

A NEW GENERALIZATION OF GRÜSS INEQUALITY IN INNER PRODUCT SPACES

NENAD UJEVIĆ

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Abstract. A new generalization of Grüss inequality in real inner product spaces is derived. Applications to Lebesgue integrals are given.

1. Introduction

In 1935, G. Grüss proved the following integral inequality

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{(\Phi - \varphi)(\Gamma - \gamma)}{4} \quad (1)$$

provided that f and g are two square integrable functions on $[a, b]$ and satisfying the condition

$$\varphi \leq g(t) \leq \Phi \quad \text{and} \quad \gamma \leq f(t) \leq \Gamma, \quad \forall t \in [a, b]. \quad (2)$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(t) = g(t) = \operatorname{sgn} \left(t - \frac{a+b}{2} \right). \quad (3)$$

In [2] and [3] we can find the next generalization of (1).

THEOREM 1. *Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and $e \in X$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real numbers and x, y are vectors in X so that the condition*

$$\langle \Phi e - x, x - \varphi e \rangle \geq 0 \quad \text{and} \quad \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \quad (4)$$

holds, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|. \quad (5)$$

The constant $\frac{1}{4}$ is the best possible.

In this paper we further generalize (5) in real inner product spaces. Applications to Lebesgue integrals are also given.

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2. Main result

Let X be a real linear space and let $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbf{R}$ be such that the following is valid:

- (i) $\langle x, x \rangle > 0$, $x \neq 0$, $x \in X$;
- (ii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, $x, y \in X$, $\lambda \in \mathbf{R}$;
- (iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, $x, y, z \in X$;
- (iv) $\langle x, y \rangle = \langle y, x \rangle$, $x, y \in X$.

As we know the space $(X, \langle \cdot, \cdot \rangle)$ is called a real inner product space.

Let n be a positive integer. Here we suppose that there are, at least, $2n$ elements $e_1, e_2, \dots, e_{2n} \in X$ such that $\langle e_i, e_j \rangle = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$.

In $(X, \langle \cdot, \cdot \rangle)$ the Cauchy-Schwarz inequality,

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad (1)$$

holds, where $\|x\| = \sqrt{\langle x, x \rangle}$.

THEOREM 2. Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and $\{e_i\}_1^n \subset X$, $\langle e_i, e_j \rangle = \delta_{ij}$. If $\varphi_i, \gamma_i, \Phi_i, \Gamma_i$, $i = 1, 2, \dots, n$, are real numbers and $x, y \in X$ such that the conditions

$$\left\langle x - \sum_{i=1}^n \gamma_i e_i, \sum_{i=1}^n \Gamma_i e_i - x \right\rangle \geq 0, \quad (2)$$

and

$$\left\langle y - \sum_{i=1}^n \varphi_i e_i, \sum_{i=1}^n \Phi_i e_i - y \right\rangle \geq 0 \quad (3)$$

hold, then we have the inequality

$$\left| \langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle y, e_i \rangle \right| \leq \frac{1}{4} \sqrt{\sum_{i=1}^n (\Phi_i - \varphi_i)^2 \sum_{i=1}^n (\Gamma_i - \gamma_i)^2}. \quad (4)$$

The constant $\frac{1}{4}$ is the best possible.

Proof. We have

$$\begin{aligned} & \left\| x - \sum_{i=1}^n \langle x, e_i \rangle e_i \right\|^2 \\ &= \left\langle x - \sum_{i=1}^n \langle x, e_i \rangle e_i, x - \sum_{i=1}^n \langle x, e_i \rangle e_i \right\rangle \\ &= \|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2 \end{aligned} \quad (5)$$

and

$$\begin{aligned}
 & \left\langle x - \sum_{i=1}^n \langle x, e_i \rangle e_i, y - \sum_{j=1}^n \langle y, e_j \rangle e_j \right\rangle \tag{6} \\
 &= \langle x, y \rangle - \sum_{j=1}^n \langle y, e_j \rangle \langle x, e_j \rangle - \sum_{i=1}^n \langle y, e_i \rangle \langle x, e_i \rangle \\
 & \quad + \sum_{i=1}^n \sum_{j=1}^n \langle x, e_i \rangle \langle y, e_j \rangle \langle e_i, e_j \rangle \\
 &= \langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle y, e_i \rangle
 \end{aligned}$$

since $\langle e_i, e_j \rangle = \delta_{ij}$. Using Cauchy-Schwarz inequality and (5) we get

$$\begin{aligned}
 & \left| \left\langle x - \sum_{i=1}^n \langle x, e_i \rangle e_i, y - \sum_{j=1}^n \langle y, e_j \rangle e_j \right\rangle \right| \tag{7} \\
 & \leq \left\| x - \sum_{i=1}^n \langle x, e_i \rangle e_i \right\| \left\| y - \sum_{j=1}^n \langle y, e_j \rangle e_j \right\| \\
 &= \sqrt{\|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2} \sqrt{\|y\|^2 - \sum_{i=1}^n \langle y, e_i \rangle^2}.
 \end{aligned}$$

We also have

$$\begin{aligned}
 & \sum_{i=1}^n (\langle x, e_i \rangle - \gamma_i) (\Gamma_i - \langle x, e_i \rangle) - \left\langle x - \sum_{i=1}^n \gamma_i e_i, \sum_{i=1}^n \Gamma_i e_i - x \right\rangle \tag{8} \\
 &= \|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2 \geq 0.
 \end{aligned}$$

From (2) and (8) it follows

$$\|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2 \leq \sum_{i=1}^n (\langle x, e_i \rangle - \gamma_i) (\Gamma_i - \langle x, e_i \rangle). \tag{9}$$

In a similar way we get

$$\|y\|^2 - \sum_{i=1}^n \langle y, e_i \rangle^2 \leq \sum_{i=1}^n (\langle y, e_i \rangle - \varphi_i) (\Phi_i - \langle y, e_i \rangle). \tag{10}$$

If we now use the elementary inequality $a_i b_i \leq \frac{1}{4}(a_i + b_i)^2$, with $a_i = \Gamma_i - \langle x, e_i \rangle$, $b_i = \langle x, e_i \rangle - \gamma_i$, then we have

$$(\langle x, e_i \rangle - \gamma_i) (\Gamma_i - \langle x, e_i \rangle) \leq \frac{1}{4}(\Gamma_i - \gamma_i)^2. \tag{11}$$

From (9) and (11) it follows

$$\|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2 \leq \frac{1}{4} \sum_{i=1}^n (\Gamma_i - \gamma_i)^2. \quad (12)$$

In a similar way we get

$$\|y\|^2 - \sum_{i=1}^n \langle y, e_i \rangle^2 \leq \frac{1}{4} \sum_{i=1}^n (\Phi_i - \varphi_i)^2. \quad (13)$$

From (6), (7), (12) and (13) we easily get (4).

To prove that $\frac{1}{4}$ is the best possible constant we choose the numbers $\varphi_i, \gamma_i, \Phi_i, \Gamma_i, i = 1, 2, \dots, n$ such that

$$\Gamma_i - \gamma_i = \lambda(\Phi_i - \varphi_i), \quad \lambda \in \mathbf{R}, \quad i = 1, 2, \dots, n. \quad (14)$$

We also choose $f_i \in X, i = 1, 2, \dots, n$ such that

$$\langle e_i, f_j \rangle = 0, \quad \langle f_i, f_j \rangle = \delta_{ij}, \quad i, j = 1, 2, \dots, n. \quad (15)$$

If we now define

$$x = \sum_{i=1}^n \frac{\Gamma_i + \gamma_i}{2} e_i + \sum_{i=1}^n \frac{\Gamma_i - \gamma_i}{2} f_i \quad (16)$$

and

$$y = \sum_{i=1}^n \frac{\Phi_i + \varphi_i}{2} e_i + \sum_{i=1}^n \frac{\Phi_i - \varphi_i}{2} f_i \quad (17)$$

then we have

$$\langle x, y \rangle = \sum_{i=1}^n \frac{(\Gamma_i + \gamma_i)(\Phi_i + \varphi_i)}{4} + \sum_{i=1}^n \frac{(\Gamma_i - \gamma_i)(\Phi_i - \varphi_i)}{4} \quad (18)$$

and

$$\sum_{i=1}^n \langle x, e_i \rangle \langle y, e_i \rangle = \sum_{i=1}^n \frac{(\Gamma_i + \gamma_i)(\Phi_i + \varphi_i)}{4}. \quad (19)$$

If x is given by (16) then we have

$$\begin{aligned} & \left\langle x - \sum_{i=1}^n \gamma_i e_i, \sum_{i=1}^n \Gamma_i e_i - x \right\rangle \quad (20) \\ &= \frac{1}{2} \sum_{i=1}^n \Gamma_i (\Gamma_i + \gamma_i) - \frac{1}{4} \sum_{i=1}^n (\Gamma_i + \gamma_i)^2 - \frac{1}{4} \sum_{i=1}^n (\Gamma_i - \gamma_i)^2 \\ & \quad - \sum_{i=1}^n \Gamma_i \gamma_i + \frac{1}{2} \sum_{i=1}^n \gamma_i (\Gamma_i + \gamma_i) \\ &= 0. \end{aligned}$$

Hence, the element x satisfies the condition (2).

In a similar way we can show that the element y , given by (17), satisfies the condition (3). From (18) and (19) it follows

$$\langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle y, e_i \rangle = \sum_{i=1}^n \frac{(\Gamma_i - \gamma_i)(\Phi_i - \varphi_i)}{4}. \tag{21}$$

We also have

$$\left(\sum_{i=1}^n \frac{(\Gamma_i - \gamma_i)(\Phi_i - \varphi_i)}{4} \right)^2 = \sum_{i=1}^n \frac{(\Gamma_i - \gamma_i)^2}{4} \sum_{i=1}^n \frac{(\Phi_i - \varphi_i)^2}{4}, \tag{22}$$

since (14) holds.

From (21) and (22) we get

$$\left| \langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle y, e_i \rangle \right| = \frac{1}{4} \sqrt{\sum_{i=1}^n (\Gamma_i - \gamma_i)^2 \sum_{i=1}^n (\Phi_i - \varphi_i)^2}. \tag{23}$$

This completes the proof. \square

3. Special cases

Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space. If we substitute $n = 1$ in (4) then we get

$$|\langle x, y \rangle - \langle x, e_1 \rangle \langle y, e_1 \rangle| \leq \frac{1}{4} |\Gamma_1 - \gamma_1| |\Phi_1 - \varphi_1| \tag{1}$$

provided

$$\langle x - \gamma_1 e_1, \Gamma_1 e_1 - x \rangle \geq 0 \quad \text{and} \quad \langle y - \varphi_1 e_1, \Phi_1 e_1 - y \rangle \geq 0. \tag{2}$$

If we now compare (1)–(2) and (5)–(4) then we see that (1) is equivalent to (5). Hence, Theorem 2 is a generalization of Theorem 1.

Applications to integrals are also natural.

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a σ -algebra Σ of subsets of Ω and a countable additive and positive measure μ on Σ with values in $\mathbf{R} \cup \{\infty\}$. Denote $L_2(\Omega)$ the Hilbert space of all real valued functions f defined on Ω and 2-integrable on Ω , i.e. $\int_{\Omega} |f(t)|^2 d\mu(t) < \infty$.

PROPOSITION 1. *Let $f, g \in L_2(\Omega)$, $\{\Psi_i\}_1^n \subset L_2(\Omega)$, $(\Psi_i, \Psi_j) = \delta_{ij}$, $i, j = 1, 2, \dots, n$. If $\gamma_i, \varphi_i, \Gamma_i, \Phi_i$, $i = 1, 2, \dots, n$ are real numbers such that the conditions*

$$\int_{\Omega} \left[f(t) - \sum_{i=1}^n \gamma_i \Psi_i(t) \right] \left[\sum_{i=1}^n \Gamma_i \Psi_i(t) - f(t) \right] d\mu(t) \geq 0,$$

and

$$\int_{\Omega} \left[g(t) - \sum_{i=1}^n \varphi_i \Psi_i(t) \right] \left[\sum_{i=1}^n \Phi_i \Psi_i(t) - g(t) \right] d\mu(t) \geq 0$$

hold, then we have the inequality

$$\begin{aligned} & \left| \int_{\Omega} f(t)g(t)d\mu(t) - \sum_{i=1}^n \int_{\Omega} f(t)\Psi_i(t)d\mu(t) \int_{\Omega} g(t)\Psi_i(t)d\mu(t) \right| \\ & \leq \frac{1}{4} \left[\sum_{i=1}^n (\Gamma_i - \gamma_i)^2 \sum_{i=1}^n (\Phi_i - \varphi_i)^2 \right]^{1/2}. \end{aligned} \quad (3)$$

The constant $\frac{1}{4}$ is best possible.

Proof. We simply apply Theorem 2 with

$$\langle f, g \rangle = (f, g) = \int_{\Omega} f(t)g(t)d\mu(t). \quad \square$$

We now show that (3) is a direct generalization of (1).

Let $\Omega = [a, b]$ and $\Psi_1(t) = 1/\sqrt{b-a}$, $n = 1$. Proposition 1 implies

$$\left| \int_a^b f(t)g(t)dt - \frac{1}{(b-a)} \int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{1}{4} |\Gamma_1 - \gamma_1| |\Phi_1 - \varphi_1| \quad (4)$$

provided

$$\begin{aligned} & \int_a^b (f(t) - \gamma_1 \Psi_1(t))(\Gamma_1 \Psi_1(t) - f(t))dt \geq 0, \\ & \int_a^b (g(t) - \varphi_1 \Psi_1(t))(\Phi_1 \Psi_1(t) - g(t))dt \geq 0. \end{aligned} \quad (5)$$

Suppose that there exist constants $\gamma, \varphi, \Gamma, \Phi \in \mathbf{R}$ such that

$$\gamma \leq f(t) \leq \Gamma \quad \text{and} \quad \varphi \leq g(t) \leq \Phi, \quad t \in [a, b].$$

If we define

$$\gamma_1 = \gamma\sqrt{b-a}, \quad \Gamma_1 = \Gamma\sqrt{b-a}, \quad \varphi_1 = \varphi\sqrt{b-a}, \quad \Phi_1 = \Phi\sqrt{b-a}$$

then we have

$$\begin{aligned} & \int_a^b (f(t) - \gamma_1 \Psi_1(t))(\Gamma_1 \Psi_1(t) - f(t))dt \\ & = \int_a^b (f(t) - \gamma)(\Gamma - f(t))dt \geq 0, \end{aligned}$$

since $f(t) - \gamma \geq 0$ and $\Gamma - f(t) \geq 0$, $t \in [a, b]$. Similar reasons give

$$\int_a^b (g(t) - \varphi_1 \Psi_1(t))(\Phi_1 \Psi_1(t) - g(t)) dt \geq 0.$$

Thus,

$$\left| \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \int_a^b g(t) dt \right| \leq (b-a) \frac{(\Phi - \varphi)(\Gamma - \gamma)}{4}.$$

The above inequality is equivalent to (1).

Hence, Proposition 1 is a generalization of (1).

Finally, let us mention that Grüss type inequalities are used in many recently published papers. For example, see [1], [4], [5], [7] and [10]. Inequalities considered in this paper can also be applied to problems in the mentioned papers. See also remarks in [2].

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Nenad Ujević
Department of Mathematics
University of Split
Testlina 12/III
21000 Split
Croatia
e-mail: ujevic@pmfst.hr