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PARTITIONED FUNCTIONAL INEQUALITIES IN BANACH MODULES AND APPROXIMATE ALGEBRA HOMOMORPHISMS

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(*communicated by Th. M. Rassias*)

Abstract. We prove the generalized Hyers-Ulam-Rassias stability of partitioned functional equations in Banach modules over a unital C^*-algebra. It is applied to show the stability of algebra homomorphisms between Banach algebras associated with partitioned functional equations in Banach algebras.

1. Partitioned functional equations

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geqslant 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th. M. Rassias [9] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\epsilon}{2-2^p}||x||^p$$

for all $x \in E_1$.

Recently, T. Trif [17, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n\ {}_{n-2}C_{k-2} f\left(\frac{x_1 + \ldots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) = k \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right)$$

for all $x_1, \ldots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Throughout this paper, let n and k be positive integers.

The following lemma is a special case of the conjecture in [8].

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LEMMA A. *Let V and W be vector spaces. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation*

$$2^n f\left(\frac{x_1 + \ldots + x_{2^n}}{2^n}\right) + 2(k-1)\sum_{i=1}^{2^{n-1}} f\left(\frac{x_{2i-1} + x_{2i}}{2}\right) \tag{A}$$
$$= k\sum_{i=1}^{2^n} f\left(\frac{x_i + \ldots + x_{i+k-1}}{k}\right)$$

for all $x_1 = x_{2^n+1}, \ldots, x_{k-1} = x_{2^n+k-1}, x_k, \ldots, x_{2^n} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that a mapping $f : V \to W$ satisfies (A). Put $x_1 = x$, $x_2 = y$ and $x_3 = \ldots = x_{2^n} = 0$ in (A). Then

$$2^n f\left(\frac{x+y}{2^n}\right) + 2(k-1)f\left(\frac{x+y}{2}\right) \tag{1.1}$$
$$= k\left[(k-1)f\left(\frac{x+y}{k}\right) + f\left(\frac{x}{k}\right) + f\left(\frac{y}{k}\right)\right].$$

Putting $y = 0$ in (1.1),

$$2^n f\left(\frac{x}{2^n}\right) + 2(k-1)f\left(\frac{x}{2}\right) = k^2 f\left(\frac{x}{k}\right). \tag{1.2}$$

Replacing x by kx and y by ky in (1.1),

$$2^n f\left(\frac{kx+ky}{2^n}\right) + 2(k-1)f\left(\frac{kx+ky}{2}\right) \tag{1.3}$$
$$= k[(k-1)f(x+y) + f(x) + f(y)].$$

Replacing x by $kx + ky$ in (1.2),

$$2^n f\left(\frac{kx+ky}{2^n}\right) + 2(k-1)f\left(\frac{kx+ky}{2}\right) = k^2 f(x+y). \tag{1.4}$$

From (1.3) and (1.4),

$$0 = -kf(x+y) + k[f(x) + f(y)].$$

Hence f is additive.

The converse is obvious. □

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $\mathscr{U}(A)$ the unitary group of A, $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. Let ${}_A\mathscr{B}$ and ${}_A\mathscr{C}$ be left Banach A-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Let d and r be positive integers.

The following lemma is useful to prove the stability of the functional equation (A).

LEMMA B [7, Theorem 1]. *Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer m greater than 2. Then there are m elements $u_1, \ldots, u_m \in \mathscr{U}(A)$ such that $ma = u_1 + \ldots + u_m$.*

The main purpose of this paper is to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra, and to prove the Hyers-Ulam-Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

2. Stability of partitioned functional equations in Banach modules over a C^*-algebra associated with its unitary group

We are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra associated with its unitary group.

From now on, let $2 \leqslant k < n$. For a given mapping $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ and a given $a \in A$, we set

$$\begin{aligned} &D_a f(x_1, \ldots, x_{2^n}) \\ &:= 2^n a f\left(\frac{x_1 + \ldots + x_{2^n}}{2^n}\right) + 2(2k-1)a \sum_{i=1}^{2^{n-1}} f\left(\frac{x_{2i-1} + x_{2i}}{2}\right) \\ &\quad - 2k \sum_{i=1}^{2^n} f\left(\frac{ax_i + \ldots + ax_{i+2k-1}}{2k}\right) \end{aligned} \tag{B}$$

for all $x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in {}_A\mathscr{B}$.

THEOREM 2.1. *Let $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathscr{B}^{2^n} \to [0, \infty)$ such that*

$$\begin{aligned} &\widetilde{\varphi}(x_1, \ldots, x_{2^n}) \\ &:= \sum_{j=0}^{\infty} 2^j \varphi\left(-\frac{x_1}{2^j}, \frac{x_2}{2^j}, \frac{x_3}{2^j}, \frac{x_4}{2^j}, \ldots, -\frac{x_{2^n-3}}{2^j}, \frac{x_{2^n-2}}{2^j}, \frac{x_{2^n-1}}{2^j}, \frac{x_{2^n}}{2^j}\right) < \infty \end{aligned} \tag{2.i}$$

and

$$\|D_u f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n}) \tag{2.ii}$$

for all $u \in \mathscr{U}(A)$ and all $x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in {}_A\mathscr{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ such that

$$\|f(x) - T(x)\| \leqslant \frac{1}{(2k-1) \cdot 2^{n-1}} \widetilde{\varphi}(x, \ldots, x) \tag{2.iii}$$

for all $x \in {}_A\mathscr{B}$.

Proof. Put $u = 1 \in \mathscr{U}(A)$. Let $x_1 = -x$, $x_2 = x_3 = x_4 = x$, $\ldots$, $x_{2^n-3} = -x$, $x_{2^n-2} = x_{2^n-1} = x_{2^n} = x$ in (2.ii). Then we get

$$\left\| 2^n f\left(\frac{x}{2}\right) + (2k-1)2^{n-1} f(x) - 2k \cdot 2^n f\left(\frac{x}{2}\right) \right\| \leqslant \varphi(-x, x, x, x, \ldots, -x, x, x, x) \tag{2.1}$$

for all $x, y \in {}_A\mathscr{B}$. So one can obtain

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leqslant \frac{1}{(2k-1)\cdot 2^{n-1}} \varphi(-x, x, x, x, \ldots, -x, x, x, x)$$

for all $x \in {}_A\mathscr{B}$. We prove by induction on j that

$$\left\| 2^j f\left(\frac{1}{2^j}x\right) - 2^{j+1} f\left(\frac{1}{2^{j+1}}x\right) \right\| \leqslant \frac{2^j}{(2k-1)\cdot 2^{n-1}} \varphi\left(-\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \ldots, -\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}\right)$$

for all $x \in {}_A\mathscr{B}$. So we get

$$\left\| f(x) - 2^j f\left(\frac{1}{2^j}x\right) \right\| \leqslant \frac{1}{(2k-1)\cdot 2^{n-1}} \sum_{m=0}^{j-1} 2^m \varphi\left(-\frac{x}{2^m}, \frac{x}{2^m}, \frac{x}{2^m}, \frac{x}{2^m}, \ldots, -\frac{x}{2^m}, \frac{x}{2^m}, \frac{x}{2^m}, \frac{x}{2^m}\right) \tag{2.2}$$

for all $x \in {}_A\mathscr{B}$.

Let x be an element in ${}_A\mathscr{B}$. For positive integers l and m with $l > m$,

$$\left\| 2^l f\left(\frac{1}{2^l}x\right) - 2^m f\left(\frac{1}{2^m}x\right) \right\| \leqslant \frac{1}{(2k-1)\cdot 2^{n-1}} \sum_{j=m}^{l-1} 2^j \varphi\left(-\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \ldots, -\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}\right),$$

which tends to zero as $m \to \infty$ by (2.i). So $\{2^j f(\frac{1}{2^j}x)\}$ is a Cauchy sequence for all $x \in {}_A\mathscr{B}$. Since ${}_A\mathscr{C}$ is complete, the sequence $\{2^j f(\frac{1}{2^j}x)\}$ converges for all $x \in {}_A\mathscr{B}$. We can define a mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ by

$$T(x) = \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}x\right) \tag{2.3}$$

for all $x \in {}_A\mathscr{B}$.

By (2.i), (2.ii) and (2.3), we get

$$\begin{aligned} \|D_1 T(x_1, \ldots, x_{2^n})\| &= \lim_{j\to\infty} 2^j \left\| D_1 f\left(\frac{1}{2^j}x_1, \ldots, \frac{1}{2^j}x_{2^n}\right) \right\| \\ &\leqslant \lim_{j\to\infty} 2^j \varphi\left(\frac{1}{2^j}x_1, \ldots, \frac{1}{2^j}x_{2^n}\right) \\ &= 0 \end{aligned}$$

for all $x_1, \ldots, x_{2^n} \in {}_A\mathscr{B}$. Hence $D_1T(x_1, \ldots, x_{2^n}) = 0$ for all $x_1, \ldots, x_{2^n} \in {}_A\mathscr{B}$. By Lemma A, T is additive. Moreover, by passing to the limit in (2.2) as $j \to \infty$, we get the inequality (2.iii).

Now let $L : {}_A\mathscr{B} \to {}_A\mathscr{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \leqslant \frac{1}{(2k-1) \cdot 2^{n-1}} \widetilde{\varphi}(x, \ldots, x)$$

for all $x \in {}_A\mathscr{B}$.

$$\begin{aligned}
\|T(x) - L(x)\| &= 2^j \left\| T\left(\frac{1}{2^j}x\right) - L\left(\frac{1}{2^j}x\right) \right\| \\
&\leqslant 2^j \left\| T\left(\frac{1}{2^j}x\right) - f\left(\frac{1}{2^j}x\right) \right\| + 2^j \left\| f\left(\frac{1}{2^j}x\right) - L\left(\frac{1}{2^j}x\right) \right\| \\
&\leqslant \frac{2}{(2k-1) \cdot 2^{n-1}} 2^j \widetilde{\varphi}\left(\frac{1}{2^j}x, \ldots, \frac{1}{2^j}x\right),
\end{aligned}$$

which tends to zero as $j \to \infty$ by (2.i). Thus $T(x) = L(x)$ for all $x \in {}_A\mathscr{B}$. This proves the uniqueness of T.

By the assumption, for each $u \in \mathscr{U}(A)$,

$$2^j \left\| D_u f\left(\frac{1}{2^j}x, \ldots, \frac{1}{2^j}x\right) \right\| \leqslant 2^j \varphi\left(\frac{1}{2^j}x, \ldots, \frac{1}{2^j}x\right)$$

for all $x \in {}_A\mathscr{B}$, and

$$2^j \left\| D_u f\left(\frac{1}{2^j}x, \ldots, \frac{1}{2^j}x\right) \right\| \to 0$$

as $j \to \infty$ for all $x \in {}_A\mathscr{B}$. So

$$D_u T(x, \ldots, x) = \lim_{j \to \infty} 2^j D_u f\left(\frac{1}{2^j}x, \ldots, \frac{1}{2^j}x\right) = 0$$

for all $u \in \mathscr{U}(A)$ and all $x \in {}_A\mathscr{B}$. Hence

$$D_u T(x, \ldots, x) = 2^n uT(x) + (2k-1)2^n uT(x) - 2k \cdot 2^n T(ux) = 0$$

for all $u \in \mathscr{U}(A)$ and all $x \in {}_A\mathscr{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathscr{U}(A)$ and all $x \in {}_A\mathscr{B}$.

Now let $a \in A$ $(a \neq 0)$ and M an integer greater than $4|a|$. Then

$$\left|\frac{a}{M}\right| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By Lemma B, there exist three elements $u_1, u_2, u_3 \in \mathscr{U}(A)$ such that $3\frac{a}{M} = u_1+u_2+u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_A\mathscr{B}$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathscr{B}$. Thus

$$\begin{aligned} T(ax) &= T\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot T\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) = \frac{M}{3}T\left(3\frac{a}{M}x\right) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}\big(T(u_1x) + T(u_2x) + T(u_3x)\big) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) \\ &= aT(x) \end{aligned}$$

for all $x \in {}_A\mathscr{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathscr{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathscr{B}$. So the unique additive mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ is an A-linear mapping, as desired. $\square$

Applying the unital C^*-algebra $\mathbb{C}$ to Theorem 2.1, one can obtain the following.

COROLLARY 2.2. *Let E_1 and E_2 be complex Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1^{2^n} \to [0, \infty)$ such that* (2.i) *and*

$$\|D_\lambda f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n})$$

for all $\lambda \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x_1, \ldots, x_{2^n} \in E_1$. Then there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{1}{(2k-1)2^{n-1}}\widetilde{\varphi}(x, \ldots, x)$$

for all $x \in E_1$.

From now on, assume that $\varphi : {}_A\mathscr{B}^{2^n} \to [0, \infty)$ is a function satisfying (2.i).

THEOREM 2.3. *Let $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ be a continuous mapping with $f(0) = 0$ such that* (2.i) *and* (2.ii) *for all $u \in \mathscr{U}(A)$ and all $x_1, \ldots, x_{2^n} \in {}_A\mathscr{B}$. If the sequence $\{2^j f(\frac{1}{2^j}x)\}$ converges uniformly on ${}_A\mathscr{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ satisfying* (2.iii).

Proof. Put $u = 1 \in \mathscr{U}(A)$. By Theorem 2.1, there exists a unique A-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ satisfying (2.iii). By the continuity of f, the uniform convergence and the definition of T, the A-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ is continuous, as desired. $\square$

THEOREM 2.4. *Let* $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *be a mapping with* $f(0) = 0$ *such that* (2.i) *and*

$$\|D_1 f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n})$$

for all $x_1, \ldots, x_{2^n} \in {}_A\mathscr{B}$. *Then there exists a unique additive mapping* $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *satisfying* (2.iii). *Further, if* $f(\lambda x)$ *is continuous in* $\lambda \in \mathbb{R}$ *for each fixed* $x \in {}_A\mathscr{B}$, *then the additive mapping* $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *is* $\mathbb{R}$*-linear.*

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ satisfying (2.iii).

Assume that $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathscr{B}$. The additive mapping T given above is similar to the additive mapping T given in the proof of [9, Theorem]. By the same reasoning as the proof of [9, Theorem], the additive mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ is $\mathbb{R}$-linear. $\square$

THEOREM 2.5. *Let* $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *be a continuous mapping with* $f(0) = 0$ *such that* (2.i) *and*

$$\|D_a f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n})$$

for all $a \in A_1^+ \cup \{i\}$ *and all* $x_1, \ldots, x_{2^n} \in {}_A\mathscr{B}$. *If the sequence* $\{2^j f(\frac{1}{2^j}x)\}$ *converges uniformly on* ${}_A\mathscr{B}$, *then there exists a unique continuous* A*-linear mapping* $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *satisfying* (2.iii).

Proof. Put $a = 1 \in A_1^+$. By Theorem 2.4, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ satisfying (2.iii). By the continuity of f and the uniform convergence, the $\mathbb{R}$-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ is continuous.

By the same reasoning as the proof of Theorem 2.1,

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$ and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements. Furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [2, Lemma 38.8]). So

$$\begin{aligned}
T(ax) &= T\left[\left(\frac{a+a^*}{2}\right)^+ x - \left(\frac{a+a^*}{2}\right)^- x + i\left(\frac{a-a^*}{2i}\right)^+ x - i\left(\frac{a-a^*}{2i}\right)^- x\right] \\
&= \left(\frac{a+a^*}{2}\right)^+ T(x) + \left(\frac{a+a^*}{2}\right)^- T(-x) + \left(\frac{a-a^*}{2i}\right)^+ T(ix) \\
&\quad + \left(\frac{a-a^*}{2i}\right)^- T(-ix) \\
&= \left(\frac{a+a^*}{2}\right)^+ T(x) - \left(\frac{a+a^*}{2}\right)^- T(x) + i\left(\frac{a-a^*}{2i}\right)^+ T(x) \\
&\quad - i\left(\frac{a-a^*}{2i}\right)^- T(x)
\end{aligned}$$

$$= \left[\left(\frac{a+a^*}{2} \right)^+ - \left(\frac{a+a^*}{2} \right)^- + i \left(\frac{a-a^*}{2i} \right)^+ - i \left(\frac{a-a^*}{2i} \right)^- \right] T(x)$$
$$= aT(x)$$

for all $a \in A$ and all $x \in {}_A\mathscr{B}$. Hence

$$T(ax+by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathscr{B}$, as desired. □

THEOREM 2.6. *Let $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ be a mapping with $f(0) = 0$ such that*

$$\|D_a f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n})$$

for all $a \in A_1^+ \cup \{i\}$ and all $x_1, \ldots, x_{2^n} \in {}_A\mathscr{B}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathscr{B}$, then there exists a unique A-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ satisfying (2.iii).

Proof. Put $a = 1 \in A_1^+$. By Theorem 2.4, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ satisfying (2.iii).

The rest of the proof is similar to the proof of Theorem 2.5. □

For a given mapping $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ and a given $a \in A$, we set

$$\begin{aligned} &D_a f(x_1, \ldots, x_{2^n}) \\ &:= \frac{d}{r} af\left(\frac{rx_1 + \ldots + rx_{2^n}}{d} \right) + 2(2k-1) \sum_{i=1}^{2^{n-1}} af\left(\frac{x_{2i-1} + x_{2i}}{2} \right) \\ &\quad - 2k \sum_{i=1}^{2^n} f\left(\frac{ax_i + \ldots + ax_{i+2k-1}}{2k} \right) \end{aligned}$$

for all $x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in {}_A\mathscr{B}$.

THEOREM 2.7. *Let $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathscr{B}^{2^n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \ldots, x_{2^n}) := \sum_{j=0}^{\infty} \left(\frac{d}{2^n r} \right)^j \varphi \left[\left(\frac{2^n r}{d} \right)^j x_1, \ldots, \left(\frac{2^n r}{d} \right)^j x_{2^n} \right] < \infty \tag{2.i}$$

$$\|D_u f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n}) \tag{2.ii}$$

for all $u \in \mathscr{U}(A)$ and all $x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in {}_A\mathscr{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ such that

$$\|f(x) - T(x)\| \leqslant \frac{1}{2^n} \widetilde{\varphi}(x, \ldots, x) \tag{2.iii}$$

for all $x \in {}_A\mathscr{B}$.

Proof. The proof is similar to Theorem 2.1 in [8]. □

THEOREM 2.8. *Let* $f : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *be a mapping with* $f(0) = 0$ *and*

$$D_u f(x_1, \ldots, x_{2^n}) := 2^n uf\left(\frac{x_1 + \ldots + x_{2^n}}{2^n}\right) + 2(2k-1)\sum_{i=1}^{2^{n-1}} uf\left(\frac{x_{2i-1} + x_{2i}}{2}\right)$$
$$- \frac{d}{r}\sum_{i=1}^{2^n} f\left(\frac{rux_i + \ldots + rux_{i+2k-1}}{d}\right)$$

for all $u \in \mathscr{U}(A)$ *and all* $x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in {}_A\mathscr{B}$. *Assume that there exists a function* $\varphi : {}_A\mathscr{B}^{2^n} \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x_1, \ldots, x_{2^n}) := \sum_{j=0}^{\infty}\left(\frac{d}{2kr}\right)^j \varphi\left[\left(\frac{2kr}{d}\right)^j x_1, \ldots, \left(\frac{2kr}{d}\right)^j x_{2^n}\right] < \infty \tag{2.iv}$$

and

$$\|D_u f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n}) \tag{2.v}$$

for all $u \in \mathscr{U}(A)$ *and all* $x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in {}_A\mathscr{B}$. *Then there exists a unique* A*-linear mapping* $T : {}_A\mathscr{B} \to {}_A\mathscr{C}$ *such that*

$$\|f(x) - T(x)\| \leqslant \frac{1}{2k \cdot 2^n}\widetilde{\varphi}(x, \ldots, x) \tag{2.vi}$$

for all $x \in {}_A\mathscr{B}$.

Proof. The proof is similar to Theorem 2.7 in [8]. □

For the cases corresponding to Theorem 2.7 and Theorem 2.8, one can obtain similar results to Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6 with $\frac{2^n r}{d} \notin \mathbb{Z}$.

3. Stability of partitioned functional equations in Banach algebras and approximate algebra homomorphisms

In this section, let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

D. G. Bourgin [3] proved the stability of ring homomorphisms between Banach algebras. In [1], R. Badora generalized the Bourgin's result.

We prove the generalized Hyers-Ulam-Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

THEOREM 3.1. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be real Banach algebras, and* $f : \mathscr{A} \to \mathscr{B}$ *a mapping with* $f(0) = 0$ *for which there exist functions* $\varphi : \mathscr{A}^{2^n} \to [0, \infty)$ *and* $\psi : \mathscr{A} \times \mathscr{A} \to [0, \infty)$ *such that* (2.i),

$$\|D_1 f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n}), \tag{3.i}$$

$$\widetilde{\psi}(x,y) := \sum_{j=0}^{\infty} 2^j \psi\left(\frac{1}{2^j}x, y\right) < \infty \tag{3.ii}$$

and

$$\|f(x \cdot y) - f(x)f(y)\| \leqslant \psi(x,y) \tag{3.iii}$$

for all $x, y, x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in \mathscr{A}$, *where* D_1 *is in* (B). *If* $f(\lambda x)$ *is continuous in* $\lambda \in \mathbb{R}$ *for each fixed* $x \in \mathscr{A}$, *then there exists a unique algebra homomorphism* $T : \mathscr{A} \to \mathscr{B}$ *satisfying* (2.iii). *Further, if* $\mathscr{A}$ *and* $\mathscr{B}$ *are unital, then* f *itself is an algebra homomorphism.*

Proof. By the same method as the proof of Theorem 2.4, one can show that there exists a unique $\mathbb{R}$-linear mapping $T : \mathscr{A} \to \mathscr{B}$ satisfying (2.iii). The $\mathbb{R}$-linear mapping $T : \mathscr{A} \to \mathscr{B}$ was given by

$$T(x) = \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}x\right)$$

for all $x \in \mathscr{A}$. Let

$$R(x,y) = f(x \cdot y) - f(x)f(y)$$

for all $x, y \in \mathscr{A}$. By (3.ii), we get

$$\lim_{j\to\infty} 2^j R\left(\frac{1}{2^j}x, y\right) = 0$$

for all $x, y \in \mathscr{A}$. So

$$\begin{aligned} T(x \cdot y) &= \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}(x \cdot y)\right) = \lim_{j\to\infty} 2^j f\left[\left(\frac{1}{2^j}x\right) y\right] \\ &= \lim_{j\to\infty} 2^j \left[f\left(\frac{1}{2^j}x\right) f(y) + R\left(\frac{1}{2^j}x, y\right)\right] = T(x)f(y) \end{aligned} \tag{3.1}$$

for all $x, y \in \mathscr{A}$. Thus

$$\begin{aligned} T(x)f\left(\frac{1}{2^j}y\right) &= T\left[x\left(\frac{1}{2^j}y\right)\right] = T\left[\left(\frac{1}{2^j}x\right) y\right] = T\left(\frac{1}{2^j}x\right) f(y) \\ &= \frac{1}{2^j} T(x)f(y) \end{aligned}$$

for all $x, y \in \mathscr{A}$. Hence

$$T(x) 2^j f\left(\frac{1}{2^j}y\right) = T(x)f(y) \tag{3.2}$$

for all $x, y \in \mathscr{A}$. Taking the limit in (3.2) as $j \to \infty$, we obtain

$$T(x)T(y) = T(x)f(y)$$

for all $x, y \in \mathscr{A}$. Therefore,

$$T(x \cdot y) = T(x)T(y)$$

for all $x, y \in \mathscr{A}$. So $T : \mathscr{A} \to \mathscr{B}$ is an algebra homomorphism.

Now assume that $\mathscr{A}$ and $\mathscr{B}$ are unital. By (3.1),

$$T(y) = T(1 \cdot y) = T(1)f(y) = f(y)$$

for all $y \in \mathscr{A}$. So $f : \mathscr{A} \to \mathscr{B}$ is an algebra homomorphism, as desired. □

THEOREM 3.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be complex Banach algebras. Let $f : \mathscr{A} \to \mathscr{B}$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathscr{A}^{2^n} \to [0, \infty)$ and $\psi : \mathscr{A} \times \mathscr{A} \to [0, \infty)$ such that* (2.i), (3.ii), (3.iii) *and*

$$\|D_\lambda f(x_1, \ldots, x_{2^n})\| \leqslant \varphi(x_1, \ldots, x_{2^n}) \tag{3.iv}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in \mathscr{A}$, where D_λ is in (B). *Then there exists a unique algebra homomorphism $T : \mathscr{A} \to \mathscr{B}$ satisfying* (2.iii). *Further, if $\mathscr{A}$ and $\mathscr{B}$ are unital, then f itself is an algebra homomorphism.*

Proof. Under the assumption (2.i) and (3.iv), in Corollary 2.2, we showed that there exists a unique $\mathbb{C}$-linear mapping $T : \mathscr{A} \to \mathscr{B}$ satisfying (2.iii).

The rest of the proof is the same as the proof of Theorem 3.1. □

THEOREM 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be complex Banach $*$-algebras. Let $f : \mathscr{A} \to \mathscr{B}$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathscr{A}^{2^n} \to [0, \infty)$ and $\psi : \mathscr{A} \times \mathscr{A} \to [0, \infty)$ such that* (2.i), (3.ii), (3.iii), (3.iv) *and*

$$\|f(x^*) - f(x)^*\| \leqslant \varphi(x, \ldots, x) \tag{3.v}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, x_1 = x_{2^n+1}, \ldots, x_{2k-1} = x_{2^n+2k-1}, x_{2k}, \ldots, x_{2^n} \in \mathscr{A}$. Then there exists a unique $*$-algebra homomorphism $T : \mathscr{A} \to \mathscr{B}$ satisfying (2.iii). *Further, if $\mathscr{A}$ and $\mathscr{B}$ are unital, then f itself is a $*$-algebra homomorphism.*

Proof. By the same reasoning as the proof of Theorem 3.2, there exists a unique $\mathbb{C}$-linear mapping $T : \mathscr{A} \to \mathscr{B}$ satisfying (2.iii).

Now

$$2^j \left\| f\left(\frac{1}{2^j}x^*\right) - f\left(\frac{1}{2^j}x\right)^* \right\| \leqslant 2^j \varphi\left(\frac{1}{2^j}x, \ldots, \frac{1}{2^j}x\right)$$

for all $x \in \mathscr{A}$. Thus

$$2^j \left\| f\left(\frac{1}{2^j}x^*\right) - f\left(\frac{1}{2^j}x\right)^* \right\| \to 0$$

as $j \to \infty$ for all $x \in \mathscr{A}$. Hence

$$T(x^*) = \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}x^*\right) = \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}x\right)^* = T(x)^*$$

for all $x \in \mathscr{A}$.

The rest of the proof is the same as the proof of Theorem 3.1. □

Similarly, for the cases corresponding to Theorem 2.7 and Theorem 2.8, one can obtain similar results to the theorems given above with $\frac{2^n r}{d} \notin \mathbb{Z}$.

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