Mathematical
Inequalities
& Applications
Volume 6, Number 3 (2003), 397–411

A CLASS OF EXPONENTIAL INEQUALITIES

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(*communicated by J. Pečarić*)

Abstract. We prove that for reals x_i with $\sum x_i \geqslant 0$, the estimate $\sum x_i\, e^{x_i} \geqslant \frac{C_N}{N} \sum x_i^2$ holds, where $C_N = \max\{2, e\,(1 - 1/N)\}$. We also prove analogues for the 1-norm and for Lebesgue-integrable functions.

1. Introduction

Lemma 5.3 in [2] says: *Suppose that for reals x_i we have $\sum_{i=1}^{N} x_i = 0$. Then*

$$\sum_{i=1}^{N} x_i\, e^{x_i} \geqslant \frac{1}{2N} \sum_{i=1}^{N} x_i^2 .$$

This is a technical lemma in a paper with an otherwise different and much broader focus. This estimate has, however, caught our attention, since it can be read as an estimate of $\mathbf{x} \cdot \exp(\mathbf{x})$ against $\|\mathbf{x}\|_2^2$ on a hyperplane. The restriction $\sum x_i = 0$ is needed, of course: without it, the left-hand side of the inequality can be negative. Even with the restriction, it is not immediately clear that $\sum x_i\, e^{x_i}$ must even be nonnegative! (In fact, its nonnegativity follows from Chebysheff's inequality, see [3], Chapter IX).

This inequality is the starting point for our paper. Some questions immediately arise: Is the constant $1/2N$ best possible? Can we obtain similar results by estimating against $\|\mathbf{x}\|_p^2$? Are there integral analogues? What do we get when the exp function is replaced by other functions; what properties of these functions lead to interesting estimates?

In Section 2 we prove the following theorem, which is an improvement of the Kostant-Michor inequality: *Suppose that for reals x_i we have $\sum_{i=1}^{N} x_i \geqslant 0$. Then*

$$\sum_{i=1}^{N} x_i\, e^{x_i} \geqslant \frac{C_N}{N} \sum_{i=1}^{N} x_i^2 .$$

Mathematics subject classification (2000): 26D15.
Key words and phrases: Inequalities, exponential function, p-norm.
Research supported by NSERC and by the Canada Research Chair Programme.

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Paper MIA-06-37

where $C_N = \max\{2, e\left(1 - \frac{1}{N}\right)\}$.

In Section 3 of this paper, we discuss the 1-norm analogue of the Kostant-Michor estimate. In fact, consider the inequality with $\|\mathbf{x}\|_1^2$ on the right-hand side: $\sum x_i e^{r_i} \geqslant \frac{C_N}{N}\left(\sum |x_i|\right)^2$ if $\sum x_i \geqslant 0$. We shall prove that this is true for $C_N = e/4$. But more importantly, dividing the inequality by an additional factor of N, we expose Riemann sums for the integral inequality $\int_0^1 x(t)\, e^{x(t)}\, dt \geqslant C\left(\int_0^1 |x(t)|\, dt\right)^2$ if $\int_0^1 x(t)\, dt \geqslant 0$. Again, we shall prove that this is true with $C = e/4$. We shall also consider a more general case, replacing the function exp by a function ϕ with certain properties. We shall prove that then the best possible statement is: the inequality is true for $C = \phi'(0)/4$. While there are functions ϕ for which this is the best possible constant, for $\phi = \exp$ it certainly is not!

Finally, in Section 4 we conclude the paper with some remarks and extensions, regarding p-norms and possible integral analogues for the 2-norm case.

We shall use the following notation: We write $\mathbf{x}$ for the vector $\mathbf{x} = (x_1, \ldots, x_N)$, and we shall always abbreviate the positive and negative parts by $\mathbf{y} = \mathbf{x}^+$ and $\mathbf{z} = \mathbf{x}^-$; and similarly for the components of these vectors or for functions $y(t) = x^+(t)$ and $z(t) = x^-(t)$ (a.e.).

2. The 2-norm case

Our tool in dealing with these inequalities will be the Karush-Kuhn-Tucker theorem. The first step is to show that the inequality is coercive on the specified domain. This is the point of the first two lemmas. Since there is no additional difficulty involved, we formulate them for the general p-norm case, but we use them mainly for $p = 2$ and $p = 1$.

LEMMA 1. *Assume that* $\mathbf{x} \in \mathbb{R}^N$ *is such that* $\sum x_i \geqslant 0$. *Assume further that* $\mathbf{x}$ *has exactly* M *strictly positive components. Then for* $1 \leqslant p < \infty$,

$$\|\mathbf{x}\|_p \leqslant \left(1 + M^{p-1}\right)^{1/p} \|\mathbf{x}^+\|_p.$$

Proof. We use the estimate $\|\mathbf{x}\|_p \leqslant \|\mathbf{x}\|_1 \leqslant M^{(p-1)/p}\|\mathbf{x}\|_p$ for every $\mathbf{x} \in \mathbb{R}^N$ which has exactly M non-zero components. Also note that $\sum x_i \geqslant 0$ is equivalent to $\sum y_i \geqslant \sum z_i$. Now,

$$\sum_{i=1}^N z_i^p \leqslant \left(\sum_{i=1}^N z_i\right)^p \leqslant \left(\sum_{i=1}^N y_i\right)^p \leqslant M^{p-1} \sum_{i=1}^N y_i^p,$$

which implies

$$\left(\sum_{i=1}^N |x_i|^p\right)^{1/p} = \left(\sum_{i=1}^N y_i^p + \sum_{i=1}^N z_i^p\right)^{1/p} \leqslant (1 + M^{p-1})^{1/p} \left(\sum_{i=1}^N y_i^p\right)^{1/p}. \qquad \square$$

LEMMA 2. *Let* $F(\mathbf{x}) := \sum_{i=1}^{N} x_i \exp(x_i) - C\left(\sum_{i=1}^{N}(x_i^+)^p\right)^{2/p}$ *with* $1 \leqslant p < \infty$. *Then* $F(\mathbf{x}) \to \infty$ *when* $\|\mathbf{x}\|_1 \to \infty$, *uniformly for all* $\mathbf{x}$ *such that* $\sum_{i=1}^{N} x_i \geqslant 0$.

Proof. We have, using convexity of $\psi(x) := x\exp(x)$ on $\mathbb{R}_+$ and $\psi(-z) > 1/2$ for $z \in \mathbb{R}_+$ and $\|\mathbf{y}\|_p \leqslant \|\mathbf{y}\|_1$ for $1 \leqslant p < \infty$,

$$\begin{aligned} F(\mathbf{x}) &= \sum_{i=1}^{N} y_i \exp(y_i) - \sum_{i=1}^{N} z_i \exp(-z_i) - C\left(\sum_{i=1}^{N} y_i^p\right)^{2/p} \\ &\geqslant N\,\psi\left(\frac{1}{N}\sum_{i=1}^{N} y_i\right) - N/2 - C\|\mathbf{y}\|_1^2 \\ &= \|\mathbf{y}\|_1 \cdot \left(\exp\left(\frac{1}{N}\|\mathbf{y}\|_1\right) - C\|\mathbf{y}\|_1\right) - N/2, \end{aligned}$$

which tends to infinity when $\|\mathbf{y}\|_1$ tends to infinity. Because of Lemma 1, this happens when $\|\mathbf{x}\|_1 \to \infty$. $\square$

In the next step we look at functions which take at most three different values, one of them zero (or non-existent), one positive and one negative (or non-existent).

LEMMA 3. *a) For all* $x, t > 0$ *and* $0 \leqslant r \leqslant 1$ *we have the estimate*

$$e^x - re^{-tx} \geqslant ex\frac{t}{t+r}.$$

b) For all $s_1 > 0$, $s_2 \geqslant 0$ *and* $x_1 > 0$, $x_2 < 0$ *with* $s_1x_1 + s_2x_2 \geqslant 0$ *and* $s_1 + s_2 \leqslant 1$ *the inequality*

$$s_1x_1e^{x_1} + s_2x_2e^{x_2} \geqslant e(s_1x_1)^2$$

holds.

Proof. a) We distinguish two cases:

(1) Assume that $x \leqslant \ln(t+r)/(t+1)$. Then note the following two facts:

(i) The function $y \mapsto (1 - r\,e^{-y})/y$ for $y > 0$ is decreasing for every $0 \leqslant r \leqslant 1$.

(ii) The inequality $(t+1)/\ln(t+r) \geqslant \exp(1)$ holds for all $t > 0$, $0 \leqslant r \leqslant 1$, *provided that* $\ln(t+r) > 0$. This is satisfied here since $0 < x(t+1) \leqslant \ln(t+r)$.

Now write the inequality to be proved as

$$\exp(x-1)\frac{1 - r\exp(-x(t+1))}{x(t+1)}\,(t+1) \geqslant 1 - r\frac{1}{t+r}.$$

The first observation together with the assumption gives us

$$\frac{1 - r\exp(-x(t+1))}{x(t+1)} \geqslant \frac{1 - r\exp(-\ln(t+r))}{\ln(t+r)}.$$

Thus the left-hand side of the assertion is bigger than

$$\exp(x-1)\frac{1-r/(t+r)}{\ln(t+r)}(t+1) \;\geqslant\; \exp(-1)\left(1-r\frac{1}{t+r}\right)\exp(1) = 1-r\frac{1}{t+r},$$

using $x \geqslant 0$ and the second observation.

(2) Assume that $x \geqslant \ln(t+r)/(t+1)$. Then write the inequality to be proved as

$$\exp(x-1)(1-r\exp(-x(t+1))) \geqslant x\left(1-r\frac{1}{t+r}\right).$$

Since $\exp(x-1) \geqslant x$ for all $x \geqslant 0$, we have to show that $1-r\exp(-x(t+1)) \geqslant 1-r/(t+r)$. This is equivalent to $e^{-x(t+1)} \leqslant 1/(t+r)$, which is in turn equivalent to $-x(t+1) \leqslant -\ln(t+r)$, as was assumed.

b) This inequality is in fact equivalent to part a). To see this, set $x_2 =: -tx_1$ with $t > 0$ and $r := s_2 t/s_1 \geqslant 0$. Then the inequality to be shown becomes

$$s_1 x_1 e^{x_1} - s_1 r x_1 e^{-tx_1} \geqslant e(s_1 x_1)^2,$$

which is equivalent to

$$e^{x_1} - re^{-tx_1} \geqslant e s_1 x_1.$$

The boundary conditions become $s_1 x_1 - s_1 r x_1 \geqslant 0$, which is equivalent to $r \leqslant 1$, and $s_1 + s_1 r/t \leqslant 1$, which is equivalent to $s_1 \leqslant t/(t+r)$. Thus we are finished if we show that

$$e^{x_1} - re^{-tx_1} \geqslant e\frac{t}{t+r}x_1$$

for all $0 \leqslant r \leqslant 1$, and that is precisely the result of a). $\square$

THEOREM 1. *For $N \geqslant 1$, let real x_i be given such that $\sum_{i=1}^{N} x_i \geqslant 0$. Let $C_N :=$* $\max\{2, e\left(1-\frac{1}{N}\right)\}$. *Then*

$$F(\mathbf{x}) := \sum_{i=1}^{N} x_i e^{x_i} - \frac{C_N}{N}\sum_{i=1}^{N} x_i^2 \geqslant 0.$$

REMARK. $C_N = 2$ for $N = 2,3,4$ and $C_N = e\left(1-\frac{1}{N}\right)$ for $N \geqslant 5$.

Proof. Without loss we assume $N > 1$. Putting Lemmas 1 and 2 together, we see that for any $c \in \mathbb{R}$, the set $\{\mathbf{x} : F(\mathbf{x}) \leqslant c,\ \sum x_i \geqslant 0\}$ is compact. Hence F is minimized at some point $\overline{\mathbf{x}}$.

We can apply the Karush-Kuhn-Tucker theorem (see [1], Theorem 7.2.9) to the problem

$$\text{“Minimize} \quad F(\mathbf{x}) \quad \text{subject to} \quad -\sum x_i \leqslant 0.\text{”}$$

Since F is differentiable, we get the following necessary conditions for $\overline{\mathbf{x}}$: There exists a $\lambda \geqslant 0$ such that

$$(\overline{x}_i + 1)\, e^{\overline{x}_i} - 2\frac{\alpha(N)}{N}\overline{x}_i = \lambda \quad \text{for all} \quad i, \tag{1}$$

where moreover $\lambda = 0$ if $\sum \overline{x}_i > 0$. This condition is satisfied when $\overline{\mathbf{x}} = \mathbf{0}$; then $F(\mathbf{x}) = 0$.

From now on we assume that $\overline{\mathbf{x}} \neq \mathbf{0}$. This implies that $\overline{\mathbf{x}}$ has at least one strictly positive component.

We consider the function $g(x) := (x+1)e^x - 2\frac{\alpha(N)}{N}x$. It has a unique critical point $0 \geqslant \hat{x} > -2$ and is strictly decreasing to the left of $\hat{x}$ and strictly increasing to the right of $\hat{x}$. (Actually $\hat{x} = \mathrm{W}\,(2\alpha(N)e^2/N) - 2$ where W is the *Lambert W* function. We have $0 = \hat{x}$ if $2\alpha(N)/N = 2$ which happens exactly when $N = 2$; otherwise we have $2\alpha(N)/N < 2$ and $\hat{x} < 0$. See Figure 1 for the two extreme cases $(x+1)e^x$ and $(x+1)e^x - 2x$.)

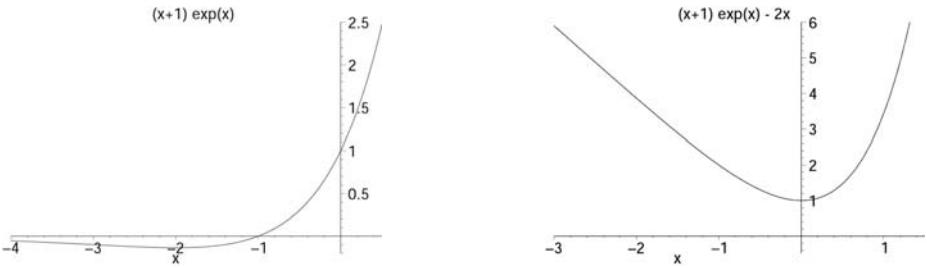

Figure 1. The functions $(x+1)e^x$ *and* $(x+1)e^x - 2x$.

Thus (1) has at most two solutions; since $\hat{x} \leqslant 0$, precisely one of them is positive, the other one (if it exists) must be negative. Therefore the components of a minimizing vector $\overline{\mathbf{x}}$ can achieve only at most two different values. We can in fact assume that at least one negative component exists, because if not, then the inequality becomes $Nae^a - C_N a^2 \geqslant 0$ for $a > 0$, which is true whenever $C_N \leqslant Ne$; certainly it will be true for our C_N.

This observation implies that the strictly positive components are all equal with value a occurring M times, say, and likewise the negative ones have some constant value b occurring $N-M$ times with $1 \leqslant M \leqslant N-1$. Moreover, we have $g(-x) \leqslant g(x)$ for $x \geqslant 0$ (indeed, this inequality is equivalent to $2\frac{\alpha(N)}{N} \leqslant \frac{1}{x}\sinh x + \cosh x$, which is true for $N > 1$). This implies that $|b| \geqslant a$, and so we can write $b = -ta$ with some $t > 1$. Also, the condition $Ma + (N-M)b \geqslant 0$ then implies $(N-M)t \leqslant M$, which in turn implies that $t \leqslant N-1$.

Thus we can write

$$\begin{aligned} F(\overline{\mathbf{x}}) &= Mae^a - (N-M)tae^{-ta} - \frac{C_N}{N}\left(Ma^2 + (N-M)t^2a^2\right) \\ &\geqslant Mae^a - Mae^{-ta} - \frac{C_N}{N}\left(Ma^2 + Mta^2\right), \end{aligned}$$

and the condition $F(\overline{\mathbf{x}}) \geqslant 0$ follows from

$$\frac{e^a - e^{-ta}}{a(1+t)} \geqslant \frac{C_N}{N} \quad \text{for all} \quad a > 0,\ 1 < t \leqslant N-1.$$

This is true for $C_N = 2$ because

$$\frac{e^a - e^{-ta}}{a(1+t)} \geqslant \frac{e^a - e^{-a}}{a(1+t)} = \frac{2\sinh a}{a(1+t)} \geqslant \frac{2}{1+t} \geqslant \frac{2}{N}.$$

It is also true for $C_N = e\left(1 - \frac{1}{N}\right)$, because by Lemma 3 with $r = 1$ we have that

$$\frac{e^a - e^{-ta}}{a(1+t)} \geqslant e\frac{t}{(1+t)^2},$$

which is greater than $\alpha(N)/N$ since $1 \leqslant t \leqslant N-1$ (and $t \mapsto \frac{t}{(1+t)^2}$ is decreasing if $t > 1$). $\square$

REMARK. a) Note that the function F in this theorem can not be interpreted as a Riemann sum. This means that there is no direct integral analogue for the inequality. However, there are slightly different but related integral inequalities of this kind. We address this in Section 4 of our paper.

b) For jointly monotone sequences a_i, b_i, Chebysheff's inequality (see also Section 3) says that

$$\sum_{i=1}^{N} a_i\, b_i \geqslant \frac{1}{N}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{i=1}^{N} b_i\right)$$

holds (with "$>$" instead of "$\geqslant$" if $a_i \not\equiv a_1$ or $b_i \not\equiv b_1$). Applying this with $a_i = x_i$ and $b_i = e^{x_i}$, we get

$$\sum_{i=1}^{N} x_i\, e^{x_i} \geqslant \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} e^{x_i}\right).$$

This together with the convexity of $\exp$, i.e.,

$$\frac{1}{N}\sum_{i=1}^{N} e^{x_i} \geqslant e^{\frac{1}{N}\sum x_i},$$

implies that

$$\begin{aligned} \tfrac{1}{N}\sum_{i=1}^{N} \psi(x_i) \ &\geqslant\ \psi\left(\tfrac{1}{N}\sum_{i=1}^{N} x_i\right) && \text{if } \sum x_i \geqslant 0, \\ &> && \text{if additionally } x_i \not\equiv x_1, \end{aligned} \tag{2}$$

with $\psi(x) = x\,e^x$. Note that ψ itself is not convex; by inequality (2), it is "conditionally convex". In any case, this shows directly that

$$\sum_{i=1}^{N} \psi(x_i) > 0 \quad \text{if} \quad \sum x_i > 0.$$

3. The 1-norm and its integral analogue

Consider

$$\int_I x(t)\, e^{x(t)}\, dt \geqslant \alpha\, \|x\|_p^2,$$

for all integrable functions x with $\int_I x(t)\, dt \geqslant 0$. Then $\alpha = 0$ if $p > 1$. Indeed, consider

$$x_n(t) := \chi_{[1/n,1]} - (n-1)\chi_{[0,1/n]},$$

for $n \in \mathbb{N}$. Then $\int_I x_n(t)\, dt = 0$ and

$$\int_I x_n(t)\, e^{x_n(t)}\, dt = \left(1 - \frac{1}{n}\right)\left(e - e^{1-n}\right) \to e,$$

while

$$\int_I x_n(t)^p\, dt = \left(1 - \frac{1}{n}\right)\left(1 + (n-1)^{p-1}\right) \to \infty, \qquad (p > 1).$$

Hence, the ratio

$$\frac{\int_I x_n(t)\, e^{x_n(t)}\, dt}{\left(\int_I x_n(t)^p\, dt\right)^\beta} \to 0,$$

for any $p > 1$ and any $\beta > 0$, and thus $\alpha > 0$ is impossible. Therefore no comparison to the p-norm can be achieved.

Note, however, that for $p = 1$, we have

$$\left(\int_I |x_n(t)|\, dt\right)^2 = 4\left(1 - \frac{1}{n}\right)^2,$$

and now the ratio is bounded below by $e/4$. Thus, we explore

$$\int_0^1 x(t)\, e^{x(t)}\, dt \geqslant \alpha\, \|x\|_1^2 \quad \text{if} \quad \int_0^1 x(t)\, dt \geqslant 0$$

and its discrete analogue. We start, however, with the more general estimate

$$\int_0^1 x(t)\, \phi(x(t))\, dt \geqslant \alpha\, \|x\|_1^2 \quad \text{if} \quad \int_0^1 x(t)\, dt \geqslant 0,$$

where ϕ is a function with properties to be specified below. As it transpires, the estimate will then be true with $\alpha = \phi'(0)/4$, and this constant can not be improved within the given class of functions ϕ. On the other hand, we have seen above that in the case of $\phi = \exp$, α is not larger than $e/4$ (and we shall prove below that this is the right value), so that in this case there is a gap between the general and the specific constants, and this gap is closed by employing more specific methods.

In the discussion of the general case, we need the following form of Chebysheff's inequality (see [3], Chapter IX). It says: *If the real functions* $a(t)$, $b(t)$, $a(t)b(t)$ *are Lebesgue integrable on* $[0,1]$ *and*

$$\big(a(t) - a(u)\big)\big(b(t) - b(u)\big) \geqslant 0 \quad \textit{for almost all} \quad u, t \in [0,1],$$

then

$$\int_0^1 a(t)b(t)\,dt \geqslant \left(\int_0^1 a(t)\,dt\right)\left(\int_0^1 b(t)\,dt\right).$$

Thus, the inequality is satisfied when a, b are of the form $a(t) = f_1(x(t))$, $b(t) = f_2(x(t))$, and f_1, f_2 are both increasing or both decreasing.

Note that if x is an integrable function, then the condition $\int_0^1 x(t)\,dt \geqslant 0$ is equivalent to $\int_0^1 x^+(t)\,dt \geqslant \int_0^1 x^-(t)\,dt$, which implies

$$\int_0^1 |x(t)|\,dt = \int_0^1 x^+(t)\,dt + \int_0^1 x^-(t)\,dt \leqslant 2\int_0^1 x^+(t)\,dt.$$

Therefore the inequality

$$\int_0^1 x(t)\,\phi(x(t))\,dt \geqslant \alpha\left(\int_0^1 |x(t)|\,dt\right)^2 \quad \text{if} \quad \int_0^1 x(t)\,dt \geqslant 0$$

follows from the sharper inequality

$$\int_0^1 x(t)\,\phi(x(t))\,dt \geqslant 4\alpha\left(\int_0^1 x^+(t)\,dt\right)^2 \quad \text{if} \quad \int_0^1 x(t)\,dt \geqslant 0,$$

which is what we will prove.

THEOREM 2. *If x is an integrable function on $[0,1]$ such that $\int_0^1 x(t)\,dt \geqslant 0$, and ϕ is a non-negative non-decreasing function which is continuous on $[-B, A]$ and differentiable on $(0, A)$, where $A \geqslant \max_{0\leqslant t\leqslant 1} x^+(t)$, $B \geqslant \max_{0\leqslant t\leqslant 1} x^-(t)$, then*

$$\int_0^1 x(t)\phi\big(x(t)\big)\,dt \geqslant c\left(\int_0^1 x^+(t)\,dt\right)^2, \tag{3}$$

where $c = \min_{0<\xi<A} \phi'(\xi)$. If, in addition, $\phi'(0)$ exists (relative to the interval $(0, A)$) and $\phi'(\xi) \geqslant \phi'(0)$ for $0 < \xi < A$, then

$$\int_0^1 x(t)\phi\big(x(t)\big)\,dt \geqslant \phi'(0)\left(\int_0^1 x^+(t)\,dt\right)^2. \tag{4}$$

Note. In the above, max and min denote the essential maximum and essential minimum repsectively

Proof. Without loss in generality, we assume that, for every $t \in [0,1]$, $x(t)$ is defined and finite and lies in $(-B, A)$. Set $y := x^+, z := x^-$ and observe that

$$\int_0^1 x(t)\,\phi(x(t))\,dt = \int_0^1 y(t)\,\phi(y(t))\,dt - \int_0^1 z(t)\,\phi(-z(t))\,dt. \tag{5}$$

Note that, for $a(t) := y(t)$, $b(t) := \phi(y(t))$, we have, using the mean value theorem,

$$\big(a(t) - a(u)\big)\big(b(t) - b(u)\big) = \big(y(t) - y(u)\big)^2 \phi'(\xi) \geqslant 0 \text{ with } u, v \in [0,1],\ \xi \in (0, A).$$

Hence, using Chebysheff's inequality in the first term in (5) and $\phi(-z(t)) \leqslant \phi(0)$ in the second, we have

$$\int_0^1 x(t)\,\phi(x(t))\,dt \geqslant \left(\int_0^1 y(t)\,dt\right)\left(\int_0^1 \phi(y(t))\,dt\right) - \phi(0)\int_0^1 z(t)\,dt.$$

But $\int_0^1 y(t)\,dt \geqslant \int_0^1 z(t)\,dt$ and so, using the mean value theorem,

$$\int_0^1 x(t)\,\phi(x(t))\,dt \geqslant \left(\int_0^1 y(t)\,dt\right)\left(\int_0^1 \big(\phi(y(t)) - \phi(0)\big)\,dt\right) \geqslant c\left(\int_0^1 y(t)\,dt\right)^2 dt.$$

This proves the first part of the theorem; the second part follows immediately. □

Are the inequalities in Theorem 2 sharp? We give some examples.

EXAMPLES. a) If $\phi(x) := x^+$, then $\phi'(0) = 1$ (relative to $(0,\infty)$), and $\phi'(\xi) = \phi'(0)$ for all $\xi \in (0,\infty)$. Therefore formula (4) says:

$$\int_0^1 x(t)\,x^+(t)\,dt \geqslant \left(\int_0^1 x^+(t)\,dt\right)^2,$$

and this is sharp even on $\int x(t)\,dt = 0$, since for $x(t) := \left\{\begin{array}{ll} 1 & \text{if } 0 \leqslant t \leqslant a \\ -a/(1-a) & \text{if } a < t \leqslant 1 \end{array}\right\}$, the left-hand integral evaluates to a and the right-hand side of the inequality evaluates to a^2, so that both sides are asymptotically equal when a approaches 1.

b) If $\phi(x) := \exp(x^3)$, then ϕ is convex on $\mathbb{R}_+$, so that again the assumptions of Formula (4) are satisfied with $\phi'(0) = 0$. Again, this is sharp even on $\int x(t)\,dt = 0$, since we can choose $x(t) := \left\{\begin{array}{ll} b & \text{if } 0 \leqslant t \leqslant 1/2 \\ -b & \text{if } 1/2 < t \leqslant 1 \end{array}\right\}$. Then the left-hand integral evaluates to $b\sinh(b^3)$, the right-hand side evaluates to $(b/2)^2$, and the quotient of both tends to 0 if b tends to 0.

c) If $\phi(x) := \arctan(x)$, then ϕ is concave on $\mathbb{R}_+$, and moreover the constant c in (3) is 0. Again, the estimate is sharp on $\int x(t)\,dt = 0$ as the same $x(t)$ as in the previous example shows, now letting b tend to ∞. (This works for every function ϕ for which $\phi(x)/x$ tends to 0 as x tends to $\pm\infty$.)

d) The example in which we are most interested is of course $\phi(x) := \exp(x)$. This function is convex, so Formula (4) obtains with $\phi'(0) = 1$. On the other hand, from the example before Theorem 2 we know that the estimate

$$\int_0^1 x(t)\,e^{x(t)}\,dt \geqslant \alpha\left(\int_0^1 x^+(t)\,dt\right)^2$$

will be true for no constant α larger than e. Here is a gap: is $\alpha = 1$ the right constant, or can α be as large as e? In fact, we now prove that the estimate remains true for $\alpha = e$; thus, for the function $\phi = \exp$ Theorem 2 is not sharp. (It might be interesting to determine properties of a function ϕ cause Theorem 2 to be sharp.)

LEMMA 4. *Let* $F(x_1, \ldots, x_N) := \sum_{i=1}^{N} x_i \exp(x_i) - \frac{e}{N}\left(\sum_{i=1}^{N} x_i^+\right)^2$. *Then* $F(\mathbf{x}) \geqslant 0$ *whenever* $\sum x_i \geqslant 0$.

Proof. Step 1. It follows from Lemma 2 that F is coercive on $\sum x_i \geqslant 0$.

Step 2. We now apply the Karush-Kuhn-Tucker conditions (see [1], Theorem 7.2.9) to the non-smooth problem

$$\text{“Minimize} \quad F(\mathbf{x}) \quad \text{subject to} \quad -\sum x_i \leqslant 0.\text{”}$$

We get the following condition for any minimizing point $\overline{x}$: There exists a $\lambda \geqslant 0$ such that

$$0 \in \partial F(\overline{\mathbf{x}}) - \lambda \cdot \nabla\left(\sum \overline{x}_i\right),$$

where moreover $\lambda = 0$ if $\sum \overline{x}_i > 0$. Computing the generalized gradients, (see [1], Chapter 6), this is equivalent to:

$$\lambda \in (1 + \overline{x}_i)\, e^{\overline{x}_i} - \frac{e}{N} 2\|\overline{\mathbf{x}}^+\|_1 \cdot \partial \overline{x}_i{}^+ \quad \text{for} \quad i = 1, \ldots, N.$$

Since $\partial \overline{x}_i{}^+ = \left\{ \begin{array}{ll} 1 & \text{if } \overline{x}_i > 0 \\ 0 & \text{if } \overline{x}_i < 0 \\ [0,1] & \text{if } \overline{x}_i = 0 \end{array} \right\}$, we get the following *necessary* conditions for any minimizing point $\overline{\mathbf{x}}$, where we set $c := 2e\|\overline{\mathbf{x}}^+\|_1/N$:

$$\begin{array}{rcll} (\overline{x}_i + 1)\, e^{\overline{x}_i} & = & \lambda + c & \text{if } \overline{x}_i > 0, \\ (\overline{x}_i + 1)\, e^{\overline{x}_i} & = & \lambda & \text{if } \overline{x}_i < 0, \\ \lambda & \in & [1 - c, 1] & \text{if } \overline{x}_i = 0. \end{array}$$

The vector $\overline{\mathbf{x}} = \mathbf{0}$ satisfies these conditions; then $F(\overline{\mathbf{x}}) = 0$.

We may now assume that $\overline{\mathbf{x}} \neq \mathbf{0}$. Since $\lambda \geqslant 0$, this implies that the vector $\overline{\mathbf{x}}$ has (at most) three distinct values: one positive value $x_1 > 0$, one negative value $x_2 \in [-1, 0)$ (since only for those arguments does the function $(x+1)\, e^x$ have values $\lambda \geqslant 0$), and possibly $x_3 = 0$.

Step 3. It now suffices to prove the theorem for those vectors $\overline{\mathbf{x}}$ among which the minima of F must be in accord with the result of Step 2. Therefore, assume that the vector $\overline{\mathbf{x}}$ consists of M_1 entries $x_1 > 0$, M_2 entries $x_2 < 0$ and $N - M_1 - M_2$ entries $x_3 = 0$ with $M_1, M_2 \geqslant 0$ and $M_1 + M_2 \leqslant N$; also, the condition $\sum \overline{x}_i \geqslant 0$ translates to $M_1 x_1 + M_2 x_2 \geqslant 0$. Then

$$\frac{1}{N} F(\overline{\mathbf{x}}) = \frac{M_1}{N}\, x_1\, e^{x_1} + \frac{M_2}{N}\, x_2\, e^{x_2} - e\left(\frac{M_1}{N}\, x_1\right)^2 \geqslant 0$$

according to Lemma 3. $\square$

REMARK. If we fix the dimension N, then the constant e in Lemma 4 can be replaced by some bigger constant; for example, if $N = 2$, then 4 instead of e works. To find these constants for general N (and so probably also to improve the constant $\alpha(N)$ in Theorem 1), we would need discrete versions of Lemma 3 (taking into account that s_1 and s_2 are of the form M/N with $1 \leqslant M \leqslant N-1$). However, our estimate is asymptotically not far off: if we choose $x_1 := 1$, $x_2 := -1$, $s_1 := (N-1)/N$, $s_2 := 1/N$, then

$$\frac{s_1 x_1 e^{x_1} + s_2 x_2 e^{x_2}}{(s_1 x_1)^2} = e\left(1 + \frac{1}{N-1}\right)\left(1 - \frac{1}{N-1} e^{-2}\right),$$

and (at least numerically) this value is quite close to the true infimum of the left-hand side. Note that we actually get smaller values with $\sum x_i > 0$ than if we restrict $\mathbf{x}$ by $\sum x_i = 0$.

The final step now is to prove the integral analogue. To make things easier, we note that it suffices to prove the estimate for functions which satisfy $x(t) \geqslant -1$ for all t. Because if not, we define

$$\tilde{x}(t) := \left\{ \begin{array}{ll} x(t) & \text{if } x(t) \geqslant -1 \\ -1 & \text{if } x(t) \leqslant -1 \end{array} \right\}.$$

Then we have $x(t)\, e^{x(t)} \geqslant \tilde{x}(t)\, e^{\tilde{x}(t)}$ and $x(t) \leqslant \tilde{x}(t)$ and $x^+(t) = \tilde{x}^+(t)$ for all t, so that the statement for x follows from that for $\tilde{x}$.

THEOREM 3. *If $x : [0,1] \to \mathbb{R}$ is Lebesgue-integrable and $\int_0^1 x(t)\, dt \geqslant 0$, then*

$$\int_0^1 x(t)\, e^{x(t)}\, dt \geqslant e \left(\int_0^1 x^+(t)\, dt \right)^2.$$

Proof. Step 1. We first prove the estimate for Riemann-integrable functions. The idea is, of course, to approximate the integrals by Riemann sums and then use Lemma 4. But it makes sense to write this down explicitly.

Fix $N \in \mathbb{N}$ and let $t_i := i/N$ for $i = 0, \ldots, N$. Define the vector $v = (v_1, \ldots, v_N)$ by $v_i := \sup x([t_{i-1}, t_i])$. Then, since $x e^x$ is increasing for $x \geqslant -1$, we also have $v_i e^{v_i} = \sup\{x(\tau)\, e^{x(\tau)} : \tau \in [t_{i-1}, t_i]\}$, so that $\frac{1}{N} \sum v_i e^{v_i}$ is a Riemann sum for $\int x(t)\, e^{x(t)}\, dt$ (see [4], Theorem 6.28). Moreover, Lemma 4 is applicable since $\frac{1}{N} \sum v_i \geqslant \int x(t)\, dt \geqslant 0$. We also have $\frac{1}{N} \sum v_i^+ \geqslant \int x^+(t)\, dt$. Thus we get

$$\frac{1}{N} \sum_{i=1}^N v_i e^{v_i} \geqslant e \left(\frac{1}{N} \sum_{i=1}^N v_i^+ \right)^2 \geqslant e \left(\int_0^1 x^+(t)\, dt \right)^2$$

by Lemma 4, and the left-hand side of this inequality tends to $\int_0^1 x(t)\, e^{x(t)}\, dt$ as N goes to infinity.

Step 2. We now prove the estimate for essentially bounded Lebesgue-integrable functions. Thus assume that $|x(t)| \leqslant M$ a.e., say. Take an $\varepsilon > 0$ and set $x_\varepsilon := x + \varepsilon$. Then $\int_0^1 x_\varepsilon(t)\, dt > 0$ and $|x_\varepsilon(t)| \leqslant M + \varepsilon$. Now use Luzin's theorem (see [4],

Theorems 6.76 and 6.77) to find a sequence of continuous functions g_n such that $g_n(t) \to x(t)$ a.e. and $|g_n(t)| \leqslant M+\varepsilon$. Then we also have $g_n(t)\, e^{g_n(t)} \to x_\varepsilon(t)\, e^{x_\varepsilon(t)}$ and $g_n^+(t) \to x_\varepsilon^+(t)$ a.e. Now using Lebesgue's dominated convergence theorem (see [4], Theorem 6.22), we get $\int g_n(t)\,dt \to \int x_\varepsilon(t)\,dt$, $\int g_n(t)\, e^{g_n(t)}\,dt \to \int x_\varepsilon(t)\, e^{x_\varepsilon(t)}\,dt$ and $\int g_n^+(t)\,dt \to \int x_\varepsilon^+(t)\,dt$. Since by part a) the inequality holds for g_n (if n is large enough such that $\int g_n(t)\,dt > 0$), it must also hold for x_ε.

Finally letting ε tend to 0, we obtain $\int x_\varepsilon(t)\, e^{x_\varepsilon(t)}\,dt \to \int x(t)\, e^{x(t)}\,dt$ and $\int x_\varepsilon^+(t)\,dt \to \int x^+(t)\,dt$, and the estimate then follows for x.

Step 3. Finally, we prove the estimate for unbounded Lebesgue-integrable functions. In this case, there are two possibilities. Either $\int x(t)\, e^{x(t)}\,dt = \infty$ (then we have nothing left to prove), or that integral is finite. In this case choose $\varepsilon > 0$. For every $M \in \mathbb{R}_+$, define $x_M(t) := \left\{ \begin{array}{ll} x(t) & \text{if } x(t) \leqslant M \\ M & \text{if } x(t) \geqslant M \end{array} \right\}$. Then choose M such that $0 \leqslant \int x(t)\, e^{x(t)}\,dt - \int x_M(t)\, e^{x_M(t)}\,dt \leqslant \varepsilon$ and $0 \leqslant \int x(t)\,dt - \int x_M(t)\,dt \leqslant \varepsilon$. Further, set $x_\varepsilon := x_M + \varepsilon$. Then $0 \leqslant \int x(t)\,dt \leqslant \int x_\varepsilon(t)\,dt$, and the estimate therefore holds for x_ε. Now, when ε tends to 0, then

$$\int_0^1 x_\varepsilon(t)\, e^{x_\varepsilon(t)}\,dt = e^\varepsilon \int_0^1 x_M(t)\, e^{x_M(t)}\,dt + \varepsilon\, e^\varepsilon \int_0^1 e^{x_M(t)}\,dt \to \int_0^1 x(t)\, e^{x(t)}\,dt$$

(using that $\int e^{x_M(t)}\,dt$ is bounded above by $\int e^{x(t)}\,dt < \infty$) and $\int x_\varepsilon^+(t)\,dt \to \int x^+(t)\,dt$, so that x again inherits the estimate. $\square$

REMARK. What property of the function $\exp(x)$ causes the improvement from the general constant $C_\phi = \phi'(0)$ ($= 1$ for $\phi = \exp$) in Theorem 2 to the specific constant $C_{\exp} = e$ in Theorem 3? We do not know, but we note that if we choose, instead of $\phi = \exp$, the function $\phi(x) := \left\{ \begin{array}{ll} x+1 & \text{if } x \geqslant 0 \\ \exp(x) & \text{if } x < 0 \end{array} \right\}$, then we have $C_\phi = 1$ again, as can be seen by choosing $x(t) := \left\{ \begin{array}{ll} a & \text{if } t \leqslant s \\ -as/(1-s) & \text{if } s < t \leqslant 1 \end{array} \right\}$ and then setting $s = 1 - 1/a$ and letting a tend to infinity. On the other hand, if we choose $\phi(x) := \left\{ \begin{array}{ll} \exp(x) & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{array} \right\}$, then we seem to retain $C_\phi = e$.

4. Remarks and extensions

Inequalities for the p-norm. So far we have only treated estimates for the 1- and the 2-norms. The method in both cases consisted of two principal steps: First, we argued that the function F is minimized when the vector $\mathbf{x}$ takes at most two different values. We did this by employing the Karush-Kuhn-Tucker theorem. Second, we computed the minimum of F for these vectors. We did this in Lemma 3 by elementary estimates.

We believe that such problems provide very nice applications of the KKT theorem. Moreover, we have tried to find a way to do the reduction in the first step by elementary (ad hoc) methods; with no success. Thus, KKT really is the method of choice here.

Thus to prove similar estimates for a p-norm on the right-hand side of the inequality, we first have to apply KKT in the first step. However, in general this does not give us the desired reduction to the two-valued case, since the analysis is then much more difficult. In fact, after applying KKT, we arrived at a certain function g in the 2-norm case. In general, this function has a more complicated structure; it even may have cusps. As a visual example, take $p = 3/2$, so that the function g is of the form $g(x) = (x+1)e^x - a\,|x|^{1/2} \cdot \mathrm{signum}(x)$ for some constant a; see Figure 2.

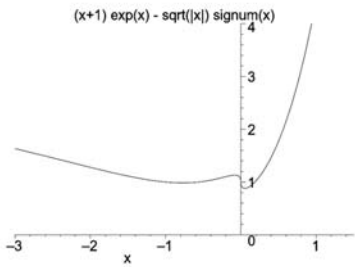

Figure 2. A function $g(x)$ for $p = 3/2$.

Integral analogues. Let us look again at the $p = 2$ case. Theorem 1 says that for all vectors $\mathbf{x} = (x_1, \dots, x_N)$ with $\sum x_i \geqslant 0$ the inequality

$$\sum_{i=1}^{N} x_i\, e^{x_i} \geqslant C_N \frac{1}{N} \sum_{i=1}^{N} x_i^2$$

is true with some constants C_N. How large can those C_N be? All we currently know is that the method still works if we set C_N equal to $\max\{2, e\left(1 - \frac{1}{N}\right)\}$. Numerically, it seems that the C_N can even be slightly bigger than that.

One way to approach this question is to consider right-hand sides which majorize $\frac{1}{N}\sum x_i^2$, but which at the same time have a simpler structure. Then if the new inequality is true with a factor of $\widetilde{C}_N$, then we can infer that $\widetilde{C}_N \leqslant C_N$. We looked at right-hand sides where the sum over all squares is replaced by a sum where only the squares of the positive components of $\mathbf{x}$ are used. (Remember the $p = 1$ case, where we used the positive components in a similar way.) Unfortunately, we have only found inequalities with very bad $\widetilde{C}_N$. However, an advantage of this method is that we arrive at functions which can (as in the $p = 1$ case) be interpreted as Riemann sums, so there will be again be an integral analogue. We think that interesting enough to describe our attempts here.

The first possibility is to observe that if $\sum x_i \geqslant 0$, then $\frac{1}{N}\|\mathbf{x}\|_2^2 \leqslant \|\mathbf{x}^+\|_2^2$. (Indeed, this is trivially true if all components of $\mathbf{x}$ are nonnegative; if at least one component is negative, then we can use Lemma 1 with $M \leqslant N-1$.) Thus if we can prove that

$$\sum_{i=1}^{N} x_i\, e^{x_i} \geqslant C_N^{(1)} \sum_{i=1}^{N} (x_i^+)^2 \quad \text{if} \quad \sum_{i=1}^{N} x_i \geqslant 0,$$

for some constants $C_N^{(1)} > 0$, then for the original estimate we get $C_N \geqslant C_N^{(1)}$. However, this is not a very good bound for C_N: the constant $C_N^{(1)}$ will for large N not be much bigger than 1. To see this, set $x_1 = \ldots = x_{N-1} = -1/(N(N-1))$ and $x_N = 1/N$. Then $\sum x_i = 0$ and

$$\begin{aligned}\frac{\sum x_i \exp(x_i)}{x_N^2} &= N^2\left(\frac{-(N-1)\exp(-1/N(N-1))}{N(N-1)} + \frac{\exp(1/N)}{N}\right)\\ &= N\left(\exp(1/N) - \exp(-1/N(N-1))\right),\end{aligned}$$

which tends to 1 if N goes to infinity. In any event, the new estimate does have an integral analogue; it is

$$\int_0^1 x(t)\, e^{x(t)}\, dt \geqslant C^{(1)} \int_0^1 (x^+(t))^2\, dt \quad \text{if} \quad \int_0^1 x(t)\, dt \geqslant 0.$$

This can be true only with some $C^{(1)} \leqslant 1$.

The second possibility is to use Lemma 1 directly to get that $\frac{1}{N}\|\mathbf{x}\|_2^2 \leqslant \frac{M+1}{N}\|\mathbf{x}^+\|_2^2$ if $\sum x_i \geqslant 0$, where M is the number of positive entries in $\mathbf{x}$. Thus if we could prove that

$$\sum_{i=1}^N x_i\, e^{x_i} \geqslant C_N^{(2)} \frac{M+1}{N} \sum_{i=1}^N (x_i^+)^2 \quad \text{if} \quad \sum_{i=1}^N x_i \geqslant 0,$$

then the original estimate follows with at least $C_N \geqslant C_N^{(2)}$. As before, for large N we have a $C_N^{(2)}$ around 1. However, there is an important difference to all previous cases: Numerical evidence suggests that in the case the corresponding function $F(\mathbf{x})$ is infimized not by a vector $\overline{\mathbf{x}}$ with two different values, but by an $\overline{\mathbf{x}}$ with three different values, one negative and two positive, one of those very close to 0. With two-valued functions, it seems we can only prove $C_N^{(2)} \leqslant e$. In any event, an integral analogue is

$$\int_0^1 x(t)\, e^{x(t)}\, dt \geqslant C^{(2)}\, s \int_0^1 (x^+(t))^2\, dt \quad \text{if} \quad \int_0^1 x(t)\, dt \geqslant 0, \text{ where } s := m(\operatorname{supp} x^+).$$

Again, we can not expect this to be true if $C^{(2)}$ is greater than 1.

Acknowledgements. We are grateful to David Borwein with whom we discussed an earlier version of this paper, and to whom Theorem 2 is due as a substantial improvement of an earlier theorem of ours. This work was done while the second author was visiting Simon Fraser University. He thanks the university for its hospitality and his home institution, the GSF-Forschungszentrum, for granting him the leave.

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(Received December 10, 2001)

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