

HARDY'S DISCRETE INEQUALITY

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Abstract. Copson's extension of Hardy's discrete inequality has been generalised in different directions by Hwang, Hwang–Yang and Pachpatte. In this paper we obtain inequalities which subsume and extend results of Pachpatte (*Libertas Math.* **14** (1994), 151–157) and Hwang–Yang (*Tamkang J. Math.* **27** (1996), 125–132).

1. Introduction

In [1] Copson established the following Hardy type inequalities involving a series of positive terms (see [3, Inequality 326] and [6, p. 145]).

THEOREM 1.1. *If $p > 1$, $\lambda_n > 0$, $a_n > 0$, $\Lambda_n = \sum_{i=1}^n \lambda_i$, $A_n = \sum_{i=1}^n \lambda_i a_i$ and $\sum_{n=1}^{\infty} \lambda_n a_n^p$ converges, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \quad (1.1)$$

The constant is the best possible.

THEOREM 1.2. *Let p_n , λ_n , a_n , Λ_n and A_n be as in Theorem 1.1 and let $H(u)$ be a real-valued positive convex function defined for $u > 0$. If $\sum_{n=1}^{\infty} \lambda_n H^{p_n}(a_n)$ converges, then*

$$\sum_{n=1}^{\infty} \lambda_n H^{p_n} \left(\frac{A_n}{\Lambda_n} \right) \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n H^{p_n}(a_n). \quad (1.2)$$

The constant is the best possible.

Generalizations of these theorems were given by Pachpatte [7] and Hwang and Yang [5]. We establish inequalities which subsume and extend those results. We also correct a proof of [5, Theorem 1].

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2. Main results

Our first result extends Copson’s inequality stated in our Theorem 1.1 and the inequality of Hwang and Yang [5, Theorem 1].

THEOREM 2.1. *Let $p > 1$, $q \geq 0$, $\beta_n > 0$, $\lambda_n > 0$, $a_n > 0$ for all $n \in \mathbb{N}$, and define*

$$\Lambda_n = \sum_{i=1}^n \beta_i \lambda_i, \quad A_n = \sum_{i=1}^n \beta_i \lambda_i a_i, \quad \rho_n = \frac{(\beta_{n+1} - \beta_n) \Lambda_n}{\beta_{n+1} \beta_n \lambda_n}, \quad n \in \mathbb{N}. \tag{2.1}$$

Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^p (A_n / \Lambda_n)^q$ converges and that $\sigma = \inf_n \rho_n > 1 - p - q$. Then

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{p+q} \leq \left(\frac{p+q}{p+q-1+\sigma} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p \left(\frac{A_n}{\Lambda_n} \right)^q. \tag{2.2}$$

Proof. The first step is to show that the series on the left in 2.2 converges. This will fill in the gap in the proof of [5, Theorem 1]. Set $\omega_n = A_n / \Lambda_n$. Following the first part of the proof of [5, Theorem 1] with p replaced by $p + q$, we obtain

$$\sum_{n=1}^N (p+q-1) \lambda_n \omega_n^{p+q} + \sum_{n=1}^{N-1} \lambda_n \rho_n \omega_n^{p+q} \leq (p+q) \sum_{n=1}^N \lambda_n a_n \omega_n^{p+q-1}. \tag{2.3}$$

Let $\kappa = (p+q)/(p+q+\sigma-1)$ and $\mu = \max\{\kappa, (p+q)/(p+q-1)\}$. We find a lower estimate of the left hand side of this inequality:

$$\begin{aligned} \sum_{n=1}^N (p+q-1) \lambda_n \omega_n^{p+q} + \sum_{n=1}^{N-1} \rho_n \lambda_n \omega_n^{p+q} &= (p+q-1) \lambda_N \omega_N^{p+q} + \sum_{n=1}^{N-1} (p+q-1+\rho_n) \lambda_n \omega_n^{p+q} \\ &\geq \frac{p+q}{\mu} \sum_{n=1}^N \lambda_n \omega_n^{p+q}; \end{aligned}$$

combining this with 2.3, we have

$$\sum_{n=1}^N \lambda_n \omega_n^{p+q} \leq \mu \sum_{n=1}^N \lambda_n a_n \omega_n^{p+q-1}.$$

Applying Hölder’s inequality with indices p and $p/(p-1)$, we get

$$\sum_{n=1}^N \lambda_n \omega_n^{p+q} \leq \mu \sum_{n=1}^N \lambda_n a_n \omega_n^{p+q-1} \leq \mu \left\{ \sum_{n=1}^N \lambda_n a_n^p \omega_n^q \right\}^{1/p} \left\{ \sum_{n=1}^N \lambda_n \omega_n^{p+q} \right\}^{(p-1)/p}. \tag{2.4}$$

Dividing by the last factor on the right and raising to the power of p , we get

$$\sum_{n=1}^N \lambda_n \omega_n^{p+q} \leq \mu^p \sum_{n=1}^N \lambda_n a_n^p \omega_n^q.$$

This proves that the series $\sum_{n=1}^\infty \lambda_n \omega_n^{p+q}$ converges. In view of 2.4, the series $\sum_{n=1}^\infty \lambda_n a_n \omega_n^{p+q-1}$ also converges. Returning to 2.3 and observing that the term $(p + q - 1)\lambda_n \omega_n$ is nonnegative, we see that

$$\sum_{n=1}^{N-1} (p + q - 1 + \rho_n) \lambda_n \omega_n^{p+q} \leq (p + q) \sum_{n=1}^N \lambda_n a_n \omega_n^{p+q-1},$$

that is,

$$\sum_{n=1}^{N-1} \lambda_n \omega_n^{p+q} \leq \kappa \sum_{n=1}^N \lambda_n a_n \omega_n^{p+q-1}.$$

Since the series on both sides converge, we have

$$\sum_{n=1}^\infty \lambda_n \omega_n^{p+q} \leq \kappa \sum_{n=1}^\infty \lambda_n a_n \omega_n^{p+q-1}.$$

Applying Hölder's inequality to the right hand side in a similar way we did above for the finite sums, we obtain 2.2. \square

Setting $q = 0$ in the preceding theorem, we get a version of [5, Theorem 1] with the constant κ given explicitly as $\kappa = p/(p - 1 + \sigma)$.

COROLLARY 2.2. *Let $p > 1$, $\beta_n > 0$, $\lambda_n > 0$, $a_n > 0$ for all $n \in \mathbb{N}$, and define*

$$A_n = \sum_{i=1}^n \beta_i \lambda_i a_i, \quad \Lambda_n = \sum_{i=1}^n \beta_i \lambda_i, \quad \rho_n = \frac{(\beta_{n+1} - \beta_n) \Lambda_n}{(\beta_{n+1} \beta_n) \lambda_n}, \quad n \in \mathbb{N}. \tag{2.5}$$

Suppose that $\sum_{n=1}^\infty \lambda_n a_n^p$ converges and that $\sigma = \inf_n \rho_n > 1 - p$. Then

$$\sum_{n=1}^\infty \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \leq \left(\frac{p}{p - 1 + \sigma} \right)^p \sum_{n=1}^\infty \lambda_n a_n^p. \tag{2.6}$$

REMARK 2.3. In the proof of [5, Theorem 1], Hwang and Yang use the inequality

$$\sum_{n=1}^{N-1} \lambda_n \omega_n^p \leq \kappa \sum_{n=1}^N \lambda_n a_n \omega_n^{p-1}, \quad N \in \mathbb{N}, \tag{2.7}$$

(for a positive κ) to prove the desired inequality by taking the limit as $N \rightarrow \infty$. However, the convergence of the two series is not established in [5]. This fact is nontrivial and depends on the relation between the two series.

From Theorem 2.1 we can deduce the following more general result.

THEOREM 2.4. *Let $p > 1$, $q > 0$, $r \geq 0$, $a_n > 0$, $\lambda_n > 0$, $\beta_n > 0$ for all $n \in \mathbb{N}$, and let A_n , Λ_n and ρ_n be as in 2.1. Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} (A_n/\Lambda_n)^r$ converges and that $\sigma = \inf_n \rho_n > 1 - p - q - r$. Then*

$$\sum_{n=1}^{\infty} \lambda_n a_n^p \left(\frac{A_n}{\Lambda_n}\right)^{q+r} \leq \left(\frac{p+q+r}{p+q+r-1+\sigma}\right)^q \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \left(\frac{A_n}{\Lambda_n}\right)^r. \tag{2.8}$$

Proof. Set $\omega_n = A_n/\Lambda_n$ and $\kappa = (p+q+r)/(p+q+r-1+\sigma)$. We apply Hölder’s inequality with indices $(p+q)/p$ and $(p+q)/q$ (for this reason we need to assume $q > 0$ rather than $q \geq 0$ as we did in Theorem 2.1), and then we apply Theorem 2.1 with q replaced by $q+r$:

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r} &= \sum_{n=1}^{\infty} (\lambda_n a_n^{p+q} \omega_n^r)^{p/(p+q)} (\lambda_n \omega_n^{p+q+r})^{q/(p+q)} \\ &\leq \left(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)} \left(\sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q+r}\right)^{q/(p+q)} \\ &\leq \left(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)} \left(\kappa^p \sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r}\right)^{q/(p+q)}, \end{aligned}$$

that is,

$$\left(\sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r}\right)^{p/(p+q)} \leq \kappa^{pq/(p+q)} \left(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)}. \tag{2.9}$$

Raising both sides of 2.9 to the power $(p+q)/p$ yields 2.8. \square

Setting $r = 0$ in the preceding theorem, we get

COROLLARY 2.5. *Let $p > 1$, $q > 0$, $a_n > 0$, $\lambda_n > 0$, $\beta_n > 0$ for all $n \in \mathbb{N}$, and let A_n , Λ_n and ρ_n be as in 2.1. Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^{p+q}$ converges and that $\sigma = \inf_n \rho_n > 1 - p - q$. Then*

$$\sum_{n=1}^{\infty} \lambda_n a_n^p \left(\frac{A_n}{\Lambda_n}\right)^q \leq \left(\frac{p+q}{p+q-1+\sigma}\right)^q \sum_{n=1}^{\infty} \lambda_n a_n^{p+q}. \tag{2.10}$$

Proof. Inequality 2.10 follows from 2.8 by setting $r = 0$. \square

We now present a multilevel version of Corollary 2.2.

THEOREM 2.6. *Let $p > 1$, $m \in \mathbb{N}$, $a_j > 0$, $\beta_{ij} > 0$, $\lambda_j > 0$ for $i \in \{1, \dots, m\}$ and all $j \in \mathbb{N}$. For $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$ define*

$$A_{0n} = a_n, \quad A_{in} = \sum_{j=1}^n \beta_{ij} \lambda_j A_{i-1,j}, \quad \Lambda_{in} = \sum_{j=1}^n \beta_{ij} \lambda_j, \quad \rho_{in} = \frac{(\beta_{i,n+1} - \beta_{in}) \Lambda_{in}}{(\beta_{i,n+1} \beta_{in}) \lambda_n}.$$

Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^p$ converges and $\sigma_i = \inf_n \rho_{in} > 1 - p$, $i \in \{1, \dots, m\}$. Then

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_{mn}}{\Lambda_{mn}} \right)^p \leq \left(\prod_{i=1}^m \frac{p}{p-1+\sigma_i} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{2.11}$$

Proof. Let $\omega_{in} = A_{in}/\Lambda_{in}$, $\kappa_i = p/(p-1+\sigma_i)$, $i \in \{0, 1, \dots, m\}$, $n \in \mathbb{N}$. Applying Corollary 2.2, we conclude that $\sum_{n=1}^{\infty} \lambda_n \omega_{1n}^p$ converges, and

$$\sum_{n=1}^{\infty} \lambda_n \omega_{1n}^p \leq \kappa_1^p \sum_{n=1}^{\infty} \lambda_n \omega_{0n}^p.$$

Continuing this way, we show that

$$\sum_{n=1}^{\infty} \lambda_n \omega_{in}^p \leq \kappa_i^p \sum_{n=1}^{\infty} \lambda_n \omega_{i-1,n}^p, \quad i \in \{1, \dots, m\}.$$

Applying this inequality m times starting with $i = m$, we obtain 2.11. \square

REMARK 2.7. Setting $\beta_{ij} = 1$, $\lambda_j = 1$ for $i \in \{1, \dots, m\}$, $j \in \mathbb{N}$, we have $\sigma_i = 0$ for all i in the preceding theorem, and recover Pachpatte's result [7, Theorem 1].

THEOREM 2.8. Let H be a real-valued positive convex function defined on $(0, \infty)$ and let $p > 1$, $q \geq 0$, $\beta_i > 0$, $\lambda_i > 0$, $a_i > 0$ for all $i \in \mathbb{N}$. Let A_n , Λ_n and ρ_n be as in 2.1, and let

$$F_n = \sum_{i=1}^n \beta_i \lambda_i H(a_i), \quad n \in \mathbb{N}. \tag{2.12}$$

Suppose that $\sum_{n=1}^{\infty} \lambda_n H^p(a_n) (F_n/\Lambda_n)^q$ converges and that $\sigma = \inf_n \rho_n > 1 - p - q$. Then

$$\sum_{n=1}^{\infty} \lambda_n H^{p+q}(A_n/\Lambda_n) \leq \left(\frac{p+q}{p+q-1+\sigma} \right)^p \sum_{n=1}^{\infty} \lambda_n H^p(a_n) (F_n/\Lambda_n)^q. \tag{2.13}$$

Proof. Write $\Phi_n = F_n/\Lambda_n$, $n \in \mathbb{N}$, $\kappa = (p+q)/(p+q-1+\sigma)$. We apply Theorem 2.1 with a_i replaced by $H(a_i)$. Then A_n is replaced by F_n , and $\omega_n = A_n/\Lambda_n$ by Φ_n . Inequality 2.2 then becomes

$$\sum_{n=1}^{\infty} \lambda_n \Phi_n^{p+q} \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n H^p(a_n) \Phi_n^q. \tag{2.14}$$

Since H is convex, we can apply Jensen's inequality to obtain

$$H(\omega_n) = H\left(\frac{A_n}{\Lambda_n}\right) = H\left(\sum_{i=1}^n \frac{\beta_i \lambda_i}{\Lambda_n} a_i\right) \leq \sum_{i=1}^n \frac{\beta_i \lambda_i}{\Lambda_n} H(a_i) = \Phi_n.$$

Substituting this in 2.14, we get 2.13. \square

The choice $H(u) = u$ in the preceding theorem yields Theorem 2.1. Setting $q = 0$, we recover [5, Theorem A]. If $q = 0$, $\beta_i = 1$ and $\sigma = 0$, the preceding theorem reduces to Copson’s result in [1] (see our Theorem 1.2).

The next result generalizes Theorem 2.8 in the same way that Theorem 2.4 generalizes Theorem 2.1. The proof is left to the reader.

THEOREM 2.9. *Let H be a real-valued positive convex function defined on $(0, \infty)$ and let $p > 1$, $q > 0$, $r \geq 0$, $\beta_i > 0$, $\lambda_i > 0$, $a_i > 0$ for all $i \in \mathbb{N}$. Let A_n , Λ_n and ρ_n be as in 2.1, and F_n as in 2.12. Suppose that $\sum_{n=1}^{\infty} \lambda_n H^r(a_n)(F_n/\Lambda_n)^{p+q}$ converges and that $\sigma = \inf_n \rho_n > 1 - p - q - r$. Then*

$$\sum_{n=1}^{\infty} \lambda_n H^{q+r}(A_n/\Lambda_n)(F_n/\Lambda_n)^p \leq \left(\frac{p+q+r}{p+q+r-1+\sigma} \right)^p \sum_{n=1}^{\infty} \lambda_n H^r(a_n)(F_n/\Lambda_n)^{p+q}. \tag{2.15}$$

3. Further inequalities

In this section we obtain extensions of Pachpatte’s results from [7].

THEOREM 3.1. *Let $p > 1$, $q \geq 0$, $b_{mn} > 0$ for $m, n \in \mathbb{N}$ and let*

$$B_{mn} = \frac{1}{mn} \sum_{s=1}^m \sum_{t=1}^n \frac{1}{st} \sum_{i=1}^s \sum_{j=1}^t b_{ij} \quad \text{for } m, n, i, j \in \mathbb{N}. \tag{3.1}$$

If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^p B_{mn}^q$ converges, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}^{p+q} \leq \left(\frac{p}{p-1} \right)^{3p} \left(\frac{p+q}{p+q-1} \right)^p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^p B_{mn}^q. \tag{3.2}$$

Proof. The proof is an adaptation of the proof of [7, Theorem 2], and extends an idea used by Elliott in [2].

Let $L, M \in \mathbb{N}$ and define

$$\omega_{mn} = mB_{mn} = \frac{1}{n} \sum_{t=1}^n \frac{1}{t} \sum_{s=1}^m \frac{1}{s} \sum_{i=1}^s \sum_{j=1}^t b_{ij}, \tag{3.3}$$

and

$$S_{ML} = \sum_{m=1}^M \sum_{n=1}^L B_{mn}^{p+q} = \sum_{m=1}^M m^{-p-q} \sum_{n=1}^L \omega_{mn}^{p+q}. \tag{3.4}$$

Thus the left hand side of 3.2 may be written as

$$\sum_{m=1}^{\infty} m^{-p-q} \sum_{n=1}^{\infty} \omega_{mn}^{p+q}. \tag{3.5}$$

By a procedure similar to the one used in [7], we obtain

$$\begin{aligned} \sum_{n=1}^L \omega^{p+q}_{mn} - \left(\frac{p+q}{p+q-1}\right) \sum_{n=1}^L \left\{ \frac{1}{n} \sum_{s=1}^m \frac{1}{s} \sum_{i=1}^s \sum_{j=1}^i b_{ij} \right\} \omega^{p+q-1}_{mn} \\ = \left(\frac{1}{p+q-1}\right) \sum_{n=1}^L \left\{ (n-1)\omega^{p+q}_{m,n-1} - n\omega^{p+q}_{mn} \right\} \\ = -\left(\frac{1}{p+q-1}\right) L\omega^{p+q}_{m,L} \leq 0. \end{aligned} \tag{3.6}$$

This is [7, Equation (24)] with p replaced by $p+q$. From 3.6, using Hölder's inequality with indices p and $p/(p-1)$, we obtain

$$\begin{aligned} \sum_{n=1}^L \omega^{p+q}_{mn} &\leq \left(\frac{p+q}{p+q-1}\right) \sum_{n=1}^L \left\{ \frac{1}{n} \sum_{s=1}^m \frac{1}{s} \sum_{i=1}^s \sum_{j=1}^i b_{ij} \right\} \omega^{p+q-1}_{mn} \\ &\leq \left(\frac{p+q}{p+q-1}\right) \left\{ \sum_{n=1}^L \left\{ \omega^{q/p}_{mn} \left\{ \frac{1}{n} \sum_{s=1}^m \frac{1}{s} \sum_{i=1}^s \sum_{j=1}^i b_{ij} \right\} \right\}^p \right\}^{1/p} \\ &\quad \cdot \left\{ \sum_{n=1}^L \left\{ \omega^{p+q-1-q/p}_{mn} \right\}^{p/(p-1)} \right\}^{(p-1)/p}. \end{aligned} \tag{3.7}$$

Dividing both sides of 3.7 by the last term on the right and raising to the p th power, we have

$$\begin{aligned} \sum_{n=1}^L \omega^{p+q}_{mn} &\leq \left(\frac{p+q}{p+q-1}\right)^p \sum_{n=1}^L \omega^q_{mn} \left\{ \frac{1}{n} \sum_{s=1}^m \frac{1}{s} \sum_{i=1}^s \sum_{j=1}^i b_{ij} \right\}^p \\ &= \left(\frac{p+q}{p+q-1}\right)^p \sum_{n=1}^L \omega^q_{mn} m^p n^{-p} \beta^p_{mn}. \end{aligned} \tag{3.8}$$

If we define

$$\beta_{mn} = \frac{1}{m} \sum_{s=1}^m \frac{1}{s} \sum_{i=1}^s \sum_{j=1}^i b_{ij}, \tag{3.9}$$

then from 3.4 and 3.8,

$$\begin{aligned} S_{ML} &\leq \left(\frac{p+q}{p+q-1}\right)^p \sum_{n=1}^L n^{-p} \sum_{m=1}^M m^{-q} \omega^q_{mn} \beta^p_{mn} \\ &= \left(\frac{p+q}{p+q-1}\right)^p \sum_{m=1}^M m^{-q} \sum_{n=1}^L n^{-p} \omega^q_{mn} \beta^p_{mn}, \end{aligned} \tag{3.10}$$

and

$$m\beta_{mn} - (m-1)\beta_{m-1,n} = \frac{1}{m} \sum_{i=1}^m \sum_{j=1}^n b_{ij} \geq 0, \tag{3.11}$$

so that $m\beta_{mn}$ is an increasing function of m . We note that

$$\omega_{mn} - \omega_{m-1,n} = \frac{1}{n} \sum_{t=1}^n \frac{1}{mt} \sum_{i=1}^m \sum_{j=1}^t b_{ij} \geq 0, \quad (3.12)$$

$$\beta_{mn} - \beta_{m,n-1} = \frac{1}{m} \sum_{i=1}^m b_{in} \geq 0. \quad (3.13)$$

From 3.9 and using the inequality

$$u^{k+1} + kv^{k+1} \geq (k+1)uv^k, \quad u, v \geq 0, k \geq 1, \quad (3.14)$$

we deduce that

$$\begin{aligned} & m^{-q} \omega_{mn}^q \beta_{mn}^p - \left(\frac{p}{p-1} \right) m^{-q} \omega_{mn}^q \left\{ \frac{1}{m} \sum_{i=1}^m \sum_{j=1}^n b_{ij} \right\} \beta_{mn}^{p-1} \\ &= m^{-q} \omega_{mn}^q \beta_{mn}^p - \left(\frac{p}{p-1} \right) m^{-q} \omega_{mn}^q (m\beta_{mn} - (m-1)\beta_{m-1,n}) \beta_{mn}^{p-1} \\ &= \left\{ 1 - \left(\frac{p}{p-1} \right) m \right\} m^{-q} \omega_{mn}^q \beta_{mn}^p \\ &\quad + \left(\frac{p}{p-1} \right) (m-1) m^{-q} \omega_{mn}^q \beta_{m-1,n} \beta_{mn}^{p-1} \\ &\leq \left\{ 1 - m - \left(\frac{1}{p-1} \right) m \right\} m^{-q} \omega_{mn}^q \beta_{mn}^p \\ &\quad + \left(\frac{p}{p-1} \right) (m-1) \frac{1}{p} m^{-q} \omega_{mn}^q \left(\beta_{m-1,n}^p + (p-1) \beta_{mn}^p \right) \\ &= \left(\frac{1}{p-1} \right) m^{-q} \omega_{mn}^q \left((m-1) \beta_{m-1,n}^p - m \beta_{mn}^p \right) \leq 0 \end{aligned} \quad (3.15)$$

by 3.11. Keeping n fixed in 3.15 and letting $m = 1, 2, \dots, M$ and adding the inequalities we have

$$\begin{aligned} & \sum_{m=1}^M m^{-q} \omega_{mn}^q \beta_{mn}^p - \left(\frac{p}{p-1} \right) \sum_{m=1}^M m^{-q} \omega_{mn}^q \left\{ \frac{1}{m} \sum_{i=1}^m \sum_{j=1}^n b_{ij} \right\} \beta_{mn}^{p-1} \\ &\leq \left(\frac{p}{p-1} \right) \sum_{m=1}^M m^{-q} \omega_{mn}^q \left((m-1) \beta_{m-1,n}^p - m \beta_{mn}^p \right) \leq 0 \end{aligned} \quad (3.16)$$

(the sum of negative terms, by 3.11). Let

$$\gamma_{mn} = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^m b_{ij}$$

so that

$$n\gamma_{mn} - (n - 1)\gamma_{m,n-1} = \sum_{i=1}^m b_{in} \geq 0. \tag{3.17}$$

Following similar procedures as used to derive 3.16 from 3.6 with a further application of Hölder's inequality we arrive at

$$S_{ML} \leq \left(\frac{p}{p-1}\right)^p \left(\frac{p+q}{p+q-1}\right)^p \sum_{m=1}^M m^{-p-q} \sum_{n=1}^L \omega_{mn}^q \gamma_{mn}^p \tag{3.18}$$

and

$$\sum_{n=1}^L \omega_{mn}^q \gamma_{mn}^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^L \omega_{mn}^q \left\{ \sum_{i=1}^m b_{in} \right\}^p. \tag{3.19}$$

From 3.18 and 3.19 we observe that

$$\begin{aligned} S_{ML} &\leq \left(\frac{p}{p-1}\right)^{2p} \left(\frac{p+q}{p+q-1}\right)^p \sum_{n=1}^L \sum_{m=1}^M m^{-p-q} m^p \omega_{mn}^q \delta_{mn}^p \\ &= \left(\frac{p}{p-1}\right)^{2p} \left(\frac{p+q}{p+q-1}\right)^p \sum_{m=1}^M m^{-q} \sum_{n=1}^L \omega_{mn}^q \delta_{mn}^p, \end{aligned} \tag{3.20}$$

where

$$\delta_{mn} = \frac{1}{m} \sum_{i=1}^m b_{in},$$

and

$$m\delta_{mn} - (m - 1)\delta_{m-1,n} = b_{mn} \geq 0. \tag{3.21}$$

Repeating the procedure above (but without a further application of Hölder's inequality), we get

$$\sum_{m=1}^M m^{-q} \omega_{mn}^q \delta_{mn}^p \leq \left(\frac{p}{p-1}\right)^p \sum_{m=1}^M m^{-q} \omega_{mn}^q b_{mn}^p. \tag{3.22}$$

From 3.20 and 3.22 we observe that

$$\begin{aligned} S_{ML} &\leq \left(\frac{p}{p-1}\right)^{3p} \left(\frac{p+q}{p+q-1}\right)^p \sum_{m=1}^M m^{-q} \sum_{n=1}^L b_{mn}^p \omega_{mn}^q \\ &= \left(\frac{p}{p-1}\right)^{3p} \left(\frac{p+q}{p+q-1}\right)^p \sum_{m=1}^M \sum_{n=1}^L b_{mn}^p B_{mn}^q. \end{aligned} \tag{3.23}$$

By letting L, M tend to infinity in 3.23 we get the desired inequality 3.2. The proof is complete. \square

If we set $q = 0$ in Theorem 3.1, we recover [7, Theorem 2].

The following result extends the preceding theorem in a similar way that Theorem 2.4 extends Theorem 2.1.

THEOREM 3.2. *Let $p > 1$, $q \geq 0$, $r \geq 0$, $b_{mn} > 0$ for $m, n \in \mathbb{N}$ and let B_{mn} be given by 3.1. If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q} B_{mn}^r$ converges, then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^p B_{mn}^{q+r} \leq \left(\frac{p}{p-1}\right)^{3(p+q)} \left(\frac{p+q+r}{p+q+r-1}\right)^{p+q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q} B_{mn}^r. \tag{3.24}$$

Proof. Using Hölder’s inequality with indices $(p+q)/p$ and $(p+q)/q$, we get

$$\begin{aligned} \sum_{n=1}^L \sum_{m=1}^M b_{mn}^p B_{mn}^{q+r} &= \sum_{n=1}^L \left\{ \sum_{m=1}^M \{b_{mn}^{p+q} B_{mn}^r\}^{p/(p+q)} \{B_{mn}^{p+q+r}\}^{q/(p+q)} \right\} \\ &\leq \sum_{n=1}^L \left\{ \left\{ \sum_{m=1}^M b_{mn}^{p+q} B_{mn}^r \right\}^{p/(p+q)} \left\{ \sum_{m=1}^M B_{mn}^{p+q+r} \right\}^{q/(p+q)} \right\} \\ &\leq \sum_{n=1}^L \left\{ \left\{ \sum_{m=1}^M b_{mn}^{p+q} B_{mn}^r \right\}^{p/(p+q)} \right. \\ &\quad \cdot \left. \left\{ \left(\frac{p}{p-1}\right)^{3p} \left(\frac{p+q+r}{p+q+r-1}\right)^p \sum_{m=1}^M b_{mn}^{p+q} B_{mn}^{q+r} \right\}^{q/(p+q)} \right\} \\ &\leq \left(\frac{p}{p-1}\right)^{3p} \left(\frac{p+q+r}{p+q+r-1}\right)^p \left\{ \sum_{n=1}^L \sum_{m=1}^M b_{mn}^{p+q} B_{mn}^r \right\}^{p/(p+q)} \\ &\quad \cdot \left\{ \sum_{n=1}^L \sum_{m=1}^M b_{mn}^p B_{mn}^{q+r} \right\}^{q/(p+q)}. \end{aligned} \tag{3.25}$$

Then

$$\begin{aligned} &\left\{ \sum_{n=1}^L \sum_{m=1}^M b_{mn}^p B_{mn}^{q+r} \right\}^{p/(p+q)} \\ &\leq \left(\frac{p}{p-1}\right)^{3p} \left(\frac{p+q+r}{p+q+r-1}\right)^p \left\{ \sum_{n=1}^L \sum_{m=1}^M b_{mn}^{p+q} B_{mn}^r \right\}^{p/(p+q)}. \end{aligned} \tag{3.26}$$

Raising both sides of 3.26 to the power $(p+q)/p$ yields 3.24. \square

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