

GENERALIZATION OF HILBERT'S INTEGRAL INEQUALITY

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(communicated by N. Elezović)

Abstract. A generalization of the well-known Hilbert's inequality is given and several other results of this type obtained in the recent years follow as a special case from our result.

1. Introduction

Throughout the paper we suppose that all integrals converge. We also assume nonnegativity of all real functions of the real variable so we omit these type of conditions in all theorems and corollaries.

First, let us recall the well known Hilbert's integral inequality:

THEOREM. *The following inequality holds*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^\infty f^2(x) dx \cdot \int_0^\infty g^2(x) dx \right)^{\frac{1}{2}},$$

where π is the best possible.

In the recent years a lot of results with generalizations of this type of inequality were obtained. Let's mention some of them which take our attention. We cite the results of Yang ([1],[2],[3]), Peachey ([4]), Gavrea ([5]) and Hong ([6]).

We start with the result of Yang:

THEOREM A (YANG). *If $\lambda > 0$, then the following inequality holds*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \cdot \int_0^\infty x^{1-\lambda} g^2(x) dx \right)^{\frac{1}{2}},$$

where B is beta-function, and $B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right)$ is the best possible constant.

Mathematics subject classification (2000): 26D15.

Key words and phrases: Hilbert's inequality, Hölder's inequality, Beta function, Gamma function.

The following generalization is given by the same author:

THEOREM B (YANG). *If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda > 2 - \min\{p, q\}$, then the following inequality holds*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \left(\int_0^\infty x^{1-\lambda} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\lambda} g^q(x) dx \right)^{\frac{1}{q}},$$

where the constant $B \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right)$ is the best possible.

Another result of the same type is given by Peachey:

THEOREM C (PEACHEY). *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $b > -\frac{1}{p}$ and $c > -\frac{1}{q}$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{b+c+1}} dx dy < B \left(b + \frac{1}{p}, c + \frac{1}{q} \right) \left(\int_0^\infty x^{p(1-b)-2} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{q(1-c)-2} g^q(x) dx \right)^{\frac{1}{q}}.$$

Yang also gave a result of the similar type:

THEOREM D (YANG). *If $\lambda > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin(\pi/p)} \left(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{(q-1)(1-\lambda)} g^q(x) dx \right)^{\frac{1}{q}},$$

where the constant $\frac{\pi}{\lambda \sin(\pi/p)}$ is the best possible.

Multiple Hilbert's integral inequality was obtained by Gavrea:

THEOREM E (GAVREA). *If $n \geq 2$ is an integer, then*

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \frac{f_1(x_1)f_2(x_2) \dots f_n(x_n)}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n \\ & \leq \frac{\Gamma^2 \left(\frac{2-n+\lambda}{2} \right)}{\Gamma(\lambda)} \left(\prod_{i=1}^n \int_0^\infty x^{n-1-\lambda} f_i^n(x) dx \right)^{\frac{1}{n}}, \end{aligned}$$

for any $\lambda \in (n-2, n-1]$, where Γ is gamma-function.

An interesting multiple Hilbert-type integral inequality is given by Hong:

THEOREM F (HONG). *If $n \geq 2$ is an integer and $\lambda > 0$, then*

$$\int_0^\infty \dots \int_0^\infty \frac{f_1(x_1)f_2(x_2) \cdot \dots \cdot f_n(x_n)}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \left(\Gamma\left(1 - \frac{b}{r_i}\right) \Gamma\left(\lambda - n + 1 + \frac{b}{r_i}\right) \int_0^\infty x^{n-1-\lambda} f_i^{p_i}(x) dx \right)^{\frac{1}{p_i}},$$

for $b \geq 0$, $p_i > 1$, $i = 1, \dots, n$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $r_i = p_1 \cdot \dots \cdot p_{i-1} \cdot p_{i+1} \cdot \dots \cdot p_n$, $r_i > b$, and $\lambda > n - 1 - \frac{b}{r_i}$, $i = 1, \dots, n$.

2. Main results

For the sake of simplicity, all the indices are taken modulo n throughout the paper. Our main result generalizes previous mentioned results:

THEOREM 1. *If $n \geq 2$ is an integer and $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$, then*

$$\int_0^\infty \dots \int_0^\infty \frac{f_1(x_1)f_2(x_2) \cdot \dots \cdot f_n(x_n)}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n (\Gamma(1 - A_{i+1}p_i)\Gamma(\lambda - n + 1 + A_{i+1}p_i))^{\frac{1}{p_i}} \times$$

$$\times \left(\int_0^\infty x^{n-1-\lambda+p_i(A_i-A_{i+1})} f_i^{p_i}(x) dx \right)^{\frac{1}{p_i}},$$

for any $\lambda > n - 2$ and $A_i \in (\frac{n-\lambda-1}{p_{i-1}}, \frac{1}{p_{i-1}})$, $i = 1, \dots, n$, while Γ is gamma-function.

Proof. Applying Hölder's inequality we obtain

$$\int_0^\infty \dots \int_0^\infty \frac{f_1(x_1)f_2(x_2) \cdot \dots \cdot f_n(x_n)}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n$$

$$= \int_0^\infty \dots \int_0^\infty \frac{f_1(x_1) \frac{x_1^{A_1}}{x_2^{A_2}} f_2(x_2) \frac{x_2^{A_2}}{x_3^{A_3}} \dots f_n(x_n) \frac{x_n^{A_n}}{x_1^{A_1}}}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n$$

$$\leq \prod_{i=1}^n \left(\int_0^\infty \dots \int_0^\infty \left(\frac{x_i^{A_i}}{x_{i+1}^{A_{i+1}}} \right)^{p_i} \frac{f_i^{p_i}(x_i)}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n \right)^{\frac{1}{p_i}}.$$

with $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$.

By putting $u_k = \frac{x_k}{x_i}$, $k = 1, \dots, n$, $k \neq i$, we have

$$\begin{aligned} & x_i^{A_i p_i} \int_0^\infty \cdots \int_0^\infty \frac{dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n}{x_{i+1}^{A_{i+1} p_i} (x_1 + \dots + x_n)^\lambda} \\ &= x_i^{n-1-\lambda+p_i(A_i-A_{i+1})} \int_0^\infty \cdots \int_0^\infty \frac{u_{i+1}^{-A_{i+1} p_i} du_1 \dots du_{i-1} du_{i+1} \dots du_n}{(u_1 + \dots + u_{i-1} + 1 + u_{i+1} + \dots + u_n)^\lambda}. \end{aligned} \quad (1)$$

By integrating

$$\int_0^\infty \frac{dv}{(a+v)^\lambda} \frac{1}{(\lambda-1)a^{\lambda-1}}$$

we obtain

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{u_{i+1}^{-A_{i+1} p_i} du_1 \dots du_{i-1} du_{i+1} \dots du_n}{(u_1 + \dots + u_{i-1} + 1 + u_{i+1} + \dots + u_n)^\lambda} \\ &= \frac{1}{(\lambda-1)(\lambda-2)\dots(\lambda-n+2)} \int_0^\infty \frac{u_{i+1}^{-A_{i+1} p_i}}{(1+u_{i+1})^{\lambda-n+2}} du_{i+1} \\ &= \frac{\Gamma(1-A_{i+1} p_i) \Gamma(\lambda-n+1+A_{i+1} p_i)}{(\lambda-1)(\lambda-2)\dots(\lambda-n+2) \Gamma(\lambda-n+2)} \\ &= \frac{1}{\Gamma(\lambda)} \Gamma(1-A_{i+1} p_i) \Gamma(\lambda-n+1+A_{i+1} p_i). \end{aligned} \quad (2)$$

where we used the well known formula for beta and gamma function

$$B(a, b) = \int_0^\infty \frac{t^{a-1} dt}{(1+t)^{a+b}} = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}.$$

Now the result follows from (1) and (2).

Also taking into account that the gamma-function is defined for positive real numbers, we obtain the conditions $\lambda > n-2$ and $A_i \in (\frac{n-\lambda-1}{p_i-1}, \frac{1}{p_i-1})$.

Also, let's note that the equality holds if and only if the numbers $(f_i(x_i) \frac{x_i^{A_i}}{x_{i+1}^{A_{i+1}}})^{p_i}$, $i = 1, \dots, n$ are proportional in what case our integrals diverge, hence the inequality is strict. \square

Theorem 1 is a generalization of the result of Theorem E. Namely, by putting $A_i = \frac{n-\lambda}{2p_i-1}$, $i = 1, \dots, n$ from Theorem 1 we obtain

COROLLARY 1. *If $n \geq 2$ is an integer, $\lambda > n-2$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$, then*

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1) f_2(x_2) \cdots f_n(x_n)}{(x_1 + x_2 + \dots + x_n)^\lambda} dx_1 dx_2 \dots dx_n \\ & < \frac{\Gamma^2(\frac{2-n+\lambda}{2})}{\Gamma(\lambda)} \prod_{i=1}^n \left(\int_0^\infty x^{(n-\lambda)(\frac{1}{2} + \frac{p_i}{2p_i-1})-1} f_i^{p_i}(x) dx \right)^{\frac{1}{p_i}}. \end{aligned}$$

Now the result of Gavrea (Theorem E) follows from Corollary 1 by putting $p_i = n, i = 1, \dots, n$.

The result of Theorem E is a multidimensional generalization of Theorem A, but the special case $n = 2$ takes our attention, especially because of the results of Theorems B and C which are not included in Gavrea's multidimensional generalization given by Theorem E.

So, our result given in Theorem 1 for the special case $n = 2$ is the following one:

COROLLARY 2. *If $\lambda > 0, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < (B(1 - A_2p, \lambda - 1 + A_2p))^{\frac{1}{p}} (B(1 - A_1q, \lambda - 1 + A_1q))^{\frac{1}{q}} \times \left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\lambda+q(A_2-A_1)} g^q(x) dx \right)^{\frac{1}{q}},$$

while $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$.

Note that we introduce beta-function by using formula $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Now we can obtain the result of Theorem B by putting $A_1 = A_2 = \frac{2-\lambda}{pq}$ in Corollary 2.

Also, if we put $A_1 = \frac{1-b}{q} - \frac{1}{pq}$, and $A_2 = \frac{1-c}{p} - \frac{1}{pq}$, we obtain the following result:

COROLLARY 3. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < b + \frac{1}{p} < \lambda$ and $0 < c + \frac{1}{q} < \lambda$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < \left(B\left(c + \frac{1}{q}, \lambda - c - \frac{1}{q}\right) \right)^{\frac{1}{p}} \left(B\left(b + \frac{1}{p}, \lambda - b - \frac{1}{p}\right) \right)^{\frac{1}{q}} \times \left(\int_0^\infty x^{(p-1)(1-b)+c-\lambda} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{(q-1)(1-c)+b-\lambda} g^q(x) dx \right)^{\frac{1}{q}}.$$

REMARK. If we add another condition $\lambda = b + c + 1$ in Corollary 3, we obtain the result of Peachey given in Theorem C.

Further it is interesting to point out the special case of Corollary 1 for $n = 2$:

COROLLARY 4. If $\lambda > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^\infty x^{p-1-\frac{p\lambda}{2}} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{q-1-\frac{q\lambda}{2}} g^q(x) dx\right)^{\frac{1}{q}}.$$

Finally, we generalize the result of Theorem D.

If we put $x_i = u_i^{\alpha_i}$, $i = 1, \dots, n$ in Theorem 1 and change $f(u_i^{\alpha_i})u_i^{\alpha_i-1}$ with $f(u_i)$ we obtain

THEOREM 2. If $n \geq 2$ is an integer and $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$, then

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \frac{f_1(x_1)f_2(x_2) \cdot \dots \cdot f_n(x_n)}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n})^\lambda} dx_1 dx_2 \dots dx_n \\ & < \frac{\prod_{i=1}^n \alpha_i^{\frac{1}{p_i}-1}}{\Gamma(\lambda)} \prod_{i=1}^n (\Gamma(1 - A_{i+1} p_i) \Gamma(\lambda - n + 1 + A_{i+1} p_i))^{\frac{1}{p_i}} \times \\ & \times \left(\int_0^\infty x^{\alpha_i(n-\lambda-p_i+p_i(A_i-A_{i+1}))+p_i-1} f_i^{p_i}(x) dx\right)^{\frac{1}{p_i}}, \end{aligned}$$

for any $\lambda > n - 2$ and $A_i \in (\frac{n-\lambda-1}{p_{i-1}}, \frac{1}{p_{i-1}})$, $i = 1, \dots, n$.

Now we obtain the result of Theorem F by putting $A_i = \frac{b}{p_1 \dots p_n}$ and $\alpha_i = 1$, $i = 1, \dots, n$.

It is also interesting to point out the special case for $n = 2$:

COROLLARY 5. If $\lambda > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\beta)^\lambda} dx dy \\ & < \alpha^{-\frac{1}{q}} \beta^{-\frac{1}{p}} (B(1 - A_2 p, \lambda - 1 + A_2 p))^{\frac{1}{p}} (B(1 - A_1 q, \lambda - 1 + A_1 q))^{\frac{1}{q}} \times \\ & \times \left(\int_0^\infty x^{\alpha(2-\lambda-p+p(A_1-A_2))+p-1} f^p(x) dx\right)^{\frac{1}{p}} \times \\ & \times \left(\int_0^\infty x^{\beta(2-\lambda-q+q(A_2-A_1))+q-1} g^q(x) dx\right)^{\frac{1}{q}}, \end{aligned}$$

while $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$.

If we put $A_1 = A_2 = \frac{1}{pq}$, $\lambda = 1$ and $\alpha = \beta = \lambda$ in Corollary 5 we obtain the result of Theorem D.

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(Received September 5, 2003)

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