq -1) \tag{1.2}$$

and

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-A}{2} \quad (z \in \mathcal{U}\ ;\ B = -1). \tag{1.3}$$

A classical result of Fekete and Szegö [4] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter μ, for functions belonging to $\mathcal{S}$. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [2,6–8,14]). More recently, Srivastava, Mishra and Das [19] have given the solution of the Fekete-Szegö problem for a certain subclass of close-to-convex functions.

Denote by $\mathcal{CS}(\beta)$ the class of strongly close-to-star functions of order β $(\beta \geqslant 0)$. Thus $f(z) \in \mathcal{CS}(\beta)$ if and only if there exists $g(z) \in \mathcal{S}^*$ such that for $z \in \mathcal{U}$,

$$\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| \leqslant \frac{\pi}{2}\beta.$$

For the case $\beta = 1$, $\mathcal{CS}(\beta)$ is the class of close-to-star functions introduced by Reade [16]. The close-to-star and similar other functions have been extensively studied by Ahuja and Mogra [1], Padmanabhan and Parvatham [12], Paravatham and Srinivasan [13], Sudharsan et. al. [20], and others.

In the present paper, we prove sharp Fekete-Szegö inequalities for functions belonging to the class $\mathcal{CS}(\beta)$. Argument properties also are investigated, which give conditions for close-to-star functions. Furthermore, we consider the integral preserving properties for functions in the class $\mathcal{CS}(\beta)$.

2. Main Results

To prove our main results, we need the following lemmas.

LEMMA 2.1 [3,15]. *Let $p(z)$ be analytic in $\mathcal{U}$ and satisfy* Re $\{p(z)\} > 0$ *for $z \in \mathcal{U}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then*

$$|p_n| \leqslant 2 \ \ (n \geqslant 1)$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \leqslant 2 - \frac{|p_1|^2}{2}.$$

LEMMA 2.2 [11]. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\Big|\arg\{p(z)\}\Big| < \frac{\pi}{2}\eta \quad \textit{for } |z| < |z_0| \tag{2.1}$$

and

$$\Big|\arg\{p(z_0)\}\Big| = \frac{\pi}{2}\eta \ \ (0 < \eta \leqslant 1). \tag{2.2}$$

Then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta, \tag{2.3}$$

where

$$k \geqslant \frac{1}{2}\Big(a + \frac{1}{a}\Big) \ \textit{when} \ \arg\{p(z_0)\} = \frac{\pi}{2}\eta, \tag{2.4}$$

$$k \leqslant -\frac{1}{2}\Big(a + \frac{1}{a}\Big) \ \textit{when} \ \arg\{p(z_0)\} = -\frac{\pi}{2}\eta, \tag{2.5}$$

and

$$\{p(z_0)\}^{\frac{1}{\eta}} = \pm ia \ \ (a > 0). \tag{2.6}$$

LEMMA 2.3 [9]. *Let $h(z)$ be convex(univalent) function in $\mathcal{U}$ and $\omega(z)$ be an analytic function in $\mathcal{U}$ with* Re $\{\omega(z)\} \geqslant 0$. *If $p(z)$ is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$

With the help of Lemma 2.1, we now derive

THEOREM 2.1. *Let $f(z) \in \mathcal{CS}(\beta)$ and be given by (1.1). Then for $\beta \geqslant 0$, we have*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} 1 + 2(1+\beta)^2(1-2\mu) & \textit{if } \mu \leqslant \frac{\beta}{2(1+\beta)}, \\ 1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)} & \textit{if } \frac{\beta}{2(1+\beta)} \leqslant \mu \leqslant \frac{1}{2}, \\ 1 + 2\beta & \textit{if } \frac{1}{2} \leqslant \mu \leqslant \frac{2+\beta}{2(1+\beta)}, \\ -1 + 2(1+\beta)^2(2\mu - 1) & \textit{if } \mu \geqslant \frac{2+\beta}{2(1+\beta)}. \end{cases}$$

For each μ, there is a function in $\mathcal{CS}(\beta)$ such that equality holds in all cases.

Proof. For $f(z) \in \mathcal{CS}(\beta)$, we may write

$$f(z) = g(z)p^{\beta}(z),$$

where $g(z)$ is starlike and $p(z)$ has positive real part. Let $g(z) = z + b_2z^2 + b_3z^3 + \cdots$, and let $p(z)$ be given as in Lemma 2.1. Then by equating coefficients, we obtain

$$a_2 = b_2 + \beta p_1 \quad \text{and} \quad a_3 = b_3 + \beta p_1 b_2 + \frac{\beta(\beta-1)}{2}p_1^2 + \beta p_2,$$

So, with $x = 1 - 2\mu$, we have

$$(a_3 - \mu a_2^2) = b_3 + \frac{1}{2}(x-1)b_2^2 + \beta\left(p_2 + \frac{1}{2}(\beta x - 1)p_1^2\right) + \beta x p_1 b_2. \tag{2.7}$$

Since rotations of $f(z)$ also belong to $\mathcal{CS}(\beta)$, we may assume, without loss of generality, that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\mathrm{Re}\,(a_3 - \mu a_2^2)$.

For some functions $h(z) = 1 + k_1 z + k_2 z^2 + \cdots$ $(z \in \mathcal{U})$ with positive real part, we have $zg'(z) = g(z)h(z)$. Hence, by equating coefficients, $b_2 = k_1$ and $b_3 = (k_2 + k_1^2)/2$. So, by using Lemma 2.1 and letting $k_1 = 2\rho e^{i\phi}$ $(0 \leqslant \rho \leqslant 1,\ 0 \leqslant \phi \leqslant 2\pi)$ and $p_1 = 2re^{i\theta}$ $(0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi)$ in (2.7), we obtain

$$\begin{aligned}\mathrm{Re}\,(a_3 - \mu a_2^2) \leqslant & 1 - \rho^2 + (1+2x)\rho^2\cos 2\phi + 2\beta((1-r^2) \\ & + \beta x r^2 \cos 2\theta + 2xr\rho\cos(\theta+\phi)),\end{aligned} \tag{2.8}$$

and we now proceed to maximize the right-hand side of (2.8). This function will be denoted by $\psi(x)$ whenever all parameters except x are held constant.

Assume that $\beta/(2(1+\beta)) \leqslant \mu \leqslant 1/2$, so that $0 \leqslant x \leqslant 1/(1+\beta)$. Since the expression $-t^2 + t^2\beta x\cos 2\theta + 2xt$ is the largest when $t = x/(1-\beta x\cos 2\theta)$, we have

$$\psi(x) \leqslant 1 + 2x + 2\beta\left(1 + \frac{x^2}{1-\beta x}\right) = 1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)}$$

and with (2.8) this estiablishes the second inequality in the theorem. Equality occurs only if

$$p_1 = \frac{2(1-2\mu)}{1-\beta(1-2\mu)},\ p_2 = b_2 = 2,\ b_3 = 3,$$

and the corresponding function $f(z)$ is defined by

$$f(z) = \frac{z}{(1-z)^2}\left(\lambda\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}\right)^{\beta},$$

where

$$\lambda = \frac{1 + (1-2\beta)(1-2\mu)}{2(1-\beta(1-2\mu)).}$$

We now prove the first inequlity. Let $\mu \leqslant \beta/(2(1+\beta))$, so that $x \geqslant 1/(1+\beta)$. With $x_0 = 1/(1+\beta)$, we see that

$$\psi(x) \leqslant \psi(x_0) + 2(x - x_0)(1+\beta)^2 \leqslant 1 + 2(1+\beta)^2(1-2\mu).$$

Equality occurs only if $p_1 = p_2 = b_2 = 2,\ b_3 = 3$, and the corresponding function $f(z)$ is defined by

$$f(z) = \frac{z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\beta}.$$

Let $x_1 = -1/(1+\beta)$. We note that $\psi(x_1) \leqslant 1 + 2\beta$. Then $\psi(x)$ satiesfies

$$\psi(x) \leqslant \psi(x_1) + 2|x - x_1|(1+\beta)^2 \leqslant -1 + 2(1+\beta)^2(2\mu - 1),$$

for $x \leqslant x_1$, i.e., $\mu \geqslant (2+\beta)/(2(1+\beta))$. Equality occurs only if $p_1 = 2i,\ p_2 = -2,\ b_2 = 2i,\ b_3 = -3$, and the corresponding function $f(z)$ is defined by

$$f(z) = \frac{z}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^{\beta}.$$

Finally, since

$$\psi(\lambda x_1) = \lambda\psi(x_1) + (1-\lambda)\psi(0) \leqslant 1 + 2\beta,$$

for $0 \leqslant \lambda \leqslant 1$, we obtain $\psi(x) \leqslant 1+2\beta$ for $x_1 \leqslant x \leqslant 0$, i.e., $1/2 \leqslant \mu \leqslant (2+\beta)/2(1+\beta)$. Equality occurs only if $p_1 = b_2 = 0,\ p_2 = 2,\ b_3 = 1$, and the corresponding function $f(z)$ is defined by

$$f(z) = \frac{z(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}.$$

Therefore we complete the proof of Theorem 2.1.

Our next result is contained in

THEOREM 2.2. *Let $f(z) \in \mathcal{A}$. If*

$$\left|\arg\left\{\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f(z)}{g(z)}\right)^{\beta}\right\}\right| < \frac{\pi}{2}\delta \quad (\alpha > 0;\ \beta \in \mathbb{R};\ 0 < \delta \leqslant 1)$$

for some $g(z) \in \mathcal{K}[A,B]$, then

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\pi}{2}\eta,$$

where $\eta\ (0 < \eta \leqslant 1)$ is the solution of the equation:

$$\delta = \begin{cases} (\alpha+\beta)\eta + \frac{2}{\pi}\alpha\tan^{-1}\left(\frac{\eta\sin[\frac{\pi}{2}\{1-t(A,B)\}]}{\frac{1+A}{1+B}+\eta\cos[\frac{\pi}{2}\{1-t(A,B)\}]}\right) & (B \neq -1) \\ (\alpha+\beta)\eta & (B = -1) \end{cases} \tag{2.9}$$

and

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB}\right). \tag{2.10}$$

Proof. Let

$$p(z) = \frac{f(z)}{g(z)} \quad \text{and} \quad q(z) = \frac{zg'(z)}{g(z)}.$$

Then, by a simple calculation, we have

$$\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\left(\frac{f(z)}{g(z)}\right)^{\beta} = (p(z))^{\alpha+\beta}\left(1 + \frac{1}{q(z)}\frac{zp'(z)}{p(z)}\right)^{\alpha}.$$

Since $g(z) \in \mathcal{K}[A,B]$, $g(z) \in \mathcal{S}^*[A,B]$. If we let

$$q(z) = \rho e^{i\frac{\pi}{2}\phi} \quad (z \in \mathcal{U}),$$

then it follows from (1.2) and (1.3) that

$$\begin{cases} \frac{1-A}{1-B} < \rho < \frac{1+A}{1+B} \\ -t(A,B) < \phi < t(A,B) \qquad (B \neq -1) \end{cases}$$

and

$$\begin{cases} \frac{1-A}{2} < \rho < \infty \\ -1 < \phi < 1 \qquad (B = -1), \end{cases}$$

where $t(A,B)$ is defined by (2.10).

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.2) we obtain (2.3) under the restrictions (2.4–6).

At first, we suppose that

$$\{p(z_0)\}^{\frac{1}{\eta}} = ia \quad (a > 0).$$

For the case $B \neq -1$, we then obtain

$$\begin{aligned} &\arg \left\{ \left(\frac{f'(z_0)}{g'(z_0)} \right)^{\alpha} \left(\frac{f(z_0)}{g(z_0)} \right)^{\beta} \right\} \\ &\geqslant (\alpha+\beta)\frac{\pi}{2}\eta + \alpha \tan^{-1} \left(\frac{\eta \sin[\frac{\pi}{2}\{1-t(A,B)\}]}{\frac{1+A}{1+B} + \eta \cos[\frac{\pi}{2}\{1-t(A,B)\}]} \right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ and $t(A,B)$ are given by (2.9) and (2.10), respectively. Similarly, for the case $B = -1$, we have

$$\arg \left\{ \left(\frac{f'(z_0)}{g'(z_0)} \right)^{\alpha} \left(\frac{f(z_0)}{g(z_0)} \right)^{\beta} \right\} \geqslant (\alpha+\beta)\frac{\pi}{2}\eta = \frac{\pi}{2}\delta.$$

These evidently contradict the assumption of the theorem.

Next, in the case $p(z_0)^{\frac{1}{\eta}} = -ia$ $(a > 0)$, applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

By setting $\alpha = 1$, $\beta = 0$, $\delta = 1$, $A = 1$ and $B = -1$ in Theorem 2.2, we have

COROLLARY 2.1. *Every close-to-convex function is close-to-star in* $\mathcal{U}$.

If we put $g(z) = z$ in Theorem 2.2, then, by letting $B \to A$ $(A < 1)$, we obtain

COROLLARY 2.2.. *If* $f(z) \in \mathcal{A}$ *and*

$$\left| \arg \left\{ (f'(z))^{\alpha} \left(\frac{f(z)}{z} \right)^{\beta} \right\} \right| < \frac{\pi}{2}\delta \ \ (\alpha > 0;\ \beta \in \mathbb{R};\ 0 < \delta \leqslant 1),$$

then

$$|\arg \{f'(z)\}| < \frac{\pi}{2}\eta,$$

where η $(0 < \eta \leqslant 1)$ *is the solution of the equation*:

$$\delta = (\alpha + \beta)\eta + \frac{2}{\pi}\alpha \tan^{-1}(\eta).$$

For a function $f(z)$ belonging to the class $\mathcal{A}$, we define the integral operator F_c as follows:

$$F_c(f) := F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt \ \ (c \geqslant 0\,;\ z \in \mathcal{U}). \tag{2.11}$$

For various interesting developments involving the operator (2.11), the reader may be referred (for example) to the recent works of Miller and Mocanu [10] and Srivastava and Owa [18].

Finally, we prove

THEOREM 2.3. *Let* $f(z) \in \mathcal{A}$. *If*

$$\left|\arg\left(\frac{f(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \ \ (0 < \gamma \leqslant 1;\ 0 < \delta \leqslant 1)$$

for some $g(z) \in \mathcal{S}^*[A,B]$, *then*

$$\left|\arg\left(\frac{F_c(f))}{F_c(g)} - \gamma\right)\right| < \frac{\pi}{2}\eta,$$

where F_c *is given by (2.11) and* η $(0 < \eta \leqslant 1)$ *is the solution of the equation*:

$$\delta = \begin{cases} \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{3\sin\frac{\pi}{2}(1-t(A,B,c))}{(\frac{1+A}{1+B}+c)+\eta\cos\frac{\pi}{2}(1-t(A,B,c))}\right) & \text{for } B \neq -1, \\ \eta & \text{for } B = -1, \end{cases}$$

when

$$t(A,B,c) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB+c(1-B^2)}\right).$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{F_c(f)}{F_c(g)} - \gamma\right) \quad \text{and} \quad q(z) = \frac{zF_c'(g)}{F_c(g)}.$$

From the assumption for $g(z)$ and an application of Briot-Bouquet differential equation [10, p. 81], we see that $F_c(g) \in \mathcal{S}^*[A,B]$. Using the equation

$$zF_c'(f)(z) + cF_c(f)(z) \ = \ (1+c)f(z)$$

and simplying, we obtain

$$\frac{1}{1-\gamma}\left(\frac{f(z)}{g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)+c}.$$

Here, we note that $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$ and Re $p(z) > 0$ in $\mathcal{U}$ by applying the assumption and Lemma 2.3 with $\omega(z) = 1/(q(z)+c)$. Hence $p(z) \neq 0$

in $\mathcal{U}$. The remaining part of the proof of Theorem 2.3 is similar to that of Theorem 2.2, and so we omit it.

REMARK. From Theorem 2.3, we see easily that every function in $\mathcal{CS}(\delta)$ $(0 < \delta \leqslant 1)$ preserves the angles under the integral operator defined by (2.11).

By letting $A = 1 - 2\beta$ $(0 \leqslant \beta \leqslant 1)$, $B = -1$, $\delta = 1$ in Theorem 2.3, we obtain

Corollary 2.3. *If $f(z) \in \mathcal{A}$ and*

$$\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > \gamma \quad (0 \leqslant \gamma < 1;\ z \in \mathcal{U}),$$

for some $g(z)$ such that

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \beta \quad (0 \leqslant \beta < 1;\ z \in \mathcal{U}),$$

then

$$\operatorname{Re}\left\{\frac{F_c(f)}{F_c(g)}\right\} > \gamma \quad (0 \leqslant \gamma < 1;\ z \in \mathcal{U}),$$

where F_c is given by (2.11).

If we take $g(z) = z$ in Theorem 2.3, then, by letting $B \to A$ $(A < 1)$, we have

COROLLARY 2.4. *If $f(z) \in \mathcal{A}$ and*

$$\left|\arg\left(\frac{f'(z)}{z} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leqslant \gamma < 1;\ 0 < \delta \leqslant 1),$$

then

$$\left|\arg\left(\frac{F_c(f)}{z} - \gamma\right)\right| < \frac{\pi}{2}\eta,$$

where F_c is given by (2.11) and η $(0 < \eta \leqslant 1)$ is the solution of the equation:

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta}{1+c}\right).$$

Acknowledgements. The authors would like to thank Professor H. M. Srivastava (Department of Mathematics and Statistics, University of Victoria, Canada) for his thoughtful encouragement and numerous helpful advices for this paper.

REFERENCES

[1] O. P. AHUJA AND M. L. MOGRA, *Effect of second coefficients on close-to-convex and close-to-star functions*, Rend. Mat. (7) **4** (1985), 21–37.

[2] H. R. ABDEL-GAWAD AND D. K. THOMAS, *The Fekete-Szegö problem for strongly close-to-convex functions*, Proc. Amer. Math. Soc. **114** (1992), 345–349.

[3] C. CARATHÉODORY, *Über den variabilitätsbereich der fourier'schen konstanten von positiven harmonischen funktionen*, Rend. Circ. Mat. Palermo **32** (1911), 193–217.

[4] M. FEKETE AND G. SZEGÖ, *Eine Bermerkung über ungerade schlichte function*, J. London Math. Soc. **8** (1933), 85–89.
[5] W. JANOWSKI, *Some extremal problems for certain families of analytic functions*, Bull. Acad. Polon. Sci. Sér. Sci. Phys. Astronom. **21** (1973), 17–25.
[6] F. R. KEOGH AND E. P. MERKES, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.
[7] W. KOEPF, *On the Fekete-Szegö problem for close-to-convex functions*, Arch. Math. (Basel) **49** (1987), 420–433.
[8] R. R. LONDON, *Fekete-Szegö inequalities for close-to-convex functions*, Proc. Amer. Math. Soc. **117** (1993), 947–950.
[9] S. S. MILLER AND P. T. MOCANU, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.
[10] S. S. MILLER AND P. T. MOCANU, *Differential Subordinations*, Marcel Dekker Inc. New York and Basel, 1999.
[11] M. NUNOKAWA, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci. **69** (1993), 234–237.
[12] K. S. PADMANABHAN AND R. PARVATHAM, *On certain generalized close-to-star functions in the unit disc*, Ann. Polon. Math. **37** (1980), 1–11.
[13] R. PARVATHAM AND S. SRINIVASAN, *On Pascu type α-close-to-star functions*, Publ. Inst. Math. (Beograd)(N. S.) **49(63)** (1991), 71–75.
[14] A. PFLUGER, *On the functional $|a_3 - \lambda a_2^2|$ in the class S*, Complex Variables Theory Appl. **10** (1988), 83–95.
[15] CH. POMMERENKE, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.
[16] M. O. READE, *On close-to-convex univalent functions*, Michigan Math. J. **3** (1955–56), 59–62.
[17] H. SILVERMAN AND E. M. SILVIA, *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math. **37** (1985), 48–61.
[18] H. M. SRIVASTAVA AND S. OWA (EDITORS), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.
[19] H. M. SRIVASTAVA, A. K. MISHRA AND M. K. DAS, *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Variables Theory Appl. **44** (2001), 145–163.
[20] T. V. SUDHARSAN, P. BALASUBRAHMANYAM AND K. G. SUBRAMANIAN, *On a subclass of close-to-star functions*, J. Math. Phys. Sci. **25** (1991), 343–350.

(Received February 19, 2001)

Nak Eun Cho
Division of Mathematical Sciences
Pukyong National University
Pusan 608-737, Korea
e-mail: necho@pknu.ac.kr

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
e-mail: owa@math.kindai.ac.jp