Mathematical
Inequalities
& Applications
Volume 8, Number 1 (2005), 89–99

A FUNCTIONAL DIFFERENTIAL EQUATION IN BANACH ALGEBRAS

B. C. DHAGE, S. N. SALUNKHE, RAVI P. AGARWAL AND W. ZHANG

(*communicated by V. Lakshmikantham*)

Abstract. In this paper an existence theorem for the first order functional differential equations in Banach algebras is proved under the mixed generalized Lipschitz and Carathéodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions.

1. Statement of The Problem

Let $\mathbb{R}$ denote the real line and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $J = I_0 \cup I$, then J is a closed and bounded interval in $\mathbb{R}$. Let C denote the Banach space of all continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|.$$

Clearly C is a Banach algebra with respect to this norm and the multiplication "$\cdot$"' defined by

$$(xy)(t) = x(t)y(t), \ t \in I_0.$$

Consider the first order functional differential equation (in short FDE)

$$\left.\begin{aligned} &\left(\frac{x(t)}{f(t, x(t))}\right)' = g(t, x_t) \ \text{ a.e. } \ t \in I \\ &x(t) = \phi(t), t \in I_0, \end{aligned}\right\} \tag{1.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous, $g : I \times C \to \mathbb{R}$ and the function $x_t(\theta) : I_0 \to C$ is defined by $x_t(\theta) := x(t + \theta)$ for all $\theta \in I_0$.

By a solution of FDE (1.1) we mean a function $x \in C(J, \mathbb{R}) \cap AC(I, \mathbb{R}) \cap C(I_0, \mathbb{R})$ that satisfies the equations in (1.1), where $AC(I, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

The functional differential equations have been the most active area of research since long time. See Hale [13], Henderson [14] and the references therein. But the study of functional differential equations in Banach algebra is very rare in the literature. Very recently the study along this line has been initiated via fixed point theorems. See Dhage

Mathematics subject classification (2000): 34K10.

Key words and phrases: functional differential equation, existence theorem, $\mathscr{D}$-Lipschitzian, Carathéodory condition, fixed point theorem.

© ELEMENT, Zagreb
Paper MIA-08-09

and Regan [9] and Dhage [6] and the references therein. The FDE (1.1) is new to the literature and the study of this problem will definitely contribute immensely to the area of functional differential equations. The fixed point theorems of Dhage [4, 5] we will be using are given in the following section.

2. Auxiliary Results

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called *$\mathscr{D}$-Lipschitzian* if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leqslant \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ ($\alpha > 0$), A is called a Lipchitzian with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear contraction on X. Sometimes we call the function ψ a *$\mathscr{D}$-function* for convenience.

An operator $T : X \to X$ is called *compact* if $\overline{T(S)}$ is a compact for any subset S of X. Similarly $T : X \to X$ is called *totally bounded* if T maps a bounded subset of X into the totally bounded subset of X. Finally $T : X \to X$ is called *completely continuous* operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [7] is embodied in th following theorem.

THEOREM 2.1. (Dhage[7]). *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) A is a $\mathscr{D}$-Lipschitzian with a $\mathscr{D}$-function ψ,
(b) B is compact and continuous, and
(c) $M\psi(r) < r$ whenever $r > 0$, where $M = \|B(X)\| = \sup\{\|Bx\| : x \in X\}$. Then either
(i) the equation $\lambda A(\frac{x}{\lambda})Bx = x$ has a solution for $\lambda = 1$, or
(ii) the set $\mathscr{E} = \{u \in X \mid \lambda A(\frac{u}{\lambda})Bu = u, 0 < \lambda < 1\}$ is unbounded.

It is known that Theorem 2.1 is useful for proving the existence theorems for the integral equations of mixed type. See [5] and the references therein. The method is commonly known as *priori bound method* for the nonlinear equations. See, for example, Dugundji and Granas [10], Zeidler [16] and the references therein.

An interesting corollary to Theorem 2.1 in its applicable form is

COROLLARY 2.1. *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) A is Lipschitzian with a Lipschitz constant α,
(b) B is compact and continuous, and
(c) $\alpha M < 1$, where $M = \|B(X)\| := \sup\{\|Bx\| : x \in X\}$. Then either
(i) the equation $\lambda A(\frac{x}{\lambda})Bx = x$ has a solution for $\lambda = 1$, or
(ii) the set $\mathscr{E} = \{u \in X \mid \lambda A(\frac{u}{\lambda})Bu = u, 0 < \lambda < 1\}$ is unbounded.

3. Existence Theory

Let $M(J,\mathbb{R})$ and $B(J,\mathbb{R})$ respectively denote the spaces of measurable and bounded real-valued functions on J. We shall seek the existence of a solution of FDE (1.1) in the space $C(J,\mathbb{R})$, of all absolutely continuous real-valued functions on J. Define a norm $\|\cdot\|$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with respect to this norm and the multiplication "$\cdot$ "' defined by

$$(x\cdot y)(t) = x(t)y(t),\ t\in J.$$

Note that $C(J,\mathbb{R}) \subset AC(J,\mathbb{R})$. We need the following definition in the sequel.

DEFINITION 3.1. A mapping $\beta : I\times C \to \mathbb{R}$ is said to satisfy a condition of L^1_X*-Carathéodory* or simply, is said to be L^1_X*-Carathéodory* if

(i) $t\to\beta(t,x)$ is measurable for each $x\in C$.
(ii) $x\to\beta(t,x)$ is continuous almost everywhere for $t\in I$, and
(iii) there exists a function $h\in L^1(I,\mathbb{R})$ such that

$$|\beta(t,x)| \leqslant h(t), \quad \text{a.e. } t\in I$$

for all $x\in C$. For convenience, the function h is referred to as a *bound function* of β.

We will need the following hypotheses in the sequel.

(H_1) The function $f : I\times\mathbb{R}\to\mathbb{R}-\{0\}$ is continuous and there exists a function $k\in B(I,\mathbb{R})$ such that $k(t)>0$, a.e. $t\in I$ and

$$|f(t,x)-f(t,y)| \leqslant k(t)|x-y|, \quad \text{a.e. } t\in I$$

for all $x,y\in\mathbb{R}$.

(H_2) $f(0,\phi(0)) = 1$.

(H_3) The function g is L^1_X-Carathéodory with bound function h.

(H_4) There exists a continuous and nondecreasing function $\Omega : [0,\infty)\to(0,\infty)$ and a function $\gamma\in L^1(I,\mathbb{R})$ such that $\gamma(t)>0$, a.e. $t\in J$ and

$$|g(t,x)| \leqslant \gamma(t)\Omega\Big(\|x\|_C\Big), \quad \text{a.e. } t\in I,$$

for all $x\in C$.

THEOREM 3.1. *Assume that the hypotheses (H_1)-(H_4) hold. Suppose that*

$$\int_{C_1}^{\infty} \frac{ds}{\Omega(s)} > C_2\|\gamma\|_{L^1}, \tag{3.1}$$

where

$$C_1 = \frac{F\|\phi\|_C}{1-\|k\|(\|\phi\|_C+\|h\|_{L^1})},\ C_2 = \frac{1}{1-\|k\|[\|\phi\|_C+\|h\|_{L^1}]},\ \|k\|(\|\phi\|_C+\|h\|_{L^1})<1,$$

$F = \max_{t\in J}|f(t,0)|$, *and* $\|k\| = \max_{t\in J}|k(t)|$. *Then the FDE (1.1) has a solution on* J.

Proof. Now the FDE (1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = [f(t, x(t))] \left(\phi(0) + \int_0^t g(s, x_s)\, ds \right), \text{ if } t \in I \tag{3.2}$$

and

$$x(t) = \phi(t), \text{ if } \quad t \in I_0. \tag{3.3}$$

Define the two mappings A and B on $C(J, \mathbb{R})$ by

$$Ax(t) = \begin{cases} f(t, x(t)), & \text{if} \quad t \in I, \\ 1, & \text{if} \quad t \in I_0. \end{cases} \tag{3.4}$$

and

$$Bx(t) = \begin{cases} \phi(0) + \int_0^t g(s, x_s)\, ds, & \text{if} \quad t \in I, \\ \phi(t), \text{ if} \quad t \in I_0. \end{cases} \tag{3.5}$$

Obviously A and B define the operators $A, B : C(J, \mathbb{R}) \to C(J, \mathbb{R})$. Then the FDE (1.1) is equivalent to the operator equation

$$x(t) = Ax(t)\, Bx(t), \quad t \in J. \tag{3.6}$$

We shall show that the operators A and B satisfy all the hypotheses of Corollary 2.1.

We first show that A is a Lipschitzian on $C(J, \mathbb{R})$. Let $x, y \in C(J, \mathbb{R})$. Then by (H_1),

$$\begin{aligned} |Ax(t) - Ay(t)| &\leqslant |f(t, x(t)) - f(t, y(t))| \\ &\leqslant k(t)|x(t) - y(t)| \\ &\leqslant k(t)\|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \leqslant \|k\| \|x - y\|.$$

for all $x, y \in C(J, \mathbb{R})$. So A is a Lipschitzian on $C(J, \mathbb{R})$ with a Lipschitz constant $\|k\|$. Next we show that B is completely continuous on $C(J, \mathbb{R})$. Using the standard arguments as in Granas et al. [11], it is shown that B is a continuous operator on $C(J, \mathbb{R})$. Let S be any subset of $C(J, \mathbb{R})$. We shall show that $B(C(J, \mathbb{R}))$ is a uniformly bounded and equicontinuous set in $C(J, \mathbb{R})$. Since $g(t, x_t)$ is L^1_X-Carathéodory, we have

$$\begin{aligned} |Bx(t)| &\leqslant \|\phi\|_C + \int_0^t |g(s, x_s)|\, ds \\ &\leqslant \|\phi\|_C + \int_0^t h(s)\, ds \\ &\leqslant \|\phi\|_C + \|h\|_{L^1}. \end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \leqslant M$ for all $x \in S$, where $M = \|\phi\|_C + \|h\|_{L^1}$. This shows that $B(C(J,\mathbb{R}))$ is a uniformly bounded set in $C(J,\mathbb{R})$. Now we show that $B(C(J,\mathbb{R}))$ is an equicontinuous set. Let $t,\tau \in I$. Then for any $x \in C(J,\mathbb{R})$ we have by (3.4),

$$\begin{aligned}
|Bx(t) - Bx(\tau)| &\leqslant \left| \int_0^t g(s,x_s)\,ds - \int_0^\tau g(s,x_s)\,ds \right| \\
&\leqslant \left| \int_\tau^t |g(s,x_s)|\,ds \right| \\
&\leqslant \left| \int_\tau^t h(s)\,ds \right| \\
&\leqslant |p(t) - p(\tau)|
\end{aligned}$$

where $p(t) = \displaystyle\int_0^t h(s)\,ds$.

Therefore

$$|Bx(t) - Bx(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Again let $\tau \in I_0, t \in I$. Then we obtain

$$\begin{aligned}
|Bx(t) - Bx(\tau)| &\leqslant |\phi(\tau) - \phi(0)| + \left| \int_0^t g(s,x_s)\,ds \right| \\
&\leqslant |\phi(\tau) - \phi(0)| + |p(t) - p(\tau)|
\end{aligned}$$

where the function p is defined above. Similarly if $\tau, t \in I_0$, then we get

$$|Bx(t) - Bx(\tau)| \leqslant |\phi(t) - \phi(\tau)|.$$

Therefore in all above three cases

$$|Bx(t) - Bx(\tau)| \to 0 \ \text{ as } \ \tau \to t, \ \forall\, t,\tau \in J.$$

Hence $B(C(J,\mathbb{R}))$ is an equicontinuous set and consequently $B(C(J,\mathbb{R}))$ is relatively compact by Arzela-Ascoli theorem. As a result B is a compact and continuous operator on $C(J,\mathbb{R})$. Thus all the conditions of Theorem 2.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $x \in X$ be any solution to FDE (1.1). Then we have, for any $\lambda \in (0,1)$,

$$\begin{aligned}
x(t) &= \lambda A\left(\frac{x}{\lambda}\right)(t)Bx(t) \\
&= \begin{cases} \lambda \left[f\left(t, \dfrac{x(t)}{\lambda}\right)\right] \left(\phi(0) + \displaystyle\int_0^t g(s,x_s)\,ds\right), & t \in I \\ \lambda\phi(t), \qquad t \in I_0 \end{cases}
\end{aligned}$$

for $t \in J$. Then we have

$$\begin{aligned}
|x(t)| &\leqslant \lambda \left| f\left(s, \frac{x(t)}{\lambda}\right)\right| \left(\|\phi\|_C + \left| \int_0^t g(s, x_s)\, ds \right| \right) \\
&\leqslant \lambda \left(\left| f\left(s, \frac{x(t)}{\lambda}\right) - f(t,0) \right| + |f(t,0)| \right) \left(\|\phi\|_C + \int_0^t |g(s,x_s)|\, ds \right) \\
&\leqslant [k(t)|x(t)| + F] \left(\|\phi\|_C + \int_0^t |g(s,x_s)|\, ds \right) \\
&\leqslant k(t)\|x\| \left(\|\phi\|_C + \int_0^t |g(s,x_s)|\, ds \right) + F \left(\|\phi\|_C + \int_0^t |g(s,x_s)|\, ds \right) \\
&\leqslant \|k\|\|x\|(\|\phi\|_C + \|h\|_{L^1}) + F\|\phi\|_C + F \int_0^t \gamma(s)\Omega(\|x_s\|_C)\, ds. \qquad (3.7)
\end{aligned}$$

Put $u(t) = \sup_{s\in[-r,t]} |x(s)|$, for $t \in J$. Then we have

$$|x(t)| \leqslant u(t) \ \forall t \in J \quad \text{and} \quad \|x_t\|_C \leqslant u(t) \quad \forall \quad t \in I,$$

and so, there is a point $t^* \in [-r, t]$ such that $u(t) = |x(t^*)|$. From (3.7) it follows that

$$\begin{aligned}
u(t) &= |x(t^*)| \\
&\leqslant \|k\||x(t^*)|(\|\phi\|_C + \|h\|_{L^1}) + F \left(\|\phi\|_C + \int_0^{t^*} \gamma(s)\Omega(\|x_s\|_C)\, ds \right) \\
&\leqslant \|k\|u(t)(\|\phi\|_C + \|h\|_{L^1}) + F \left(\|\phi\|_C + \int_0^t \gamma(s)\Omega(u(s))\, ds \right) \\
&= C_1 + C_2 \int_0^t \gamma(s)\Omega(u(s)))\, ds \qquad (3.8)
\end{aligned}$$

where

$$C_1 = \frac{F\|\phi\|_C}{1 - \|k\|[\|\phi\|_C + \|h\|_L^1]} \quad \text{and} \quad C_2 = \frac{1}{1 - \|k\|[\|\phi\|_C + \|h\|_L^1]}.$$

Let

$$w(t) = C_1 + C_2 \int_0^t \gamma(s)\Omega(u(s))\, ds.$$

Then $u(t) \leqslant w(t)$ and a direct differentiation of $w(t)$ yields

$$\left.\begin{array}{rcl} w'(t) & \leqslant & C_2\gamma(t)\Omega(w(t)) \\ w(0) & = & C_1, \end{array}\right\} \qquad (3.9)$$

that is,

$$\int_0^t \frac{w'(s)}{\Omega(w(s))}\, ds \leqslant C_2 \int_0^t \gamma(s)\, ds \leqslant C_2\|\gamma\|_{L^1}$$

A change of variables in the above integral gives that

$$\int_{C_1}^{w(t)} \frac{ds}{\Omega(s)} \leqslant C_2\|\gamma\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Omega(s)}.$$

Now an application of mean value theorem yields that there is a constant $M > 0$ such that $w(t) \leqslant M$ for all $t \in J$. This further implies that

$$|x(t)| \leqslant u(t) \leqslant w(t) \leqslant M.$$

for all $t \in J$. Thus the conclusion (ii) of Corollary 2.1 does not hold. Therefore the operator equation $AxBx = x$ and consequently the FDE (1.1) has a solution on J. This completes the proof. $\square$

4. Existence of Extremal Solutions

A non-empty closed set K in a Banach algebra X is called a *cone* if (i) $K+K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geqslant 0$ and (iii) $\{-K\} \cap K = \{0\}$, where 0 is the zero element of X. A cone K is called to be *positive* if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leqslant$ in K as follows. Let $x, y \in X$. Then $x \leqslant y$ if and only if $y - x \in K$. A cone K is called to be *normal* if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [12].

We equip the space $C(J,\mathbb{R})$ with the order relation $\leqslant$ with the help of the cone defined by

$$K = \{x \in C(J,\mathbb{R}) : x(t) \geqslant 0, \forall\, t \in J\}. \tag{4.1}$$

It is well known that the cone K is positive and normal in $C(J,\mathbb{R})$. As a result of positivity of the cone K in $C(J,\mathbb{R})$ we have:

LEMMA 4.1. (Dhage [6]). Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leqslant v_1$ and $u_2 \leqslant v_2$. Then $u_1u_2 \leqslant v_1v_2$.

For any $a, b \in X = C(J,\mathbb{R}), a \leqslant b$, the order interval $[a,b]$ is a set in X given by

$$[a,b] = \{x \in X : a \leqslant x \leqslant b\}.$$

We use the following fixed point theorem of Dhage [6] for proving the existence of extremal solutions of the FDE (1.1) under certain monotonicity conditions.

THEOREM 4.1. (Dhage [6], Corollary 3.1). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a,b] \to K$ are two operators such that*

- *(a) A is Lipschitzian with a Lipschitz constant α,*
- *(b) B is completely continuous,*
- *(c) $AxBx \in [a,b]$ for each $x \in [a,b]$, and*
- *(d) A and B are nondecreasing. Further if the cone K is positive and normal, then the operator equation $AxBx = x$ has a maximal and a minimal positive solution in $[a,b]$, whenever $\alpha M < 1$, where $M = \|B([a,b])\| := \sup\{\|Bx\| : x \in [a,b]\}$.*

We need the following definitions in the sequel.

DEFINITION 4.1. A function $u \in C(J,\mathbb{R})$ is called a lower solution of the FDE (1.1) on J if

$$\left(\frac{u(t)}{f(t,u(t))}\right)' \leqslant g(t,u_t), \quad \text{a.e } t \in I$$

and

$$u(t) \leqslant \phi(t) \text{ for all } t \in I_0.$$

Again a function $v \in C(J,\mathbb{R})$ is called an upper solution of the BVP (1.1) on J if

$$\left(\frac{v(t)}{f(t,v(t))}\right)' \geqslant g(t,v_t), \quad \text{a.e } t \in J$$

and

$$v(t) \geqslant \phi(t) \text{ for all } t \in I_0.$$

DEFINITION 4.2. A solution x_M of the FDE (1.1) is said to be maximal if for any other solution x to FDE (1.1) one has $x(t) \leqslant x_M(t), \forall t \in J$. Again a solution x_m of the FDE (1.1) is said to be minimal if $x_m(t) \leqslant x(t), \forall t \in J$, where x is any solution of the FDE (1.1) on J.

We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R}^+ \to \mathbb{R}^+ - \{0\}$, $g : J \times C \to \mathbb{R}^+$ and $\phi(t) \geqslant 0$ on I_0.

(B_1) $g(t,x)$ is L^1_X-Carathéodory.

(B_2) The functions $f(t,x)$ and $g(t,y)$ are nondecreasing in x and y almost everywhere for $t \in J$.

(B_3) The FDE (1.1) has a lower solution u and an upper solution v on J with $u \leqslant v$.

REMARK 4.1. Assume that (B_1)–(B_3) hold. Define a function $h : J \to \mathbb{R}^+$ by

$$h(t) = |g(t,u_t)| + |g(t,v_t)| = g(t,u_t) + g(t,v_t), \text{ for all } t \in I.$$

Then h is Lebesgue integrable and

$$|g(t,x_t)| = g(t,x_t) \leqslant h(t), \quad \text{a.e. } t \in I,$$

for all $x \in [u,v]$.

THEOREM 4.2. *Suppose that the assumptions (H_1)–(H_3) and (B_0)–(B_3) hold. Further if $\|k\|(\|\phi\|_C + \|h\|_{L^1}) < 1$, and h is given in Remark 4.1, then FDE (1.1) has a minimal and a maximal positive solution on J.*

Proof. Now FDE (1.1) is equivalent to FIE (3.2)–(3.3) on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (3.4) and (3.5) respectively. Then FIE (1.1) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_1) implies $A, B : [u,v] \to K$. Since the cone K in X is normal, $[u,v]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitzian with a Lipschitz constant $\|\alpha\|$ and B is completely continuous operator

on $[u,v]$. Again the hypothesis (B_2) implies that A and B are nondecreasing on $[u,v]$. To see this, let $x,y \in [u,v]$ be such that $x \leqslant y$. Then by (B_2),

$$Ax(t) = f(t,x(t)) \leqslant f(t,y(t)) = Ay(t), \ \forall t \in I,$$

and

$$Ax(t) = 1 = Ay(t), \quad \text{for all} \quad t \in I_0.$$

Similarly

$$\begin{aligned} Bx(t) &= \phi(0) + \int_0^t g(s,x_s)\,ds \\ &\leqslant \phi(0) + \int_0^t g(s,y_s)ds \\ &= Ay(t), \ \forall t \in I, \end{aligned}$$

and

$$Bx(t) = \phi(t) = By(t) \quad \text{for all} \quad t \in I_0.$$

So A and B are nondecreasing operators on $[u,v]$. Again Lemma 4.1 and hypothesis (B_3) implies that

$$\begin{aligned} u(t) &\leqslant [f(t,u(t))]\left(\phi(0) + \int_0^t g(s,u_s))\,ds\right) \\ &\leqslant [f(t,x(t))]\left(\phi(0) + \int_0^t g(s,x_s)\,ds\right) \\ &\leqslant [f(t,v(t))]\left(\phi(0) + \int_0^t g(s,v_s))\,ds\right) \\ &\leqslant v(t), \end{aligned}$$

for all $t \in I$ and $x \in [u,v]$. As a result $u(t) \leqslant Ax(t)\,Bx(t) \leqslant v(t), \forall t \in J$, and so, $x \in [u,v]$. Hence $Ax\,Bx \in [u,v], \forall\, x \in [u,v]$.

Again

$$\begin{aligned} M &= \|B([u,v])\| \\ &= \sup\{\|Bx\| : x \in [u,v]\} \\ &\leqslant \sup\left\{\|\phi\|_C + \sup_{t\in J}\int_0^t |g(s,x_s)|\,ds \mid x \in [u,v]\right\} \\ &\leqslant \|\phi\|_C + \int_0^a h(s)ds \\ &= \|\phi\|_C + \|h\|_{L^1}. \end{aligned}$$

Since $\alpha M \leqslant \|k\|(\|\phi\|_C + \|h\|_{L^1}) < 1$, we apply Theorem 4.1 to the operator equation $Ax(t)\,Bx(t) = x(t)$ to yield that the FDE (1.1) has a minimal and a maximal positive solution on J. This completes the proof. $\square$

5. An Example

Given the closed and bounded intervals $I_0 = [-\pi/2, 0]$ and $I = [0, \pi/2]$ in $\mathbb{R}$, consider the nonlinear FDE

$$\left.\begin{aligned} &\left(\frac{x(t)}{f(t, x(t))}\right)' = \frac{p(t)}{1 + \|x_t\|_C}, \quad \text{a.e. } t \in I \\ &x(t) = \sin t, \ t \in I_0. \end{aligned}\right\} \tag{5.1}$$

where $p \in L^1(I, \mathbb{R})$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t, x(t)) = 1 + \alpha|x(t)|, \ \alpha > 0$$

for all $t \in I$. Obviously $f : I \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$. Define a function $g : I \times C \to \mathbb{R}$ by $g(t, x_t) = \dfrac{p(t)}{1 + \|x_t\|_C}$. It is easy to verify that f is continuous and Lipschitzian on $J \times \mathbb{R}$ with a Lipschitz constant α. Further $g(t, x)$ is L^1_X-Carathéodory with the bound function $h(t) = p(t)$ on I. Therefore if $\alpha(1 + \|p\|_{L^1}) < 1$, then by Theorem 3.1, the FDE (5.1) has a solution on J, because the function Ω satisfies condition (3.1) with $\gamma(t) = p(t)$ for all $t \in I$ and $\Omega(r) = 1$, for all $r \in \mathbb{R}^+$.

REFERENCES

[1] D. W. BOYD AND J. S. W. WONG, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 456–444.

[2] T. A. BURTON AND C. KIRK, *A fixed point theorem of Krasnoselskii-Schaefer type*, Math. Nachr. **189** (1998), 23–31.

[3] K. DEIMLING, *Nonlinear Functional Analysis*, Springer Verlag, New York, 1985.

[4] B. C. DHAGE, *On some variants of Schauder's fixed point principle and applications to nonlinear integral equations*, J. Math. Phy. Sci. **25** (1988), 603–611.

[5] B. C. DHAGE, *On α-condensing mappings in Banach algebras*, The Math. Student **63** (1994), 146–152.

[6] B. C. DHAGE, *Fixed point theorems in ordered Banach algebras and applications*, Panamer. Math. J. **9** 4 (1999), 93–102.

[7] B. C. DHAGE, *Some nonlinear alternatives in Banach algebras with applications I*, Nonlinear Studies (accepted)

[8] B. C. DHAGE, *A functional integro-differential equation in Banach algebras*, Functional Diff. Equations **11** 3–4 (2004), 321–332.

[9] B. C. DHAGE AND D. O'REGAN, *A fixed point theorem in Banach algebras with applications to nonlinear integral equations*, Functional Diff. Equations **7** 3–4 (2000), 259–267.

[10] J. DUGUNDJI AND A. GRANAS, *Fixed Point Theory*, Monographie Math., Warsaw, 1982.

[11] A. GRANAS, R. B. GUENTHER AND J. W. LEE, *Some general existence principles for Carathèodory theory of nonlinear differential equations*, J. Math. Pures et Appl. **70** (1991), 153–196.

[12] D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Spaces*, Academic Press, New York, 1988.

[13] J. K. HALE, *Theory of Functional Differential Equations*, Springer Verlag, New York, 1977.

[14] J. HENDERSON, *Boundary Value Problems for Functional Differential Equations*, World Scientific, Singapore, 1995.

[15] D. R. SMART, *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1980.
[16] E. ZEIDLER, *Nonlinear Functional Analysis: Part I,* Springer Verlag, New York, 1985.

(Received August 15, 2003)

B. C. Dhage
Kasubai, Gurukul Colony
Ahmedpur-413 515
Dist: Latur Maharashtra, India
e-mail: bcd20012001@yahoo.co.in

S. N. Salunkhe
Kasubai, Gurukul Colony
Ahmedpur-413 515
Dist: Latur Maharashtra, India

Ravi P. Agarwal
Department of Applied Mathematics
Florida Institute of Technology
150 West University Boulevard
Melbourne, FL 32901-6975, USA
e-mail: agarwal@fit.edu

W. Zhang
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P. R. China
e-mail: matzwn@sohu.com