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ANTI–PERIODIC BOUNDARY VALUE PROBLEM FOR NONLINEAR FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We prove several new existence results for a nonlinear anti-periodic first order problem using a Leray-Schauder alternative. Two definitions of lower and upper solutions are presented and we show in this paper the validity of the lower and upper solution method. Also, we give a method to generate a sequence of approximate solutions converging to a solution of the anti-periodic problem.

1. Introduction

In this paper we study an anti-periodic problem for first order differential equations. Anti-periodic problems have been studied extensively in the last ten years. For example, for first order ordinary differential equations, a Massera's type criterion is presented in [10] and in [12, 21, 22] the validity of the monotone iterative technique is shown. Also for higher order ordinary differential equations existence and uniqueness results based on a Leray-Schauder type argument are presented in [2, 3]. Anti-periodic boundary conditions for partial differential equations and abstract differential equations are considered in [5, 6, 7, 15, 16, 19, 20]. Anti-periodic trigonometric polynomials are important in the study of interpolation problems [11] and anti-periodic wavelets are discussed in [9]. Finally we note that anti-periodic boundary conditions appear in physics in a variety of situations [1, 4, 13, 18].

Sometimes we have a connection between anti-periodic and periodic problems. For example any T-antiperiodic solution gives rise to a $2T$-periodic solution if the nonlinearity f satisfies some symmetry condition.

Let $T > 0$ and $I = [0,T]$. Consider the following nonlinear anti-periodic boundary value problem

$$\begin{aligned} u'(t) &= f(t,u(t)), \quad \text{a.e.} \quad t \in I, \\ u(0) &= -u(T), \end{aligned} \tag{1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function, i.e., f satisfies

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- For every $x \in \mathbb{R}$, $f(\cdot, x)$ is Lebesgue measurable on I.
- For a.e. $t \in I$, $f(t, \cdot)$ is continuous on $\mathbb{R}$.
- For every $R > 0$ there exists $\phi \in L^1(I)$ such that

$$|f(t,x)| \leqslant \phi(t) \text{ for a.e. } t \in I \text{ and all } x \in \mathbb{R} \text{ with } |x| \leqslant R.$$

Throughout this paper, $C(I)$ denotes the space of continuous functions on I and $AC(I)$ the subspace of absolutely continuous functions on I. For $u \in C(I)$ we consider the usual norm

$$\|u\|_0 = \sup_{t \in I} |u(t)|.$$

In the space $C(I)$ we also consider the usual pointwise partial ordering. In such a case we define the interval

$$[v, w] = \{u \in C(I) : v \leqslant u \leqslant w\}.$$

We say that a function $u : I \to \mathbb{R}$ is a solution to (1) if $u \in AC(I)$ and it solves (1).

The paper is organized as follows. In Section 2 we establish two existence results based on a nonlinear alternative of Leray-Schauder type [17]. In Section 3 we prove the existence of a solution using the notion of upper and lower solution; in particular our result does not assume any type of monotonicity condition on f as is customary in the literature. Finally, in Section 4 we introduce a more general concept of upper and lower solution following the ideas of [8]. We first prove it generalizes the definition (see [21]), and then we obtain a new existence result via the monotone iterative technique [14].

2. Basic Existence Theory

Let $\lambda \in \mathbb{R}$, $F : I \times \mathbb{R} \to \mathbb{R}$ a L^1-Carathéodory function and consider the problem

$$\begin{aligned} &u'(t) + \lambda u(t) = F(t, u(t)), \quad \text{a.e.} \quad t \in I, \\ &u(0) = -u(T). \end{aligned} \tag{2}$$

Evidently if $F(t,u) = f(t,u) + \lambda u$ and u is a solution to (2) then u is a solution to (1). Furthermore, it is easy to show that solving (2) is equivalent to finding a $u \in C(I)$ that satisfies $u = Au$. Here $A : C(I) \to C(I)$ is given by

$$[Au](t) = \int_0^T g(t,s) F(s, u(s))\, ds, \tag{3}$$

where g is the Green's function

$$g(t,s) = \begin{cases} \dfrac{e^{\lambda(T-t+s)}}{e^{\lambda T}+1}, & 0 \leqslant s \leqslant t \leqslant T \\[2ex] \dfrac{-e^{\lambda(s-t)}}{e^{\lambda T}+1}, & 0 \leqslant t < s \leqslant T. \end{cases}$$

Note that if $F(t,u) = \sigma(t)$ problem (2) is linear and solvable for each $\lambda \in \mathbb{R}$ and the solution is given by expression (3).

Using a nonlinear alternative of Leray-Schauder type we now establish two existence principles for (2).

We include the statement of the Leray-Schauder alternative here for the sake of completeness (Theorem 2.5 of [17]).

THEOREM 1. *Let C be a complete convex subset of a locally convex Hausdorff linear topological space E and U an open subset of C with $p \in U$. In addition let $F : \overline{U} \to C$ be a continuous, compact map. Then either*

(A1). F has a fixed point in $\overline{U}$; or

(A2). there is a $u \in \partial U$ and $\mu \in (0,1)$, with $u = \mu F(u) + (1-\mu)p$.

THEOREM 2. *Suppose that there exists a constant M independent of μ, with $\|u\|_0 \neq M$ for any solution $u \in AC(I)$ to*

$$\begin{aligned} &u'(t) + \lambda u(t) = \mu F(t, u(t)), \quad a.e. \quad t \in I, \\ &u(0) = -u(T), \end{aligned} \tag{4}$$

for each $\mu \in (0,1)$. Then (2) has at least one solution in $AC(I)$.

Proof. A function $u \in C(I)$ is a solution to (4) if and only if

$$u = \mu A u$$

where A is defined in (3), i.e.

$$[Au](t) = e^{-\lambda t} \int_0^t e^{\lambda s} F(s, u(s))\, ds - \frac{e^{-\lambda T}}{1 + e^{-\lambda T}} e^{-\lambda t} \int_0^T e^{\lambda s} F(s, u(s))\, ds.$$

Since F is L^1-Carathéodory it is easy to check that A is continuous and completely continuous. Let $U = \{u \in C(I) : \|u\|_0 < M\}$, $C = E = C(I)$, $p = 0$ and apply Theorem 1. $\square$

THEOREM 3. *Suppose that there exist a continuous and nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $q \in L^1(I)$ with*

$$|F(t,u)| \leqslant q(t)\psi(|u|), \quad \text{for a.e. } t \in I \text{ and all } u \in \mathbb{R}.$$

In addition suppose that

$$\sup_{c \geqslant 0} \frac{c}{\psi(c)} > k_0 \tag{5}$$

with

$$k_0 = \sup_{t \in I} \int_0^T |g(t,s)| q(s) ds.$$

Then (2) has at least one solution in $AC(I)$.

Proof. From (5) there exists $M > 0$ with

$$\frac{M}{\psi(M)} > k_0. \tag{6}$$

For $\mu \in (0,1)$, let $u \in AC(I)$ be any solution of (4). Then, for $t \in I$ we have

$$u(t) = \mu \int_0^T g(t,s)F(s,u(s))\,ds$$

and so

$$\begin{aligned} |u(t)| &\leqslant \mu \int_0^T |g(t,s)F(s,u(s))|\,ds \leqslant \int_0^T |g(t,s)|q(s)\psi(|u(s)|)\,ds \\ &\leqslant \psi(\|u\|_0) \int_0^T |g(t,s)|q(s)\,ds. \end{aligned}$$

Consequently, $\|u\|_0 \leqslant k_0\psi(\|u\|_0)$ and so $\|u\|_0 \neq M$ from (6). Now, we use Theorem 2 to deduce that (2) has a solution in $AC(I)$. $\square$

3. Upper and Lower Solutions

In [12, 21, 22] the following definition of related lower and upper solution is presented.

DEFINITION 1. We say that a pair of functions

$$\alpha, \beta \in AC(I)$$

are related lower and upper solutions for the anti-periodic problem (1) if

$$\alpha(t) \leqslant \beta(t), \quad t \in I, \tag{7}$$

$$\alpha'(t) \leqslant f(t,\alpha(t)), \quad \text{a.e. } t \in I, \ \alpha(0) \leqslant -\beta(T), \tag{8}$$

and

$$\beta'(t) \geqslant f(t,\beta(t)), \quad \text{a.e. } t \in I, \ \beta(0) \geqslant -\alpha(T). \tag{9}$$

With this definition several authors developed a monotone iterative technique assuming several monotone conditions in the nonlinearity f. The following is the first result, to our knowledge, that establishes the validity of the lower and upper solution method for (1) without monotone criteria.

THEOREM 4. *Suppose that there exist $\alpha, \beta \in AC(I)$ related lower and upper solutions for (1). Then (1) has at least one solution between α and β.*

Proof. Let $\lambda > 0$ and consider the modified problem

$$\begin{aligned} &u'(t) + \lambda u(t) = F^*(t,u(t)), \ \text{a.e. } t \in I, \\ &u(0) = -u(T), \end{aligned} \tag{10}$$

with

$$F^*(t,u) = \begin{cases} f(t,\beta(t)) + \lambda\beta(t), & \text{if} \quad \beta(t) < u \\ f(t,u) + \lambda u, & \text{if} \quad \alpha(t) \leqslant u \leqslant \beta(t) \\ f(t,\alpha(t)) + \lambda\alpha(t), & \text{if} \quad u < \alpha(t). \end{cases}$$

Then, by the Schauder fixed point theorem, we conclude that (10) has a solution u, since in this case the operator A defined in (3) is continuous and compact.

Now we will show that this solution u satisfies $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ for $t \in I$. Assume that $u - \beta$ attains a positive maximum on I at s_0. We shall consider two cases:

Case 1. $s_0 \in (0,T]$.
Then there exists $\tau \in (0,s_0)$ such that

$$0 \leqslant u(t) - \beta(t) \leqslant u(s_0) - \beta(s_0), \qquad \text{for all} \quad t \in [\tau, s_0].$$

This yields a contradiction, since

$$\begin{aligned} \beta(s_0) - \beta(\tau) &\leqslant u(s_0) - u(\tau) = \int_\tau^{s_0} [f(s,\beta(s)) - \lambda(u(s) - \beta(s))]ds \\ &< \int_\tau^{s_0} \beta'(s)ds = \beta(s_0) - \beta(\tau). \end{aligned}$$

Case 2. $s_0 = 0$.
Then $0 < u(0) - \beta(0)$. Note also that $u(T) - \alpha(T) < 0$ since

$$u(T) = -u(0) < -\beta(0) \leqslant \alpha(T).$$

Moreover by hypothesis $\alpha(0) \leqslant \beta(0) < u(0)$. Therefore, there exists $\tau \in (0,T)$ with $u(t) - \alpha(t) < 0$ for all $t \in (\tau, T]$ and $u(\tau) - \alpha(\tau) = 0$. Now we have

$$u(T) - u(\tau) = \int_\tau^T [f(s,\alpha(s)) + \lambda(\alpha(s) - u(s))]\, ds > \int_\tau^T \alpha'(s)\, ds = \alpha(T) - \alpha(\tau)$$

which contradicts $u(T) - \alpha(T) < 0$.

Consequently, $u(t) \leqslant \beta(t)$ for all $t \in I$. Similarly, we can show that $\alpha \leqslant u$ on I. $\square$

4. Coupled Upper and Lower Solutions

Now we return to the problem (2) with $F(t,u) = f(t,u) + \lambda u$ and we consider again the operator A defined in (3). Note that g is not of constant sign on $I \times I$. Hence, $g = g^+ - g^-$ with

$$g^+(t,s) = \max\{g(t,s), 0\} \qquad \text{and} \qquad g^-(t,s) = \max\{-g(t,s), 0\}$$

and we can write the operator given in (3) as

$$[Au](t) = \int_0^T g^+(t,s)F(s,u(s))\, ds - \int_0^T g^-(t,s)F(s,u(s))\, ds, \tag{11}$$

or equivalently as

$$[Au](t) = \int_0^t \frac{e^{\lambda(T-t+s)}}{e^{\lambda T}+1} F(s,u(s))\,ds - \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}+1} F(s,u(s))\,ds.$$

Motivated by the expression (11) and the results of [8] we introduce the following operators. For $\eta \in C(I)$, $t \in I$, we define

$$[A^+\eta](t) = \int_0^T g^+(t,s)F(s,\eta(s))\,ds,$$

and

$$[A^-\eta](t) = \int_0^T g^-(t,s)F(s,\eta(s))\,ds.$$

Note that $A^+ : C(I) \to C(I)$ and $A^- : C(I) \to C(I)$ are continuous and completely continuous.

DEFINITION 2. We say that a pair of functions $\alpha, \beta \in C^1(I)$ are coupled lower and upper solutions for the anti-periodic problem (1) if (7) holds and

$$\alpha \leqslant A^+\alpha - A^-\beta, \tag{12}$$

and

$$\beta \geqslant A^+\beta - A^-\alpha. \tag{13}$$

The relation between both definitions is given by the following result.

THEOREM 5. *Suppose that α, β are a pair of related lower and upper solutions for the anti-periodic problem (1). Then α, β are a pair of coupled lower and upper solutions for (1). In other words, if α, β are lower and upper solutions in the sense of Definition 1, then they are lower and upper solutions in the sense of Definition 2.*

Proof. For every $t \in I$, we have that

$$\begin{aligned}
&[A^+\alpha](t) - [A^-\beta](t) \\
&\quad = \int_0^T g^+(t,s)F(s,\alpha(s))\,ds - \int_0^T g^-(t,s)F(s,\beta(s))\,ds \\
&\quad = \int_0^T g^+(t,s)[f(s,\alpha(s)) + \lambda\alpha(s)]\,ds - \int_0^T g^-(t,s)[f(s,\beta(s)) + \lambda\beta(s)]\,ds \\
&\quad \geqslant \int_0^t \frac{e^{\lambda(T-t+s)}}{e^{\lambda T}+1}[\alpha'(s) + \lambda\alpha(s)]\,ds - \int_t^T \frac{e^{\lambda(-t+s)}}{e^{\lambda T}+1}[\beta'(s) + \lambda\beta(s)]\,ds \\
&\quad = \frac{e^{\lambda(T-t)}}{e^{\lambda T}+1}\int_0^t \frac{d}{ds}(\alpha(s)e^{\lambda s})\,ds - \frac{e^{-\lambda t}}{e^{\lambda T}+1}\int_t^T \frac{d}{ds}(\beta(s)e^{\lambda s})\,ds \\
&\quad = \frac{e^{\lambda(T-t)}}{e^{\lambda T}+1}[\alpha(t)e^{\lambda t} - \alpha(0)] - \frac{e^{-\lambda t}}{e^{\lambda T}+1}[\beta(T)e^{\lambda T} - \beta(t)e^{\lambda t}] \\
&\quad = \frac{e^{\lambda T}}{e^{\lambda T}+1}\alpha(t) - \frac{e^{\lambda(T-t)}}{e^{\lambda T}+1}[\alpha(0) + \beta(T)] + \frac{1}{e^{\lambda T}+1}\beta(t) \\
&\quad \geqslant \frac{e^{\lambda T}}{e^{\lambda T}+1}\alpha(t) + \frac{1}{e^{\lambda T}+1}\alpha(t) = \alpha(t).
\end{aligned}$$

Therefore, (12) holds. The validity of (13) is proved analogously. □

Now we develop the monotone iterative technique for (1) using Definition 2.

THEOREM 6. *Let $\alpha, \beta \in AC(I)$ be a pair of coupled lower and upper solutions for (1). Suppose that f satisfies for a.e. $t \in J$:*

$$f(t,x) - f(t,y) \geqslant -\lambda(x-y), \qquad \alpha(t) \leqslant y \leqslant x \leqslant \beta(t). \tag{14}$$

Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\{\alpha_n\} \nearrow \phi$ and $\{\beta_n\} \searrow \psi$ uniformly on I, and any solution to (1) such that $u \in [\alpha, \beta]$ satisfies $u \in [\phi, \psi]$.

In addition, if we suppose that there exists $k > 0$ such that for a.e. $t \in J$

$$f(t,x) - f(t,y) \leqslant -\lambda(x-y) + k(x-y), \qquad \alpha(t) \leqslant y \leqslant x \leqslant \beta(t), \tag{15}$$

and

$$k\frac{e^{\lambda T} - 1}{\lambda(1 + e^{\lambda T})} < 1.$$

Then problem (1) has a unique solution $u \in [\alpha, \beta]$.

Proof. We define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ by $\alpha_0 = \alpha$, $\beta_0 = \beta$ and for each $n \geqslant 1$

$$\begin{aligned} \alpha_n &= A^+\alpha_{n-1} - A^-\beta_{n-1} \\ \beta_n &= A^+\beta_{n-1} - A^-\alpha_{n-1}. \end{aligned} \tag{16}$$

Using condition (14) it is easy to show that A^+ and A^- are nondecreasing operators on $[\alpha, \beta]$. Hence it is easy to check that $\{\alpha_n\}$ is nondecreasing, $\{\beta_n\}$ is nonincreasing and $\alpha_n \leqslant \beta_n$ for each $n \geqslant 0$.

In view of the fact that A^+ and A^- are completely continuous and $\alpha \leqslant \alpha_n \leqslant \beta_n \leqslant \beta$ for all $n \geqslant 0$ we can deduce that $\{\alpha_n\}$ converges to ϕ uniformly on I, and $\{\beta_n\}$ converges to ψ uniformly on I.

Now suppose that u is solution to (1) and $u \in [\alpha, \beta]$. Again, by the monotonicity of A^+ and A^-, we get for each $n \geqslant 0$

$$\alpha_n \leqslant u \leqslant \beta_n.$$

Thus, passing to the limit when $n \to \infty$ we obtain $\phi \leqslant u \leqslant \psi$.

In order to prove the second part of the result we pass to the limit in expression (16) to obtain that ϕ and ψ satisfy

$$\phi = A^+\phi - A^-\psi, \qquad \psi = A^+\psi - A^-\phi.$$

If $\psi = \phi$, then ψ is a solution to (1) since $A = A^+ - A^-$.

To show that $\psi = \phi$ we consider

$$\psi(t) - \phi(t) = [A^+\psi](t) - [A^+\phi](t) - [A^-\psi](t) + [A^-\phi](t).$$

Using condition (15) we obtain

$$\psi(t) - \phi(t) \leqslant \int_0^T k[g^+(t,s) + g^-(t,s)][\psi(s) - \phi(s)]ds.$$

This together with

$$\begin{aligned}\int_0^T \left[g^+(t,s) + g^-(t,s)\right] ds &= \frac{1}{1+e^{\lambda T}}\left[\int_0^t e^{\lambda(T-t+s)}ds + \int_t^T e^{\lambda(s-t)}ds\right] \\ &= \frac{e^{\lambda T}-1}{\lambda(1+e^{\lambda T})}\end{aligned}$$

implies

$$\|\phi - \psi\|_0 \leqslant k\frac{e^{\lambda T}-1}{\lambda(1+e^{\lambda T})}\|\phi - \psi\|_0.$$

Hence $\phi = \psi$. $\square$

Finally we remark that Theorem 6 generalizes the results of [21, 22], since the definition of coupled upper and lower solutions is more general and, in addition, we avoid the regularity hypotheses that appear in those papers.

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