

ON THE STABILITY OF HOMOGENEOUS FUNCTIONAL EQUATIONS WITH DEGREE t AND n -VARIABLES

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Abstract. In this paper, we obtain a generalization of Hyers-Ulam-Rassias stability for the family of the functional equation $f(\circ(x_1, x_2, \dots, x_n)) = H\left(f(x_1)^{\frac{1}{t}}, f(x_2)^{\frac{1}{t}}, \dots, f(x_n)^{\frac{1}{t}}\right)$, where H is a homogeneous function of degree t and \circ is an n -times-symmetric operation on the set S . As a consequence we can obtain the Hyers-Ulam stability.

1. Introduction

In 1940, S. M. Ulam [12] raised the following problem for the stability of Cauchy equations: Given a group G_1 , a metric group G_2 with a metric $d(\cdot, \cdot)$, and an $\epsilon > 0$, fixed a $\delta > 0$ such that, if $f : G_1 \rightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then there exists a homomorphism $g : G_1 \rightarrow G_2$ such that $d(f(x), g(x)) \leq \epsilon$ for all $x \in G_1$.

In 1941, D. H. Hyers [3] answered this question in the affirmative when G_1 and G_2 are Banach spaces. In 1978, Rassias [9] generalized the result of Hyers. This property is called the Hyers-Ulam-Rassias stability of Cauchy equation. This terminology is also applied to other functional equations.

Throughout this paper, let S be a nonempty set and G be a multiplicative subsemigroup of the real or complex field with $1 \in G$ and the property that $u^t, u^{\frac{1}{t}} \in G$ for all $u \in G$ and for a fixed $t > 0$. Let $f : S \rightarrow G$ be a function and let $H : G \times \dots \times G \rightarrow G$ be a G -homogeneous function of degree $t > 0$, that is, H satisfies

$$H(uu_1, uu_2, \dots, uu_n) = u^t H(u_1, u_2, \dots, u_n) \quad (u, u_1, u_2, \dots, u_n \in G) \quad (1.1)$$

for some fixed natural number n .

An operation $\circ : S \times \dots \times S \rightarrow S$ will be called n -times-symmetric if \circ satisfies the following identity:

$$\begin{aligned} & \circ[\circ(x_1, x_2, \dots, x_n), \dots, \circ(x_1, x_2, \dots, x_n)] \\ &= \circ[\circ(x_1, \dots, x_1), \circ(x_2, \dots, x_2), \dots, \circ(x_n, \dots, x_n)]. \end{aligned}$$

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Páles, Volkman and Luce [8] proved the Hyers-Ulam stability for the functional equation $f(x \circ y) = H(f(x), f(y))$ ($x, y \in S$), where $\circ : S \times S \rightarrow S$ is a square-symmetric operation, and H is a continuous G-homogeneous function of degree 1 and two variables on the set S .

In this paper, we extend the above result to n -variables and homogeneous of degree t , and obtain a generalization of Hyers-Ulam-Rassias stability for the following functional equation:

$$f(\circ(x_1, x_2, \dots, x_n)) = H(f(x_1)^{\frac{1}{t}}, f(x_2)^{\frac{1}{t}}, \dots, f(x_n)^{\frac{1}{t}}) \quad (x_1, x_2, \dots, x_n \in S). \quad (1.2)$$

A particular case of (1.2) is the Cauchy functional equation for n -variables

$$f(x_1 + x_2 + \dots + x_n) = f(x_1) + f(x_2) + \dots + f(x_n) \quad (x_1, x_2, \dots, x_n \in S),$$

where S is a semigroup with the operation $+$ and $f : S \rightarrow \mathbb{C}$.

As a corollary we obtain the Hyers-Ulam stability and Hyers-Ulam-Rassias stability for this functional equation without the direct proof.

2. n -times-symmetric operation

In the following result we show that if the equation (1.2) has sufficiently many solutions then \circ is necessarily n -times-symmetric.

LEMMA 1. *Assume that the set of solutions of the functional equation (1.2) separates the points of S , that is, if $u, v \in S$ and $u \neq v$, then there exists a solution $f : S \rightarrow G$ of (1.2) such that $f(u) \neq f(v)$. Then the operation \circ is n -times-symmetric.*

Proof. Let $x_1, x_2, \dots, x_n \in S$, and let $f : S \rightarrow G$ be an arbitrary solution of (1.2). Then, using the degree t -homogeneity of H and (1.2) several times, we obtain

$$\begin{aligned} & f(\circ[\circ(x_1, x_2, \dots, x_n), \dots, \circ(x_1, x_2, \dots, x_n)]) \\ &= H\left(f(\circ(x_1, x_2, \dots, x_n))^{\frac{1}{t}}, \dots, f(\circ(x_1, x_2, \dots, x_n))^{\frac{1}{t}}\right) \\ &= f(\circ(x_1, x_2, \dots, x_n))H(1, \dots, 1) \\ &= H\left(f(x_1)^{\frac{1}{t}}, f(x_2)^{\frac{1}{t}}, \dots, f(x_n)^{\frac{1}{t}}\right)H(1, \dots, 1) \\ &= H\left(\left(f(x_1)H(1, \dots, 1)\right)^{\frac{1}{t}}, \left(f(x_2)H(1, \dots, 1)\right)^{\frac{1}{t}}, \dots, \left(f(x_n)H(1, \dots, 1)\right)^{\frac{1}{t}}\right) \\ &= H\left(H\left(f(x_1)^{\frac{1}{t}}, \dots, f(x_1)^{\frac{1}{t}}\right)^{\frac{1}{t}}, H\left(f(x_2)^{\frac{1}{t}}, \dots, f(x_2)^{\frac{1}{t}}\right)^{\frac{1}{t}}, \dots, \right. \\ &\quad \left. H\left(f(x_n)^{\frac{1}{t}}, \dots, f(x_n)^{\frac{1}{t}}\right)^{\frac{1}{t}}\right) \\ &= H\left(f(\circ(x_1, \dots, x_1))^{\frac{1}{t}}, f(\circ(x_2, \dots, x_2))^{\frac{1}{t}}, \dots, f(\circ(x_n, \dots, x_n))^{\frac{1}{t}}\right) \\ &= f(\circ[\circ(x_1, \dots, x_1), \circ(x_2, \dots, x_2), \dots, \circ(x_n, \dots, x_n)]). \end{aligned}$$

By the assumption of separability, \circ is n -times-symmetric. \square

The next result describes a set of n -times-symmetric operations.

COROLLARY 1. *Let G be a multiplicative subsemigroup of \mathbb{C} , let $H : G \times \cdots \times G \rightarrow G$ satisfy (1.1), and let $\phi : S \rightarrow G$ be an arbitrary bijective function. Then the operation $\circ : S \times \cdots \times S \rightarrow S$ defined by*

$$\circ(x_1, x_2, \dots, x_n) := \phi^{-1}\left(H(\phi(x_1)^{\frac{1}{n}}, \phi(x_2)^{\frac{1}{n}}, \dots, \phi(x_n)^{\frac{1}{n}})\right) \quad (x_1, x_2, \dots, x_n \in S) \tag{2.1}$$

is n -times-symmetric.

Proof. Clearly, ϕ is a solution of the functional equation (1.2) (with the operation \circ defined in (2.1)). By its injectivity, it separates the points of S . Thus, due to the previous lemma, \circ must be an n -times-symmetric operation. \square

LEMMA 2. *Let \circ be an n -times-symmetric operation on S . Define, for $x \in S$, the sequence $x[n^k]$ ($k = 0, 1, 2, \dots$) by*

$$x[1] = x[n^0] := x, \quad x[n^{k+1}] := \circ(x[n^k], \dots, x[n^k]), \quad k \in \mathbb{N}. \tag{2.2}$$

Then, for each $k \in \mathbb{N}$, the mapping $x \mapsto x[n^k]$ is an endomorphism of (S, \circ) , that is

$$\circ(x_1, x_2, \dots, x_n)[n^k] = \circ(x_1[n^k], x_2[n^k], \dots, x_n[n^k]) \quad \forall x_1, x_2, \dots, x_n \in S. \tag{2.3}$$

Proof. We prove by induction on $k \in \mathbb{N}$. For $k = 1$, the statement is equivalent to the n -times-symmetry of the operation \circ . Assume that (2.3) is true for k . Then, using (2.2), the n -times-symmetry, and the inductive hypothesis,

$$\begin{aligned} &\circ(x_1, x_2, \dots, x_n)[n^{k+1}] \\ &= \circ[\circ(x_1, x_2, \dots, x_n)[n^k], \dots, \circ(x_1, x_2, \dots, x_n)[n^k]] \\ &= \circ[\circ(x_1[n^k], \dots, x_n[n^k]), \dots, \circ(x_1[n^k], \dots, x_n[n^k])] \\ &= \circ[\circ(x_1[n^k], \dots, x_1[n^k]), \circ(x_2[n^k], \dots, x_2[n^k]), \dots, \circ(x_n[n^k], \dots, x_n[n^k])] \\ &= \circ(x_1[n^{k+1}], x_2[n^{k+1}], \dots, x_n[n^{k+1}]). \end{aligned}$$

Thus, (2.3) also holds for $k + 1$. \square

3. A generalization of the Hyers-Ulam-Rassias stability of (1.2)

In this section, let G be a closed multiplicative subsemigroup of \mathbb{C} . In each theorem of this paper, φ and Φ_i ($i = 1, 2, 3, 4$) are real mappings from $S \times \cdots \times S$ into G .

We will investigate the generalization of Hyers-Ulam-Rassias stability for the functional equation (1.2). By using an idea in Găvruta [1], and Pales, Volkmann and Luce [8] we can obtain the following results:

THEOREM 1. *Let \circ be an n -times-symmetric operation on S , and $H : G \times \cdots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t such that $|H(1, \dots, 1)| \neq 0$ and $\frac{1}{H(1, \dots, 1)} \in G$. Assume that a function $g : S \rightarrow G$ satisfies the inequality*

$$\left| g(\circ(x_1, x_2, \dots, x_n)) - H(g(x_1)^{\frac{1}{t}}, g(x_2)^{\frac{1}{t}}, \dots, g(x_n)^{\frac{1}{t}}) \right| \leq \varphi(x_1, x_2, \dots, x_n) \quad (3.1)$$

for all $x_1, x_2, \dots, x_n \in S$.

If the function φ satisfies

$$\liminf_{k \rightarrow \infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{\varphi(x_1[n^k], x_2[n^k], \dots, x_n[n^k])} |H(1, \dots, 1)| > 1 \quad \text{or} \quad (3.2)$$

$$\limsup_{k \rightarrow \infty} \sqrt[k]{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])} < |H(1, \dots, 1)|, \quad (3.3)$$

then there exists a unique function $f : S \rightarrow G$ such that f is a solution of (1.2) and

$$|f(x) - g(x)| \leq \Phi_1(x, \dots, x) \quad (x \in S), \quad (3.4)$$

where $\Phi_1(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{|H(1, \dots, 1)|^k}$ for all $x_1, x_2, \dots, x_n \in S$.

Proof. We consider the sequence p_k defined by

$$p_k(x_1, x_2, \dots, x_n) = \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{|H(1, \dots, 1)|^k}.$$

Note that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{p_{k+1}(x_1, x_2, \dots, x_n)}{p_k(x_1, x_2, \dots, x_n)} &= \limsup_{k \rightarrow \infty} \frac{\varphi(x_1[n^k], x_2[n^k], \dots, x_n[n^k])}{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}]) |H(1, \dots, 1)|} \\ &= \frac{1}{\liminf_{k \rightarrow \infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{\varphi(x_1[n^k], x_2[n^k], \dots, x_n[n^k])} |H(1, \dots, 1)|} < 1 \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in S$ by (3.2). Similarly in case (3.3)

$$\limsup_{k \rightarrow \infty} \sqrt[k]{p_k(x_1, x_2, \dots, x_n)} = \limsup_{k \rightarrow \infty} \frac{\sqrt[k]{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}}{|H(1, \dots, 1)|} < 1.$$

By ratio and root test in each case we see the series Φ_1 converges for all $x_1, x_2, \dots, x_n \in S$, namely it is bounded.

Replacing x_1, x_2, \dots, x_n by x in (3.1) and using the degree t -homogeneity of H , we get

$$|g(x[n]) - g(x)H(1, \dots, 1)| \leq \varphi(x, \dots, x) \quad (x \in S). \quad (3.5)$$

Let $x \in S$ be fixed, and replace x by $x[n^{k-1}]$ (defined in Lemma 2) in (3.5).

Then by Lemma 2 we obtain

$$\left| \frac{g(x[n^k])}{H(1, \dots, 1)^k} - \frac{g(x[n^{k-1}])}{H(1, \dots, 1)^{k-1}} \right| \leq \frac{\varphi(x[n^{k-1}], \dots, x[n^{k-1}])}{|H(1, \dots, 1)|^k} \tag{3.6}$$

for all $x \in S$ and $k \in \mathbb{N}$.

Let $g_0 := g$ and define, for $k \in \mathbb{N}$, the function g_k by

$$g_k(x) := \frac{g(x[n^k])}{H(1, \dots, 1)^k} \quad (x \in S). \tag{3.7}$$

Since $\frac{1}{H(1, \dots, 1)} \in G$, $g_k : S \rightarrow G$ is a function and, due to (3.6), we have

$$\begin{aligned} |g_j(x) - g_i(x)| &\leq \sum_{k=i+1}^j |g_k(x) - g_{k-1}(x)| \\ &\leq \sum_{k=i+1}^j \frac{\varphi(x[n^{k-1}], \dots, x[n^{k-1}])}{|H(1, \dots, 1)|^k} \end{aligned}$$

for $j > i > 0$. Therefore, by letting $i \rightarrow \infty$ in the last inequality, the sequence $g_k(x)$ is a Cauchy sequence for each fixed $x \in S$ from the boundedness of Φ_1 . Since the set G is closed, we can define a mapping $f : S \rightarrow G$ by

$$f(x) := \lim_{k \rightarrow \infty} g_k(x) \quad (x \in S).$$

It follows from (3.6) that

$$\begin{aligned} |g_k(x) - g(x)| &\leq \sum_{j=1}^k \frac{\varphi(x[n^{j-1}], \dots, x[n^{j-1}])}{|H(1, \dots, 1)|^j} \\ &\leq \sum_{j=1}^{\infty} \frac{\varphi(x[n^{j-1}], \dots, x[n^{j-1}])}{|H(1, \dots, 1)|^j} \\ &= \Phi_1(x, \dots, x). \end{aligned}$$

Taking the limit $k \rightarrow \infty$, we obtain (3.4).

To see that f satisfies (1.2), replace x_1, x_2, \dots, x_n by $x_1[n^k], x_2[n^k], \dots, x_n[n^k]$ in (3.1). Using Lemma 2, we get

$$\begin{aligned} \left| g \left((x_1, x_2, \dots, x_n)[n^k] \right) - H \left(g(x_1[n^k])^{\frac{1}{t}}, g(x_2[n^k])^{\frac{1}{t}}, \dots, g(x_n[n^k])^{\frac{1}{t}} \right) \right| \\ \leq \varphi(x_1[n^k], x_2[n^k], \dots, x_n[n^k]). \end{aligned}$$

Hence, by the degree t -homogeneity of H and (3.7),

$$\begin{aligned} \left| g_k \left((x_1, x_2, \dots, x_n) \right) - H \left(g_k(x_1)^{\frac{1}{t}}, g_k(x_2)^{\frac{1}{t}}, \dots, g_k(x_n)^{\frac{1}{t}} \right) \right| \\ \leq \frac{\varphi(x_1[n^k], x_2[n^k], \dots, x_n[n^k])}{|H(1, \dots, 1)|^k} \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in S$ and $k \in \mathbb{N}$. Taking the limit $k \rightarrow \infty$, by the continuity of H and the boundedness of Φ_1 it follows that

$$\left| f(\circ(x_1, x_2, \dots, x_n)) - H(f(x_1)^{\frac{1}{t}}, f(x_2)^{\frac{1}{t}}, \dots, f(x_n)^{\frac{1}{t}}) \right| = 0 \quad (x_1, x_2, \dots, x_n \in S).$$

Thus (1.2) holds.

Assume that $h : S \rightarrow G$ is another function which satisfies (1.2) and (3.4). Since $f(x[n^k]) = f(x)H(1, \dots, 1)^k$ and $h(x[n^k]) = h(x)H(1, \dots, 1)^k$ from (1.2) for all $x \in S$ and $k \in \mathbb{N}$. It follows from (3.4) that

$$\begin{aligned} |h(x) - f(x)| &= \frac{1}{|H(1, \dots, 1)|^k} |h(x[n^k]) - f(x[n^k])| \\ &\leq \frac{1}{|H(1, \dots, 1)|^k} (|h(x[n^k]) - g(x[n^k])| + |g(x[n^k]) - f(x[n^k])|) \\ &\leq \frac{2}{|H(1, \dots, 1)|^k} \Phi_1(x[n^k], \dots, x[n^k]) \\ &= 2 \sum_{j=k+1}^{\infty} \frac{\varphi(x[n^{j-1}], \dots, x[n^{j-1}])}{|H(1, \dots, 1)|^j} \end{aligned}$$

for all $x \in S$ and $k \in \mathbb{N}$. By letting $k \rightarrow \infty$ in the preceding inequality, we immediately see the uniqueness of f from the boundedness of Φ_1 , and the proof of theorem is completed. \square

THEOREM 1 —'. Let \circ be an n -times-symmetric operation on S , and $H : G \times \dots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t such that $|H(1, \dots, 1)| \neq 0$ and $\frac{1}{H(1, \dots, 1)} \in G$. Assume that a function $g : S \rightarrow G$ satisfies the inequality (3.1).

Then there exists a unique function $f : S \rightarrow G$ such that f is a solution of (1.2) and satisfies (3.4) with $\Phi_1(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{|H(1, \dots, 1)|^k} < \infty$ for all $x_1, x_2, \dots, x_n \in S$.

COROLLARY 2. Let \circ be an n -times-symmetric operation S , and $H : G \times \dots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t such that $|H(1, \dots, 1)| > 1$ and $\frac{1}{H(1, \dots, 1)} \in G$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \rightarrow G$ satisfies the stability inequality

$$\left| g(\circ(x_1, x_2, \dots, x_n)) - H(g(x_1)^{\frac{1}{t}}, g(x_2)^{\frac{1}{t}}, \dots, g(x_n)^{\frac{1}{t}}) \right| \leq \varepsilon \quad (x_1, x_2, \dots, x_n \in S). \quad (3.8)$$

Then there exists a function $f : S \rightarrow G$ such that f is a solution of (1.2) and

$$|f(x) - g(x)| \leq \frac{\varepsilon}{|H(1, \dots, 1)| - 1} \quad (x \in S).$$

Proof. Apply Theorem 1 with $\varphi_1(x_1, x_2, \dots, x_n) = \varepsilon$. \square

We say that the operation \circ has the *divisibility property* if, for each $x \in S$, there exists a unique element $y \in S$ such that $\circ(y, \dots, y) = x$.

THEOREM 2. *Let \circ be an n -times-symmetric operation with the divisibility property on S , and $H : G \times \dots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t . Assume that a function $g : S \rightarrow G$ satisfies the inequality (3.1).*

If the function φ satisfies

$$\liminf_{k \rightarrow \infty} \frac{\varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])}{\varphi(x_1[n^{-k-1}], x_2[n^{-k-1}], \dots, x_n[n^{-k-1}])|H(1, \dots, 1)|} > 1, \tag{3.9}$$

then there exists a unique function $f : S \rightarrow G$ such that f is a solution of (1.2) and

$$|f(x) - g(x)| \leq \Phi_2(x, \dots, x) \quad (x \in S), \tag{3.10}$$

where $\Phi_2(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|H(1, \dots, 1)|^{k-1}$ for all $x_1, x_2, \dots, x_n \in S$.

Proof. The proof of this theorem is analogous to that of Theorem 1. We consider the sequence p_k defined by

$$p_k(x_1, x_2, \dots, x_n) = \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|H(1, \dots, 1)|^{k-1}.$$

Note that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{p_{k+1}(x_1, x_2, \dots, x_n)}{p_k(x_1, x_2, \dots, x_n)} &= \limsup_{k \rightarrow \infty} \frac{\varphi(x_1[n^{-k-1}], x_2[n^{-k-1}], \dots, x_n[n^{-k-1}])}{\varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])} |H(1, \dots, 1)| \\ &= \frac{1}{\liminf_{k \rightarrow \infty} \frac{\varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])}{\varphi(x_1[n^{-k-1}], x_2[n^{-k-1}], \dots, x_n[n^{-k-1}])} |H(1, \dots, 1)|} < 1 \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in S$ by (3.9).

By ratio test we see the series Φ_2 converges for all $x_1, x_2, \dots, x_n \in S$, namely it is bounded.

It follows from the divisibility assumption that the equation $y[n^k] = x$ has a unique solution y for each fixed $x \in S$ and $k \in \mathbb{N}$. Denote this unique element y by $x[n^{-k}]$. Clearly, the mapping $x \rightarrow x[n^{-k}]$ is also an endomorphism of (S, \circ) . Replacing each x_i ($i = 1, \dots, n$) by $x[n^{-k}]$ in (3.1), and using the degree t -homogeneity of H , we get

$$|g(x[n^{1-k}]) - g(x[n^{-k}])H(1, \dots, 1)| \leq \varphi(x[n^{-k}], \dots, x[n^{-k}]) \quad (x \in S, k \in \mathbb{N}).$$

Thus

$$\begin{aligned} |g(x[n^{1-k}])H(1, \dots, 1)^{k-1} - g(x[n^{-k}])H(1, \dots, 1)^k| \\ \leq \varphi(x[n^{-k}], \dots, x[n^{-k}]) |H(1, \dots, 1)|^{k-1} \end{aligned} \tag{3.11}$$

for $x \in S, k \in \mathbb{N}$. Let $g_0 := g$ and define, for $k \in \mathbb{N}$, the function g_k by

$$g_k(x) := g(x[n^{-k}])H(1, \dots, 1)^k \quad (x \in S).$$

Then $g_k : S \rightarrow G$ and, by (3.11), as in the proof of Theorem 1, we can deduce that the sequence $g_k(x)$ is a Cauchy sequence for all fixed $x \in S$ from the boundedness of Φ_2 . Define f as the pointwise limit function of the sequence g_k . It follows from (3.11) that (3.10) satisfies by analogous way to that of Theorem 1 by taking the limit $k \rightarrow \infty$.

We show that f satisfies (1.2). Replace x_1, x_2, \dots, x_n by $x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}]$ in (3.1). Then we get, by an endomorphism of the above mapping $x \rightarrow x[n^{-k}]$,

$$\begin{aligned} & \left| g(\circ(x_1, x_2, \dots, x_n)[n^{-k}]) - H(g(x_1[n^{-k}])^{\frac{1}{t}}, g(x_2[n^{-k}])^{\frac{1}{t}}, \dots, g(x_n[n^{-k}])^{\frac{1}{t}}) \right| \\ & \leq \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}]). \end{aligned}$$

Hence, by the definition of g_k ,

$$\begin{aligned} & \left| g_k(\circ(x_1, x_2, \dots, x_n)) - H(g_k(x_1)^{\frac{1}{t}}, g_k(x_2)^{\frac{1}{t}}, \dots, g_k(x_n)^{\frac{1}{t}}) \right| \\ & \leq \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|H(1, \dots, 1)|^k \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in S, k \in \mathbb{N}$. Taking the limit $k \rightarrow \infty$, by using the continuity of H and the boundedness of Φ_2 , it follows that

$$\left| f(\circ(x_1, x_2, \dots, x_n)) - H(f(x_1)^{\frac{1}{t}}, f(x_2)^{\frac{1}{t}}, \dots, f(x_n)^{\frac{1}{t}}) \right| = 0 \quad (x_1, x_2, \dots, x_n \in S).$$

Therefore (1.2) holds and the uniqueness can be proved as similar arguments of Theorem 1. \square

THEOREM 2 —' *Let \circ be an n -times-symmetric operation with the divisibility property on S , and $H : G \times \dots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t . Assume that a function $g : S \rightarrow G$ satisfies the inequality (3.1).*

Then there exists a function $f : S \rightarrow G$ such that f is a solution of (1.2) and satisfies (3.10) with $\Phi_2(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|H(1, \dots, 1)|^{k-1} < \infty$ for all $x_1, x_2, \dots, x_n \in S$.

COROLLARY 3. *Let \circ be an n -times-symmetric operations with the divisibility property on S , and $H : G \times \dots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t such that $|H(1, \dots, 1)| < 1$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \rightarrow G$ satisfies the inequality (3.8)*

Then there exists a function $f : S \rightarrow G$ such that f is a solution of the equation (1.2) and

$$|f(x) - g(x)| \leq \frac{\varepsilon}{1 - |H(1, \dots, 1)|}.$$

Proof. Apply Theorem 2 with $\varphi(x_1, x_2, \dots, x_n) = \varepsilon$. \square

From the results of Theorem 1 and 2, for the case when $S = G$, we have the following corollaries for functional equation

$$g(H(x_1^{\frac{1}{t}}, x_2^{\frac{1}{t}}, \dots, x_n^{\frac{1}{t}})) = H(g(x_1)^{\frac{1}{t}}, g(x_2)^{\frac{1}{t}}, \dots, g(x_n)^{\frac{1}{t}}) \quad (x_1, x_2, \dots, x_n \in G). \tag{3.12}$$

COROLLARY 4. Let G be a closed multiplicative subsemigroup of \mathbb{C} and $H : G \times \cdots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t such that $|H(1, \dots, 1)| \neq 0$ and $\frac{1}{H(1, \dots, 1)} \in G$. Assume that a function $g : G \rightarrow G$ satisfies the inequality

$$\left| g(H(x_1^{\frac{1}{t}}, x_2^{\frac{1}{t}}, \dots, x_n^{\frac{1}{t}})) - H(g(x_1)^{\frac{1}{t}}, g(x_2)^{\frac{1}{t}}, \dots, g(x_n)^{\frac{1}{t}}) \right| \leq \varphi(x_1, x_2, \dots, x_n) \tag{3.13}$$

for all $x_1, x_2, \dots, x_n \in G$.

If the function φ satisfies one of the conditions (3.2), (3.3), (3.9), then there exists a unique function $f : G \rightarrow G$ such that f is a solution of (3.12) and

$$|f(x) - g(x)| \leq \begin{cases} \Phi_1(x, \dots, x) \\ \Phi_2(x, \dots, x) \end{cases} \quad (x \in G), \tag{3.14}$$

where Φ_1 and Φ_2 are the series in Theorem 1 and 2.

Proof. In order to apply Theorem 1 or 2, let $S := G$ and $\circ(x_1, x_2, \dots, x_n) := H(x_1^{\frac{1}{t}}, x_2^{\frac{1}{t}}, \dots, x_n^{\frac{1}{t}})$. By Corollary 1, \circ is a n -times-symmetric operation. In the case $|H(1, \dots, 1)| \neq 0$, the statement of Theorem 1 is fulfilled.

For the application of Theorem 2, it suffices to show that \circ satisfies the divisibility assumption. Indeed, for arbitrary $x \in G$, we can find that there exists a unique element $y \in G$ such that the equation $x = \circ(y, \dots, y)$, which is equivalent to $yH(1, \dots, 1) = x$. Since the element $\frac{1}{H(1, \dots, 1)}$ is in G , we have $y = \frac{x}{H(1, \dots, 1)} \in G$. \square

COROLLARY 4 --'. Let G be a closed multiplicative subsemigroup of \mathbb{C} and $H : G \times \cdots \times G \rightarrow G$ be a continuous G -homogeneous function of degree t such that $|H(1, \dots, 1)| \neq 0$ and $\frac{1}{H(1, \dots, 1)} \in G$. Assume that a function $g : G \rightarrow G$ satisfies the inequality (3.13).

Then there exists a unique function $f : G \rightarrow G$ such that f is a solution of (3.12), and satisfies (3.14) with the functions Φ_1 and Φ_2 are the bounded series in Theorem 1 --' and 2 --'.

COROLLARY 5. Let G be a closed multiplicative subsemigroup of \mathbb{C} , and $H : G \times \cdots \times G \rightarrow G$ be a continuous G -homogeneous of degree t function such that $|H(1, \dots, 1)| \neq \{0, 1\}$ and $\frac{1}{H(1, \dots, 1)} \in G$. Assume that a function $g : G \rightarrow G$ satisfies the inequality

$$\left| g(H(x_1^{\frac{1}{t}}, x_2^{\frac{1}{t}}, \dots, x_n^{\frac{1}{t}})) - H(g(x_1)^{\frac{1}{t}}, g(x_2)^{\frac{1}{t}}, \dots, g(x_n)^{\frac{1}{t}}) \right| \leq \varepsilon \quad (x_1, x_2, \dots, x_n \in G).$$

Then there exists a function $f : G \rightarrow G$ such that f is a solution of the equation (3.12) and

$$|f(x) - g(x)| \leq \frac{\varepsilon}{|1 - |H(1, \dots, 1)||} \quad (x_1, x_2, \dots, x_n \in G).$$

Proof. Apply Corollary 4 with $\varphi(x_1, x_2, \dots, x_n) = \varepsilon$. By separating $|H(1, \dots, 1)| > 1$ and $|H(1, \dots, 1)| < 1$ for $\Phi_1(x_1, x_2, \dots, x_n)$ and $\Phi_2(x_1, x_2, \dots, x_n)$ respectively, the proof is completed. \square

4. Applications to the stability of homogeneous functions

In this section, we shall investigate a generalization of Hyers-Ulam-Rassias stability of the functional equation

$$f(\circ(x_1, x_2, \dots, x_n)) = a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n). \quad (4.1)$$

In the case when $X = \mathbb{C}$ or $X = \mathbb{R}$ these results are also corollaries of Theorems 1 and 2 if one takes the homogeneous function $H(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$.

THEOREM 3. *Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let \circ be an n -times-symmetric operation on S . Let $a_i \in \mathbb{K}$ ($i = 1, \dots, n$) such that $|a_1 + a_2 + \dots + a_n| \neq 0$. Assume that a function $g : S \rightarrow X$ satisfies the inequality*

$$|g(\circ(x_1, x_2, \dots, x_n)) - (a_1 g(x_1) + a_2 g(x_2) + \dots + a_n g(x_n))| \leq \varphi(x_1, x_2, \dots, x_n) \quad (4.2)$$

for all $x_1, x_2, \dots, x_n \in S$.

If the function φ satisfies

$$\liminf_{k \rightarrow \infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{\varphi(x_1[n^k], x_2[n^k], \dots, x_n[n^k])} |a_1 + a_2 + \dots + a_n| > 1 \quad (4.3)$$

or

$$\limsup_{k \rightarrow \infty} \sqrt[k]{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])} < |a_1 + a_2 + \dots + a_n|, \quad (4.4)$$

then there exists a unique function $f : S \rightarrow X$ such that f is a solution of (4.1) and

$$|f(x) - g(x)| \leq \Phi_3(x, \dots, x) \quad (x \in S), \quad (4.5)$$

where $\Phi_3(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{|a_1 + a_2 + \dots + a_n|^k}$ for all $x_1, x_2, \dots, x_n \in S$.

Proof. As in the proof of Theorem 1, putting the sequence p_k by

$$p_k(x_1, x_2, \dots, x_n) = \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{|a_1 + a_2 + \dots + a_n|^k}.$$

In each cases (4.3) and (4.4), applying the ratio test to p_k and $\sqrt[k]{p_k}$ as

$$\limsup_{k \rightarrow \infty} \sqrt[k]{p_k(x_1, x_2, \dots, x_n)} = \limsup_{k \rightarrow \infty} \frac{\sqrt[k]{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}}{|a_1 + a_2 + \dots + a_n|}.$$

We note that the series Φ_3 converges in each case for all $x_1, x_2, \dots, x_n \in G$, namely it is bounded.

Substituting each x_i by x in (4.2),

$$|g(x[n]) - (a_1 + a_2 + \dots + a_n)g(x)| \leq \varphi(x, \dots, x). \tag{4.6}$$

Replacing x by $x[n^{k-1}]$ in (4.6), we get

$$\left| \frac{g(x[n^k])}{(a_1 + a_2 + \dots + a_n)^k} - \frac{g(x[n^{k-1}])}{(a_1 + a_2 + \dots + a_n)^{k-1}} \right| \leq \frac{\varphi(x[n^{k-1}], \dots, x[n^{k-1}])}{|a_1 + a_2 + \dots + a_n|^k}.$$

Let $g_0 := g$ and define, for $k \in \mathbb{N}$, the function g_k by

$$g_k(x) := \frac{g(x[n^k])}{|a_1 + a_2 + \dots + a_n|^k} \quad (x \in S).$$

We can define the function f from the limit of g_k . By employing analogous steps in the proof of Theorem 1 with $t = 1$, we can get the desired result. \square

THEOREM 3 —'. *Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let \circ be an n -times-symmetric operation on S . Let $a_i \in \mathbb{K}$ ($i = 1, \dots, n$) such that $|a_1 + a_2 + \dots + a_n| \neq 0$. Assume that a function $g : S \rightarrow X$ satisfies the inequality (4.2).*

Then there exists a unique function $f : S \rightarrow X$ such that f is a solution of (4.1), and satisfies (4.5) with $\Phi_3(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \frac{\varphi(x_1[n^{k-1}], x_2[n^{k-1}], \dots, x_n[n^{k-1}])}{|a_1 + a_2 + \dots + a_n|^k} < \infty$ for all $x_1, x_2, \dots, x_n \in S$.

COROLLARY 6. *Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let \circ be an n -times-symmetric operation on S . Let $a_i \in \mathbb{K}$ such that $|a_1 + a_2 + \dots + a_n| > 1$. Assume that, for some $\varepsilon \geq 0$ a function $g : S \rightarrow X$ satisfies the inequality*

$$|g(\circ(x_1, \dots, x_n)) - (a_1g(x_1) + a_2g(x_2) + \dots + a_n g(x_n))| \leq \varepsilon \quad (x_1, x_2, \dots, x_n \in S). \tag{4.7}$$

Then there exists a unique function $f : S \rightarrow X$ such that f is a solution of (4.1) and

$$|f(x) - g(x)| \leq \frac{\varepsilon}{|a_1 + a_2 + \dots + a_n| - 1} \quad (x \in S).$$

THEOREM 4. *Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let \circ be an n -times-symmetric operation on S with the divisibility property. Consider that a function $g : S \rightarrow X$ satisfies the inequality (4.2).*

If the function φ satisfies

$$\liminf_{k \rightarrow \infty} \frac{\varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])}{\varphi(x_1[n^{-k-1}], x_2[n^{-k-1}], \dots, x_n[n^{-k-1}])|a_1 + a_2 + \dots + a_n|} > 1, \tag{4.8}$$

then there exists a function $f : S \rightarrow X$ such that f is a solution of (4.1) and

$$|f(x) - g(x)| \leq \Phi_4(x, \dots, x) \quad (x \in S), \tag{4.9}$$

where $\Phi_4(x_1, x_2, \dots, x_n) \sum_{k=1}^{\infty} \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|a_1 + a_2 \dots + a_n|^{k-1}$ for all $x_1, x_2, \dots, x_n \in S$.

Proof. Putting the sequence p_k by

$$p_k(x_1, x_2, \dots, x_n) = \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|a_1 + a_2 \dots + a_n|^{k-1}.$$

As in the proof of Theorem 2, we note that the series Φ_4 converges for all $x_1, x_2, \dots, x_n \in S$, namely it is bounded.

Replacing each x_i by $x[n^{-k}]$ in (4.2), by an endomorphism of the mapping $x \rightarrow x[n^{-k}]$, we have

$$|g(x[n^{1-k}]) - (a_1 + a_2 + \dots + a_n)g(x[n^{-k}])| \leq \varphi(x[n^{-k}], \dots, x[n^{-k}]) \quad (x \in S, k \in \mathbb{N}).$$

Thus we find that, for all $x \in S$ and $k \in \mathbb{N}$,

$$|g(x[n^{1-k}])|(a_1 + a_2 + \dots + a_n)^{k-1} - g(x[n^{-k}])|(a_1 + a_2 + \dots + a_n)^k| \leq \varphi(x[n^{-k}], \dots, x[n^{-k}]) |a_1 + a_2 + \dots + a_n|^{k-1}. \quad (4.10)$$

Let $g_0 := g$ and define, for $k \in \mathbb{N}$, the function g_k by

$$g_k(x) := g(x[n^{-k}])|a_1 + a_2 + \dots + a_n|^k \quad (x \in S).$$

We can define the function f from the limit of g_k .

Now, using (4.10), an analogous arguments such as Theorem 2 with $t = 1$ shows that f satisfies the desired conditions of this theorem. \square

THEOREM 4 –’. *Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let \circ be an n -times-symmetric operation on S with the divisibility property. Assume that a function $g : S \rightarrow X$ satisfies the inequality (4.2).*

Then there exists a function $f : S \rightarrow X$ such that f is a solution of (4.1), and satisfies (4.9) with $\Phi_4(x_1, x_2, \dots, x_n) = \sum_{k=1}^{\infty} \varphi(x_1[n^{-k}], x_2[n^{-k}], \dots, x_n[n^{-k}])|a_1 + a_2 \dots + a_n|^{k-1} < \infty$ for all $x_1, x_2, \dots, x_n \in S$.

COROLLARY 7. *Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let S be a nonempty set and \circ be an n -times-symmetric operation on S with the divisibility property. Let $a_i \in \mathbb{K}$ such that $|a_1 + a_2 + \dots + a_n| < 1$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \rightarrow X$ satisfies the inequality (4.7). Then there exists a function $f : S \rightarrow X$ such that f is a solution of (4.1) and*

$$|f(x) - g(x)| \leq \frac{\varepsilon}{1 - |a_1 + a_2 + \dots + a_n|} \quad (x \in S).$$

From the results of Theorem 3 and 4, for the case when $S = G$, we have the following corollaries for functional equation

$$g(a_1x_1 + a_2x_2 + \dots + a_nx_n) = a_1g(x_1) + a_2g(x_2) + \dots + a_n g(x_n). \quad (4.11)$$

COROLLARY 8. Let G be a closed multiplicative subsemigroup of \mathbb{C} and $H : G \times \dots \times G \rightarrow G$ be a continuous G -homogeneous function such that $|a_1 + a_2 + \dots + a_n| \neq 0$ and $\frac{1}{a_1 + a_2 + \dots + a_n} \in G$. Assume that a function $g : G \rightarrow G$ satisfies the inequality

$$\begin{aligned} & |g(a_1x_1 + a_2x_2 + \dots + a_nx_n) - (a_1g(x_1) + a_2g(x_2) + \dots + a_ng(x_n))| \\ & \leq \varphi(x_1, x_2, \dots, x_n) \end{aligned} \tag{4.12}$$

for all $x_1, x_2, \dots, x_n \in G$.

If the function φ satisfies one of the conditions (4.3), (4.4), (4.8), then there exists a unique function $f : G \rightarrow G$ such that f is a solution of (4.11) and

$$|f(x) - g(x)| \leq \begin{cases} \Phi_3(x, \dots, x) \\ \Phi_4(x, \dots, x) \end{cases} \quad (x \in G), \tag{4.13}$$

where the functions Φ_3 and Φ_4 are the series in Theorem 3 and 4.

Proof. Apply an analogous argument such as the proof of Corollary 4 with Theorem 3 or 4. \square

COROLLARY 9. Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let $H : X \times \dots \times X \rightarrow X$ be a continuous homogeneous function. Let $a_i \in \mathbb{K}$ such that $|a_1 + a_2 + \dots + a_n| \neq 0$. Assume that a function $g : X \rightarrow X$ satisfies the inequality (4.12).

Then there exists a unique function $f : X \rightarrow X$ such that f is a solution of (4.11) and satisfies (4.13) with the functions Φ_1 and Φ_2 are the bounded series in Theorem 3 ---', 4 ---'.

COROLLARY 10. Let X be a Banach space over \mathbb{K} , where \mathbb{K} denotes the field of real or complex numbers. Let $H : X \times \dots \times X \rightarrow X$ be a continuous homogeneous function. Let $a_i \in \mathbb{K}$ such that $|a_1 + a_2 + \dots + a_n| \neq \{0, 1\}$. Assume that a function $g : X \rightarrow X$ satisfies the inequality

$$|g(a_1x_1 + a_2x_2 + \dots + a_nx_n) - (a_1g(x_1) + a_2g(x_2) + \dots + a_ng(x_n))| \leq \varepsilon$$

for all $x_1, x_2, \dots, x_n \in X$.

Then there exists a unique function $f : X \rightarrow X$ such that f is a solution of (4.7) and

$$|f(x) - g(x)| \leq \frac{\varepsilon}{|1 - |a_1 + a_2 + \dots + a_n||} \quad (x \in X).$$

REMARK. If we apply $\varphi_1(x_1, x_2, \dots, x_n) = \theta(|x_1|^t + |x_2|^t + \dots + |x_n|^t)$ into our results, then from them we can obtain the Hyers-Ulam-Rassias stability.

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