

## ON AN ESTIMATE RELATED TO THE HESSIAN AND APPLICATION TO AN OBLIQUE DERIVATIVE PROBLEM

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(communicated by R. P. Agarwal)

*Abstract.* We prove an estimate on the  $L^2(\Omega)$ -norm of the Hessian of a function  $u \in W^{2,q}(\Omega)$ , satisfying an oblique derivative type condition on the boundary, allowing the oblique axis to be tangential at a finite number of points of  $\partial\Omega$ . Using this inequality, the solvability in Sobolev spaces  $W^{2,q}(\Omega)$ , with  $q$  close to 2, follows for a class of nonlinear differential equations in the plane with quadratic growth.

### 1. Introduction and main results

In the previous papers [4], [6] we concerned with the oblique derivative problem

$$\mathcal{A}(x, u, Du, D^2u) = f(x, u, Du) \quad \text{a.e. in } \Omega \quad (1.1)$$

$$\frac{\partial u}{\partial l} + \sigma u = 0 \quad \text{on } \partial\Omega \quad (1.2)$$

assuming  $\sigma$  to be a positive constant and the elliptic operator  $\mathcal{A}(x, z, p, \xi)$  to be discontinuous with respect to  $x$ .

It seems to be interesting to study the same problem assuming  $\sigma$  to be a positive function, allowing the operator to remain discontinuous, taking into account that until now no result is present in this case (for a survey on tangential oblique derivative problem in the plane see [6], [7]). In the sequel we solve this problem assuming the function  $\sigma$  to be of class  $C^2$ -piecewise and satisfying a differential condition. To be analytically precise, let us introduce the problem in details.

Let us consider the oblique derivative problem (1.1)-(1.2) with boundary condition (1.2) replaced by

$$\frac{\partial u}{\partial l} + \sigma(x)u = 0 \quad \text{on } \partial\Omega. \quad (1.2')$$

Here  $\Omega \subset \mathbb{R}^2$  is a bounded, convex and open set with  $C^2$ -smooth boundary  $\partial\Omega$ . We assume that  $\partial\Omega$  is a closed curve and let  $x_1 = x_1(\varphi)$ ,  $x_2 = x_2(\varphi)$  be the normal parameterization of  $\partial\Omega$ , with  $\varphi$  being a curvilinear parameter,  $\varphi \in [0, L]$ . On  $\partial\Omega$

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is defined a unit vector field  $l = (Y_1(\varphi), Y_2(\varphi))$ , such that  $Y_1(L) = Y_1(0)$ ;  $Y_2(L) = Y_2(0)$ . Setting  $n = (X_1(\varphi), X_2(\varphi))$  for the unit outward normal to  $\partial\Omega$ , and denoting by  $\theta$  the angle between the normal  $n$  and the vector  $l$ , we assume that

$$\cos \theta(\varphi) = \sum_{i=1}^2 X_i(\varphi) Y_i(\varphi) \geq 0 \quad \forall \varphi \in [0, L] \quad (1.3)$$

with  $\cos \theta(\varphi) = 0$  at a finite number of points  $\varphi_j \in ]0, L[$ ,  $j = 1, \dots, n$ , with  $\varphi_1 < \varphi_2 < \dots < \varphi_n$ , and setting  $\varphi_0 = 0$ ,  $\varphi_{n+1} = L$ ,  $\cos \theta(\varphi) > 0 \quad \forall \varphi \in ]\varphi_{j-1}, \varphi_j[$ ,  $j = 1, \dots, n+1$ ,  $\cos \theta(\varphi_0) > 0$ ,  $\cos \theta(\varphi_{n+1}) > 0$ .

Furthermore, denoting by  $\kappa$  the mean curvature of  $\partial\Omega$ , we suppose

$$\left\{ \begin{array}{l} Y_i(\varphi) \in C^1(] \varphi_{j-1}, \varphi_j[), \quad i = 1, 2, \quad j = 1, \dots, n+1 \\ \text{there exists } \lim_{\varphi \rightarrow \varphi_j^-} Y_i(\varphi) = - \lim_{\varphi \rightarrow \varphi_j^+} Y_i(\varphi) = Y_i^*(\varphi_j) \in \mathbb{R}, \quad i = 1, 2, \quad j = 1, \dots, n \\ \text{there exists } \lim_{\varphi \rightarrow \varphi_0^+} Y_i(\varphi) = Y_i(0) = \lim_{\varphi \rightarrow \varphi_{n+1}^-} Y_i(\varphi) = Y_i(L) \in \mathbb{R}, \quad i = 1, 2 \\ \text{there exists } \lim_{\varphi \rightarrow \varphi_j^+} Y_i'(\varphi) \in \mathbb{R}, \quad i = 1, 2, \quad j = 0, \dots, n \\ \text{there exists } \lim_{\varphi \rightarrow \varphi_j^-} Y_i'(\varphi) \in \mathbb{R}, \quad i = 1, 2, \quad j = 1, \dots, n+1, \\ Y_1^2(\varphi) + Y_2^2(\varphi) = 1 \quad \forall \varphi \in ] \varphi_{j-1}, \varphi_j[), \quad j = 1, \dots, n+1 \\ (Y_1^*(\varphi_j))^2 + (Y_2^*(\varphi_j))^2 = 1, \quad j = 1, \dots, n. \\ (Y_1(0))^2 + (Y_2(0))^2 = 1 \end{array} \right. \quad (1.4)$$

and

$$\left\{ \begin{array}{l} \theta \in C^1(] \varphi_{j-1}, \varphi_j[), \quad j = 1, \dots, n+1 \\ \kappa(\varphi) \leq \frac{d\theta(\varphi)}{d\varphi} \leq 0, \quad \forall \varphi \in ] \varphi_{j-1}, \varphi_j[), \quad j = 1, \dots, n+1 \\ \lim_{\varphi \rightarrow \varphi_j^-} \theta(\varphi) = \theta(\varphi_j^-), \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_j^+} \theta(\varphi) = \theta(\varphi_j^+), \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_0^+} \theta(\varphi) = \theta(0) = \lim_{\varphi \rightarrow \varphi_{n+1}^-} \theta(\varphi) + 2n\pi = \theta(L) + 2n\pi \\ \text{there exists } \lim_{\varphi \rightarrow \varphi_j^+} \frac{d\theta(\varphi)}{d\varphi} = \frac{d\theta(\varphi_j^+)}{d\varphi} < 0, \quad j = 0, \dots, n \\ \text{there exists } \lim_{\varphi \rightarrow \varphi_j^-} \frac{d\theta(\varphi)}{d\varphi} = \frac{d\theta(\varphi_j^-)}{d\varphi} < 0, \quad j = 1, \dots, n+1. \end{array} \right. \quad (1.5)$$

As it concerns  $\sigma(x)$ , we suppose that it is a positive function of class  $C^2(\overline{\Omega})$  and such that, setting  $\sigma(\varphi) = \sigma(x_1(\varphi), x_2(\varphi))$ , it results that  $\sigma(\varphi)$  is  $C^2([0, L])$ -piecewise,

namely

$$\left\{ \begin{array}{l} \sigma(\varphi) \in C^2([\varphi_{j-1}, \varphi_j]), \quad j = 1, \dots, n+1, \\ \lim_{\varphi \rightarrow \varphi_j^-} \sigma(\varphi) = \sigma(\varphi_j^-) \neq 0, \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_j^+} \sigma(\varphi) = \sigma(\varphi_j^+) \neq 0, \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_j^+} \sigma'(\varphi) = \sigma'(\varphi_j^+) \in \mathbb{R}, \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_j^-} \sigma'(\varphi) = \sigma'(\varphi_j^-) \in \mathbb{R}, \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_j^-} \sigma''(\varphi) = \sigma''(\varphi_j^-) \in \mathbb{R}, \quad j = 1, \dots, n+1 \\ \lim_{\varphi \rightarrow \varphi_j^+} \sigma''(\varphi) = \sigma''(\varphi_j^+) \in \mathbb{R}, \quad j = 0, \dots, n \\ \lim_{\varphi \rightarrow \varphi_0^+} \sigma(\varphi) = \sigma(0) = \lim_{\varphi \rightarrow \varphi_{n+1}^-} \sigma(\varphi) = \sigma(L) \\ \lim_{\varphi \rightarrow \varphi_0^+} \sigma'(\varphi) = \sigma'(0) = \lim_{\varphi \rightarrow \varphi_{n+1}^-} \sigma'(\varphi) = \sigma'(L) \end{array} \right.$$

and

$$\sigma''(\varphi) + \sigma'(\varphi) \tan \theta(\varphi) \frac{d\theta}{d\varphi} \leq 0 \quad \forall \varphi \in (\varphi_{j-1}, \varphi_j), \quad j = 1, \dots, n+1. \quad (1.6)$$

We define  $W_l^q(\Omega)$  the closure in  $W^{2,q}(\Omega)$  of the class  $W = \{u \in C^2(\overline{\Omega}) \cap C^3(\Omega) : \frac{\partial u}{\partial l} + \sigma(\varphi)u = 0 \text{ on } \partial\Omega\}$  with respect to the norm (see Lemma 2.2)

$$\|u\|_{W_l^q(\Omega)} = \left( \int_{\Omega} \left( \sum_{i,j=1}^2 (D_{ij}u)^2 \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}}.$$

We are aimed at the investigation of solvability of the oblique derivative problem

$$\begin{cases} \mathcal{A}(x, u, Du, D^2u) = f(x, u, Du) \quad \text{a.e. in } \Omega \subset \mathbb{R}^2 \\ u \in W_l^q(\Omega). \end{cases} \quad (1.7)$$

Throughout the paper we will suppose that the functions  $\mathcal{A}(x, z, p, \xi)$ ,  $f(x, z, p)$  are real valued functions which satisfy Carathéodory’s condition and a suitable ellipticity condition due to S.Campanato (see [2]):

there exist  $\alpha, \gamma, \delta > 0$ , with  $\gamma + \delta < 1$ , such that, for almost all  $x \in \Omega$ , for all  $z \in \mathbb{R}, p \in \mathbb{R}^2, \xi, \tau \in \mathbb{R}^{2 \times 2}$ , one has

$$\left| \sum_{i=1}^2 \xi_{ii} - \alpha [\mathcal{A}(x, z, p, \xi + \tau) - \mathcal{A}(x, z, p, \tau)] \right|^2 \leq \gamma \|\xi\|^2 + \delta \left| \sum_{i=1}^2 \xi_{ii} \right|^2 \quad (A)$$

where  $\xi = \{\xi_{ij}\}_{i,j=1,2}$  and  $\|\xi\| = \left(\sum_{ij=1}^2 \xi_{ij}^2\right)^{\frac{1}{2}}$ ; and

$$\mathcal{A}(x, z, p, 0) = 0.$$

As it concerns the function  $f(x, z, p)$  we assume

$$|f(x, z, p)| \leq f_1(|z|) \left[ f_2(x) + |p|^2 \right] \tag{1.8}$$

for almost all  $x \in \Omega$ , for all  $(z, p) \in \mathbb{R} \times \mathbb{R}^2$ , where  $f_1 \in C^0(\mathbb{R}^+)$  is positive, monotone nondecreasing function and  $f_2 \in L^r(\Omega)$ ,  $r > 2$ , is a positive one; and

$$(\text{sign } z)f(x, z, p) \leq 2\sqrt{\frac{g(x)\det |a_{ij}|}{h(p)}} \tag{1.9}$$

for a.a.  $x \in \Omega$ ,  $|z| \geq M = \text{const} > 0$ ,  $p \in \mathbb{R}^2$ , where

$$a_{ij}(x, z, p, \xi) = \int_0^1 \frac{\partial \mathcal{A}}{\partial \xi_{ij}}(x, z, p, s\xi) ds \in L^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2 \times 2}),$$

$g(x) \in L^1(\Omega)$  and  $h \in L^1_{\text{loc}}(\mathbb{R}^2)$  are positive functions such that  $\int_\Omega g(x)dx < \int_{\mathbb{R}^2} h(p)dp$ . Let us note that Campanato's (A) condition ensures that the derivatives  $\frac{\partial \mathcal{A}}{\partial \xi_{ij}}(x, z, p, \xi)$  exist almost everywhere and they are essentially bounded (for comments about condition (A) we refer to [8]).

We are in position to formulate our existence result.

**THEOREM 1.1.** *Let conditions (A), (1.3), (1.4), (1.5), (1.6), (1.8), (1.9) be fulfilled. Then there exists a number  $\bar{q} > 2$ , such that, for each  $q \in ]2, \bar{q}[$ , problem (1.7) is solvable.*

**REMARK 1.** We note that the values of solution  $u$  at the points of tangency are known from the boundary condition. In fact if  $u \in W$ , from condition (1.4) we get

$$\lim_{\varphi \rightarrow \varphi_j^-} \frac{\partial u}{\partial l} = - \lim_{\varphi \rightarrow \varphi_j^+} \frac{\partial u}{\partial l}, \quad j = 1, \dots, n$$

and hence from boundary condition

$$\begin{aligned} \lim_{\varphi \rightarrow \varphi_j^-} \left( \frac{\partial u}{\partial l} + \sigma(\varphi)u \right) &= \lim_{\varphi \rightarrow \varphi_j^-} \frac{\partial u}{\partial l} + \sigma(\varphi_j^-)u(\varphi_j) = 0, \quad j = 1, \dots, n \\ \lim_{\varphi \rightarrow \varphi_j^+} \left( \frac{\partial u}{\partial l} + \sigma(\varphi)u \right) &= - \lim_{\varphi \rightarrow \varphi_j^+} \frac{\partial u}{\partial l} + \sigma(\varphi_j^+)u(\varphi_j) = 0, \quad j = 1, \dots, n. \end{aligned}$$

Then

$$u(\varphi_j) = 0, \quad \frac{\partial u}{\partial l}(\varphi_j) = \frac{du}{d\varphi}(\varphi_j) = 0, \quad j = 1, \dots, n. \tag{1.10}$$

The same conditions hold in  $W_1^q(\Omega)$  in the trace sense.

REMARK 2. As it concerns the condition

$$\sigma''(\varphi) + \sigma'(\varphi) \tan \theta(\varphi) \frac{d\theta}{d\varphi} = r(\theta(\varphi)) \leq 0 \quad \forall \varphi \in (\varphi_{j-1}, \varphi_j), \quad j = 1, \dots, n+1, \quad (1.6)$$

for example, it is possible to obtain an expression of the function  $\sigma(\varphi)$  that satisfy (1.6) in terms of  $r(\theta(\varphi)) \in C^{0,\alpha}([\varphi_{j-1}, \varphi_j])$ ,  $j = 1, \dots, n+1$ , with  $\lim_{\varphi \rightarrow \varphi_j} r(\theta(\varphi)) = 0$ ,  $j = 1, \dots, n$ .

## 2. Preliminary results

A crucial step in order to obtain the existence theorem is the following Miranda-Talenti type estimate.

LEMMA 2.1. *Under assumptions (1.3), (1.4), (1.5), (1.6), for every  $u \in W$ , it results*

$$\int_{\Omega} \sum_{i,j=1}^2 (D_{ij}u)^2 dx \leq \int_{\Omega} |\Delta u|^2 dx. \quad (2.11)$$

*Proof.* We will prove the lemma supposing  $\cos \theta(\varphi) = 0$  in the unique point  $\varphi = \varphi_1$ . By the identity

$$\sum_{i,k=1}^2 (D_{iku})^2 + \sum_{i,k=1}^2 (D_{ii}uD_{kk}u - (D_{ik}u)^2) = (\Delta u)^2 \quad (2.12)$$

in order to obtain estimate (2.11), we have to prove

$$\int_{\Omega} (D_{11}uD_{22}u - (D_{12}u)^2) dx \geq 0.$$

Taking into account the identity

$$\begin{aligned} & D_{11}uD_{22}u - (D_{12}u)^2 \\ &= \frac{1}{2} \frac{\partial}{\partial x_1} (D_{1u}D_{22}u - D_{2u}D_{12}u) - \frac{1}{2} \frac{\partial}{\partial x_2} (D_{1u}D_{21}u - D_{2u}D_{11}u) \end{aligned}$$

by means of Gauss formula, we get

$$\begin{aligned} & \int_{\Omega} (D_{11}uD_{22}u - (D_{12}u)^2) dx \\ &= \frac{1}{2} \int_0^L [(D_{1u}D_{22}u - D_{2u}D_{12}u)X_1 - (D_{1u}D_{21}u - D_{2u}D_{11}u)X_2] d\varphi. \end{aligned} \quad (2.13)$$

Let us consider the system

$$\begin{cases} D_{1u}Y_1 + D_{2u}Y_2 = -\sigma(\varphi)u \\ -D_{1u}Y_2 + D_{2u}Y_1 = c(\varphi). \end{cases} \quad \forall \varphi \in (\varphi_{j-1}, \varphi_j), \quad j = 1, 2. \quad (2.14)$$

It results

$$\begin{cases} D_1u = -\sigma(\varphi)uY_1 - c(\varphi)Y_2 \\ D_2u = c(\varphi)Y_1 - \sigma(\varphi)uY_2, \end{cases} \quad \forall \varphi \in (\varphi_{j-1}, \varphi_j), \quad j = 1, 2. \quad (2.15)$$

Moreover

$$(D_1u)^2 + (D_2u)^2 = c^2(\varphi) + \sigma^2(\varphi)u^2 \quad \forall \varphi \in (\varphi_{j-1}, \varphi_j), \quad j = 1, 2$$

and

$$c(0) = \lim_{\varphi \rightarrow \varphi_0^+} [-D_1uY_2 + D_2uY_1] = \lim_{\varphi \rightarrow \varphi_{n+1}^-} [-D_1uY_2 + D_2uY_1] = c(L).$$

If we substitute in (2.13)

$$\begin{aligned} & \int_{\Omega} [D_{11}uD_{22}u - (D_{12}u)^2] dx \\ &= \frac{1}{2} \int_0^{\varphi_1} c(\varphi) [-D_{22}uX_1Y_2 - D_{12}uX_1Y_1 + D_{21}uX_2Y_2 + D_{11}uX_2Y_1] d\varphi \\ &+ \frac{1}{2} \int_0^L c(\varphi) [-D_{22}uX_1Y_2 - D_{12}uX_1Y_1 + D_{21}uX_2Y_2 + D_{11}uX_2Y_1] d\varphi \\ &+ \frac{1}{2} \int_0^{\varphi_1} \sigma(\varphi)u [-D_{22}uX_1Y_1 + D_{12}uX_1Y_2 + D_{21}uX_2Y_1 - D_{11}uX_2Y_2] d\varphi \\ &+ \frac{1}{2} \int_{\varphi_1}^L \sigma(\varphi)u [-D_{22}uX_1Y_1 + D_{12}uX_1Y_2 + D_{21}uX_2Y_1 - D_{11}uX_2Y_2] d\varphi. \end{aligned} \quad (2.16)$$

Differentiating the equations of system (2.14), we have

$$-D_{11}uX_2Y_1 + D_{12}uX_1Y_1 - D_{21}uX_2Y_2 + D_{22}uX_1Y_2 = -\sigma'(\varphi)u - \sigma(\varphi) \frac{du}{d\varphi} - [D_{11}uY_1' + D_{22}uY_2']$$

and

$$D_{11}uX_2Y_2 - D_{12}uX_1Y_2 - D_{21}uX_2Y_1 + D_{22}uX_1Y_1 = c'(\varphi) + D_{11}uY_2' - D_{22}uY_1'.$$

Bearing in mind (2.16), replacing the above identities and taking into account that

$$Y_1Y_2' - Y_1'Y_2 = \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \text{ in } (0, \varphi_1) \text{ and } (\varphi_1, L), \text{ we get}$$

$$\begin{aligned} & \int_{\Omega} [D_{11}uD_{22}u - (D_{12}u)^2] dx \\ &= \frac{1}{2} \int_0^{\varphi_1} c(\varphi)\sigma'(\varphi)ud\varphi - \frac{1}{2} \int_0^{\varphi_1} \sigma(\varphi)uc'(\varphi)d\varphi + \frac{1}{2} \int_0^{\varphi_1} c(\varphi)\sigma(\varphi) \frac{du}{d\varphi} d\varphi \\ &+ \frac{1}{2} \int_{\varphi_1}^L c(\varphi)\sigma'(\varphi)ud\varphi - \frac{1}{2} \int_{\varphi_1}^L \sigma(\varphi)uc'(\varphi)d\varphi + \frac{1}{2} \int_{\varphi_1}^L c(\varphi)\sigma(\varphi) \frac{du}{d\varphi} d\varphi \\ &+ \frac{1}{2} \int_0^{\varphi_1} ((D_1u)^2 + (D_2u)^2) \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi \\ &+ \frac{1}{2} \int_{\varphi_1}^L ((D_1u)^2 + (D_2u)^2) \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi. \end{aligned} \quad (2.7)$$

Integrating by parts and bearing in mind that  $u(\varphi_1) = 0$ ,  $u(0) = u(L)$ ,  $c(0) = c(L)$ ,  $\sigma(0) = \sigma(L)$

$$\begin{aligned}
 & \int_{\Omega} [D_{11}u D_{22}u - (D_{12}u)^2] dx \\
 &= \frac{1}{2} \int_0^{\varphi_1} ((D_1u)^2 + (D_2u)^2) \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi \\
 & \quad + \frac{1}{2} \int_{\varphi_1}^L ((D_1u)^2 + (D_2u)^2) \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi \\
 & \quad + \int_0^{\varphi_1} uc(\varphi)\sigma'(\varphi)d\varphi + \int_{\varphi_1}^L uc(\varphi)\sigma'(\varphi)d\varphi \\
 & \quad + \int_0^{\varphi_1} c(\varphi)\sigma(\varphi)\frac{du}{d\varphi}d\varphi + \int_{\varphi_1}^L c(\varphi)\sigma(\varphi)\frac{du}{d\varphi}d\varphi. \tag{2.8}
 \end{aligned}$$

Let us observe that

$$\begin{aligned}
 \int_0^{\varphi_1} u(\varphi)c(\varphi)\sigma'(\varphi) d\varphi &= \lim_{\varepsilon \rightarrow 0} \int_0^{\varphi_1 - \varepsilon} u(\varphi)c(\varphi)\sigma'(\varphi) d\varphi, \\
 \int_{\varphi_1}^L u(\varphi)c(\varphi)\sigma'(\varphi) d\varphi &= \lim_{\varepsilon \rightarrow 0} \int_{\varphi_1 + \varepsilon}^L u(\varphi)c(\varphi)\sigma'(\varphi) d\varphi, \\
 \int_0^{\varphi_1} c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} d\varphi &= \lim_{\varepsilon \rightarrow 0} \int_0^{\varphi_1 - \varepsilon} c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} d\varphi, \\
 \int_{\varphi_1}^L c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} d\varphi &= \lim_{\varepsilon \rightarrow 0} \int_{\varphi_1 + \varepsilon}^L c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} d\varphi.
 \end{aligned}$$

In the intervals  $[0, \varphi_1 - \varepsilon]$ ,  $[\varphi_1 + \varepsilon, L]$ ,  $0 < \varepsilon < \min\{\varphi_1, L - \varphi_1\}$ , we have  $\cos \theta(\varphi) > 0$ , then from the system

$$\begin{cases} D_1uY_1 + D_2uY_2 = -\sigma(\varphi)u \\ -D_1uX_2 + D_2uX_1 = \frac{du}{d\varphi}. \end{cases} \tag{2.19}$$

it results

$$\begin{cases} D_1u = \frac{1}{\cos \theta} [-\sigma(\varphi)uX_1 - \frac{du}{d\varphi}Y_2] \\ D_2u = \frac{1}{\cos \theta} [\frac{du}{d\varphi}Y_1 - \sigma(\varphi)uX_2] \\ c(\varphi) = \frac{du}{d\varphi} \frac{1}{\cos \theta} + \sigma(\varphi)u \tan \theta, \end{cases}$$

and

$$\begin{aligned}
 u(\varphi)c(\varphi)\sigma'(\varphi) &= \sigma'(\varphi)[u(\varphi)\frac{du}{d\varphi}\frac{1}{\cos \theta} + u^2(\varphi)\sigma(\varphi)\tan \theta], \\
 c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} &= \frac{1}{\cos \theta(\varphi)} \left( \frac{du}{d\varphi} \right)^2 \sigma(\varphi) + \sigma^2(\varphi)u(\varphi)\frac{du}{d\varphi} \tan \theta(\varphi).
 \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\varphi_1-\varepsilon} u(\varphi)c(\varphi)\sigma'(\varphi) d\varphi &= \int_0^{\varphi_1-\varepsilon} \sigma'(\varphi)\left[u(\varphi)\frac{du}{d\varphi}\frac{1}{\cos\theta} + u^2(\varphi)\sigma(\varphi)\tan\theta\right]d\varphi, \\ \int_{\varphi_1+\varepsilon}^L u(\varphi)c(\varphi)\sigma'(\varphi) d\varphi &= \int_{\varphi_1+\varepsilon}^L \sigma'(\varphi)\left[u(\varphi)\frac{du}{d\varphi}\frac{1}{\cos\theta} + u^2(\varphi)\sigma(\varphi)\tan\theta\right]d\varphi, \\ \int_0^{\varphi_1-\varepsilon} c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} d\varphi &= \int_0^{\varphi_1-\varepsilon} \frac{1}{\cos\theta(\varphi)}\left(\frac{du}{d\varphi}\right)^2 \sigma(\varphi)d\varphi + \frac{1}{2}\int_0^{\varphi_1-\varepsilon} \sigma^2(\varphi)\frac{du^2}{d\varphi}\tan\theta(\varphi)d\varphi \end{aligned}$$

and

$$\begin{aligned} \int_{\varphi_1+\varepsilon}^L c(\varphi)\sigma(\varphi)\frac{du}{d\varphi} d\varphi &= \int_{\varphi_1+\varepsilon}^L \frac{1}{\cos\theta(\varphi)}\left(\frac{du}{d\varphi}\right)^2 \sigma(\varphi)d\varphi + \frac{1}{2}\int_{\varphi_1+\varepsilon}^L \sigma^2(\varphi)\frac{du^2}{d\varphi}\tan\theta(\varphi)d\varphi. \end{aligned}$$

Taking into account that, integrating by parts, we get

$$\begin{aligned} \frac{1}{2}\int_0^{\varphi_1-\varepsilon} \sigma^2(\varphi)\frac{du^2}{d\varphi}\tan\theta(\varphi)d\varphi &= \frac{1}{2}[u^2(\varphi)\sigma^2(\varphi)\tan\theta(\varphi)]_0^{\varphi_1-\varepsilon} \\ &- \int_0^{\varphi_1-\varepsilon} \sigma(\varphi)\sigma'(\varphi)u^2(\varphi)\tan\theta(\varphi)d\varphi - \frac{1}{2}\int_0^{\varphi_1-\varepsilon} \sigma^2(\varphi)\frac{u^2(\varphi)}{\cos^2\theta(\varphi)}\frac{d\theta}{d\varphi}d\varphi \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\int_{\varphi_1+\varepsilon}^L \sigma^2(\varphi)\frac{du^2}{d\varphi}\tan\theta(\varphi)d\varphi &= \frac{1}{2}[u^2(\varphi)\sigma^2(\varphi)\tan\theta(\varphi)]_{\varphi_1+\varepsilon}^L \\ &- \int_{\varphi_1+\varepsilon}^L \sigma(\varphi)\sigma'(\varphi)u^2(\varphi)\tan\theta(\varphi)d\varphi - \frac{1}{2}\int_{\varphi_1+\varepsilon}^L \sigma^2(\varphi)\frac{u^2(\varphi)}{\cos^2\theta(\varphi)}\frac{d\theta}{d\varphi}d\varphi, \end{aligned}$$

the last two lines in (2.8) become

$$\begin{aligned} &\int_0^{\varphi_1} uc(\varphi)\sigma'(\varphi)d\varphi + \int_{\varphi_1}^L uc(\varphi)\sigma'(\varphi)d\varphi + \int_0^{\varphi_1} c(\varphi)\sigma(\varphi)\frac{du}{d\varphi}d\varphi + \int_{\varphi_1}^L c(\varphi)\sigma(\varphi)\frac{du}{d\varphi}d\varphi \\ &= \lim_{\varepsilon\rightarrow 0} \left[ \int_0^{\varphi_1-\varepsilon} \frac{1}{\cos\theta(\varphi)}\left(\frac{du}{d\varphi}\right)^2 \sigma(\varphi)d\varphi - \frac{1}{2}\int_0^{\varphi_1-\varepsilon} \sigma^2(\varphi)\frac{u^2(\varphi)}{\cos^2\theta(\varphi)}\frac{d\theta}{d\varphi}d\varphi \right. \\ &\quad \left. + \int_0^{\varphi_1-\varepsilon} \sigma'(\varphi)u(\varphi)\frac{du}{d\varphi}\frac{1}{\cos\theta}d\varphi \right] + \lim_{\varepsilon\rightarrow 0} \left[ \int_{\varphi_1+\varepsilon}^L \frac{1}{\cos\theta(\varphi)}\left(\frac{du}{d\varphi}\right)^2 \sigma(\varphi)d\varphi \right. \\ &\quad \left. - \frac{1}{2}\int_{\varphi_1+\varepsilon}^L \sigma^2(\varphi)\frac{u^2(\varphi)}{\cos^2\theta(\varphi)}\frac{d\theta}{d\varphi}d\varphi + \int_{\varphi_1+\varepsilon}^L \sigma'(\varphi)u(\varphi)\frac{du}{d\varphi}\frac{1}{\cos\theta}d\varphi \right] \\ &\quad + \frac{1}{2}\lim_{\varepsilon\rightarrow 0} \{ [u^2(\varphi)\sigma^2(\varphi)\tan\theta(\varphi)]_0^{\varphi_1-\varepsilon} + [u^2(\varphi)\sigma^2(\varphi)\tan\theta(\varphi)]_{\varphi_1+\varepsilon}^L \}. \end{aligned} \tag{2.10}$$

Repeating the same arguments as in [5] we obtain that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^{\varphi_1 - \varepsilon} \frac{1}{\cos \theta(\varphi)} \left( \frac{du}{d\varphi} \right)^2 \sigma(\varphi) d\varphi &= \int_0^{\varphi_1} \frac{1}{\cos \theta(\varphi)} \left( \frac{du}{d\varphi} \right)^2 \sigma(\varphi) d\varphi, \\ \lim_{\varepsilon \rightarrow 0} \int_{\varphi_1 + \varepsilon}^L \frac{1}{\cos \theta(\varphi)} \left( \frac{du}{d\varphi} \right)^2 \sigma(\varphi) d\varphi &= \int_{\varphi_1}^L \frac{1}{\cos \theta(\varphi)} \left( \frac{du}{d\varphi} \right)^2 \sigma(\varphi) d\varphi, \\ \lim_{\varepsilon \rightarrow 0} \int_0^{\varphi_1 - \varepsilon} \sigma^2(\varphi) \frac{u^2(\varphi)}{\cos^2 \theta(\varphi)} \frac{d\theta}{d\varphi} d\varphi &= \int_0^{\varphi_1} \sigma^2(\varphi) \frac{u^2(\varphi)}{\cos^2 \theta(\varphi)} \frac{d\theta}{d\varphi} d\varphi, \\ \lim_{\varepsilon \rightarrow 0} \int_{\varphi_1 + \varepsilon}^L \sigma^2(\varphi) \frac{u^2(\varphi)}{\cos^2 \theta(\varphi)} \frac{d\theta}{d\varphi} d\varphi &= \int_{\varphi_1}^L \sigma^2(\varphi) \frac{u^2(\varphi)}{\cos^2 \theta(\varphi)} \frac{d\theta}{d\varphi} d\varphi, \\ \lim_{\varepsilon \rightarrow 0} u^2(\varphi_1 - \varepsilon) \sigma^2(\varphi_1 - \varepsilon) \tan \theta(\varphi_1 - \varepsilon) &\leq k \sigma^2(\varphi_1^-) \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2}{\cos \theta(\varphi_1 - \varepsilon)} = 0 \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} u^2(\varphi_1 + \varepsilon) \sigma^2(\varphi_1 + \varepsilon) \tan \theta(\varphi_1 + \varepsilon) \leq k \sigma^2(\varphi_1^+) \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2}{\cos \theta(\varphi_1 + \varepsilon)} = 0.$$

It remains to consider in (2.10) the terms

$$\int_0^{\varphi_1 - \varepsilon} \sigma'(\varphi) u(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi \quad \text{and} \quad \int_{\varphi_1 + \varepsilon}^L \sigma'(\varphi) u(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi.$$

Taking into account Remark 1., we have for all  $\varphi \in [0, \varphi_1]$  and for suitable points  $\bar{\varphi} \in ]\varphi, \varphi_1[$ ,  $\varphi^* \in ]\varphi, \varphi_1[$

$$\begin{aligned} u(\varphi) &= u(\varphi) - u(\varphi_1) = (\varphi - \varphi_1) \frac{du(\bar{\varphi})}{d\varphi} \\ \frac{du(\varphi)}{d\varphi} &= \frac{du(\varphi)}{d\varphi} - \frac{du(\varphi_1)}{d\varphi} = (\varphi - \varphi_1) \frac{d^2 u(\varphi^*)}{d\varphi^2}. \end{aligned}$$

If we set  $M = \max_{[0, L]} \frac{du(\varphi)}{d\varphi}$ ,  $N = \max_{[0, L]} \frac{d^2 u(\varphi)}{d\varphi^2}$ , bearing in mind condition (1.5)

$$\begin{aligned} \lim_{\varphi \rightarrow \varphi_1^-} \left| \frac{u(\varphi)}{\cos \theta(\varphi)} \sigma'(\varphi) \frac{du}{d\varphi} \right| &\leq NM \lim_{\varphi \rightarrow \varphi_1^-} \frac{(\varphi - \varphi_1)^2}{\cos \theta(\varphi)} |\sigma'(\varphi)| \\ &= K \lim_{\varphi \rightarrow \varphi_1^-} \left| \frac{\theta(\varphi) - \theta(\varphi_1)}{\cos \theta(\varphi)} \right| \left| \frac{|\varphi - \varphi_1|}{\left| \frac{\theta(\varphi) - \theta(\varphi_1)}{\varphi - \varphi_1} \right|} \right| = 0 \end{aligned} \quad (2.11)$$

and then

$$\lim_{\varepsilon \rightarrow 0} \int_0^{\varphi_1 - \varepsilon} \sigma'(\varphi) u(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi = \int_0^{\varphi_1} \sigma'(\varphi) u(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi.$$

Similarly we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\varphi_1+\varepsilon}^L \sigma'(\varphi)u(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi = \int_{\varphi_1}^L \sigma'(\varphi)u(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi.$$

Then (2.8) becomes

$$\begin{aligned} \int_{\Omega} (D_{11}u D_{22}u - (D_{12}u)^2) dx &= \frac{1}{2} \int_0^{\varphi_1} [(D_1u)^2 + (D_2u)^2] \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi \\ &+ \frac{1}{2} \int_{\varphi_1}^L [(D_1u)^2 + (D_2u)^2] \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi + \int_0^{\varphi_1} \sigma(\varphi) \left( \frac{du}{d\varphi} \right)^2 \frac{1}{\cos \theta} d\varphi \\ &- \frac{1}{2} \int_0^{\varphi_1} \sigma^2(\varphi) \frac{u^2}{\cos^2 \theta} \frac{d\theta}{d\varphi} d\varphi + \int_{\varphi_1}^L \sigma(\varphi) \left( \frac{du}{d\varphi} \right)^2 \frac{1}{\cos \theta} d\varphi \\ &- \frac{1}{2} \int_{\varphi_1}^L \sigma^2(\varphi) \frac{u^2}{\cos^2 \theta} \frac{d\theta}{d\varphi} d\varphi + \int_0^{\varphi_1} u\sigma'(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi + \int_{\varphi_1}^L u\sigma'(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi \\ &+ \frac{1}{2} u^2(L) \sigma^2(L) \tan \theta(L) - \frac{1}{2} u^2(0) \sigma^2(0) \tan \theta(0). \end{aligned} \quad (2.12)$$

Taking into account that  $u(\varphi_1) = 0$  and the vanishing limit obtained in (2.11), integrating by parts, the last two integrals in (2.12) become

$$\begin{aligned} &\int_0^{\varphi_1} u\sigma'(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi \\ &= \frac{1}{2} \left[ \sigma'(\varphi) \frac{u^2}{\cos \theta} \right]_0^{\varphi_1} - \frac{1}{2} \int_0^{\varphi_1} \frac{u^2}{\cos \theta} [\sigma''(\varphi) + \sigma'(\varphi) \frac{\sin \theta}{\cos \theta} \frac{d\theta}{d\varphi}] d\varphi \\ &= -\frac{1}{2} \sigma'(0) \frac{u^2(0)}{\cos \theta(0)} - \frac{1}{2} \int_0^{\varphi_1} \frac{u^2}{\cos \theta} [\sigma''(\varphi) + \sigma'(\varphi) \frac{\sin \theta}{\cos \theta} \frac{d\theta}{d\varphi}] d\varphi \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} &\int_{\varphi_1}^L u\sigma'(\varphi) \frac{du}{d\varphi} \frac{1}{\cos \theta} d\varphi \\ &= \frac{1}{2} \sigma'(L) \frac{u^2(L)}{\cos \theta(L)} - \frac{1}{2} \int_{\varphi_1}^L \frac{u^2}{\cos \theta} [\sigma''(\varphi) + \sigma'(\varphi) \frac{\sin \theta}{\cos \theta} \frac{d\theta}{d\varphi}] d\varphi. \end{aligned} \quad (2.14)$$

Finally, since (1.5) implies  $\tan \theta(L) = \tan \theta(0)$ , from (2.12), (2.13), (2.14), we get

$$\begin{aligned} \int_{\Omega} (D_{11}uD_{22}u - (D_{12}u)^2)dx &= \frac{1}{2} \int_0^{\varphi_1} [(D_1u)^2 + (D_2u)^2] \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi \\ &+ \frac{1}{2} \int_{\varphi_1}^L [(D_1u)^2 + (D_2u)^2] \left( \frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \right) d\varphi \\ &+ \int_0^{\varphi_1} \sigma(\varphi) \left( \frac{du}{d\varphi} \right)^2 \frac{1}{\cos \theta} d\varphi - \frac{1}{2} \int_0^{\varphi_1} \sigma^2(\varphi) \frac{u^2}{\cos^2 \theta} \frac{d\theta}{d\varphi} d\varphi \\ &- \frac{1}{2} \int_0^{\varphi_1} \frac{u^2}{\cos \theta} [\sigma''(\varphi) + \sigma'(\varphi) \frac{\sin \theta}{\cos \theta} \frac{d\theta}{d\varphi}] d\varphi + \int_{\varphi_1}^L \sigma(\varphi) \left( \frac{du}{d\varphi} \right)^2 \frac{1}{\cos \theta} d\varphi \\ &- \frac{1}{2} \int_{\varphi_1}^L \sigma^2(\varphi) \frac{u^2}{\cos^2 \theta} \frac{d\theta}{d\varphi} d\varphi - \frac{1}{2} \int_{\varphi_1}^L \frac{u^2}{\cos \theta} [\sigma''(\varphi) + \sigma'(\varphi) \frac{\sin \theta}{\cos \theta} \frac{d\theta}{d\varphi}] d\varphi. \end{aligned}$$

Then by conditions (1.3), (1.5), (1.6), we obtain estimate (2.11).

Let us observe that estimate (2.11) holds true also for  $u \in W_l^2(\Omega)$ . In fact let  $\{u_n\} \subset W$ , such that  $u_n \rightarrow u$  in  $W_l^2(\Omega)$ , we have

$$\|u_n\|_{W_l^2(\Omega)} \leq \|\Delta u_n\|_{L^2(\Omega)},$$

and hence, as  $n \rightarrow \infty$ , we get

$$\|u\|_{W_l^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)}.$$

As in [6] we get the following result.

LEMMA 2.2. *Let  $u \in W_l^q(\Omega)$ ,  $q \geq 2$ . Then  $\|u\|_{W_l^q(\Omega)}$  is an equivalent norm to  $\|u\|_{W^{2,q}(\Omega)}$ .*

In some stage of the proof we will use the following existence and uniqueness result for the Laplacian.

LEMMA 2.3. *Under assumptions (1.3), (1.4), (1.5), (1.6), for each  $f \in L^q(\Omega)$ ,  $q \geq 2$ , the oblique derivative problem for the Laplacian*

$$\begin{cases} \Delta u = f(x) & \text{a.e. in } \Omega \subset \mathbb{R}^2 \\ u \in W_l^q(\Omega) \end{cases} \tag{2.25}$$

*is uniquely solvable. Moreover it results*

$$\|u\|_{W_l^q(\Omega)} \leq c(q) \|\Delta u\|_{L^q(\Omega)} \tag{2.26}$$

*where  $c(q) : [2, +\infty) \rightarrow [1, +\infty)$  is a continuous function at  $q = 2$  and  $c(2) = 1$ .*

*Proof.* As it is known from the classical theory (cf. [11] page 16), the linear oblique derivative problem in two dimensions is always a non-degenerate one if the field  $l$  is nowhere zero. That is, the problem has a finite index. On the other hand in our

situation, since the vector field  $l$  is tangential to  $\partial\Omega$  at a finite number of points, the kernel of the linear problem is nontrivial. This difficulty is avoidable by prescribing the values of the solution  $u$  at the points of tangency (cf. [11] page 14). This way a new problem arises having a trivial kernel and therefore solvable for any data. Concerning our problem, in virtue of (1.10), the values of  $u$  at the points of tangency are known from the boundary condition and, hence, it follows the uniquely solvability of problem (2.25), that is  $\Delta$  is an isomorphism from  $W_l^q(\Omega)$  in  $L^q(\Omega)$ ,  $\forall q \in [2, +\infty[$ . As a consequence the linear inverse isomorphism  $\Delta_q^{-1} : L^q(\Omega) \rightarrow W_l^q(\Omega)$  is a continuous one and the following estimate holds:

$$\|u\|_{W_l^q(\Omega)} = \|\Delta_q^{-1}f\|_{W_l^q(\Omega)} \leq c(q) \|f\|_{L^q(\Omega)} \quad \forall u \in W_l^q(\Omega).$$

In virtue of Lemma 2.1,  $c(2) = 1$  and then  $\|\Delta_2^{-1}\| \leq 1$ . If we fix  $r \in (2, +\infty)$ , by means of convexity lemma, we get,  $\forall q \in [2, r]$

$$\|\Delta_q^{-1}\| \leq \|\Delta_2^{-1}\|^a \|\Delta_r^{-1}\|^{1-a} \leq \|\Delta_r^{-1}\|^{1-a}$$

with  $\frac{1}{q} = \frac{a}{2} + \frac{1-a}{r}$ ,  $\forall a \in ]0, 1[$ . Then

$$\|\Delta_q^{-1}\| \leq \|\Delta_r^{-1}\|^{\frac{r(q-2)}{q(r-2)}}$$

If we substitute  $c(q)$  with the function  $\|\Delta_r^{-1}\|^{\frac{r(q-2)}{q(r-2)}}$ , we obtain that  $c(q)$  is a continuous function at  $q = 2$ , with  $c(2) = 1$ .

Repeating the same arguments as in [3], bearing in mind Lemma 2.3, we may easily derive the following result related to a Carathéodory operator  $\mathcal{A}(x, \xi)$ .

**THEOREM 2.1.** *Under assumptions (1.3), (1.4), (1.5), (1.6), (A), there exists a number  $q_0 > 2$ , such that  $\forall q \in [2, q_0[$ ,  $\forall f \in L^q(\Omega)$ , the problem*

$$\begin{cases} \mathcal{A}(x, D^2u) = f(x) & \text{a.e. in } \Omega \\ u \in W_l^q(\Omega) \end{cases}$$

*admits a unique solution and it results:*

$$\|u\|_{W_l^q(\Omega)} \leq \frac{\alpha c(q)}{1 - k(q)} \|f(x)\|_{L^q(\Omega)}$$

where  $k(q) = 2^{\frac{q-2}{2q}} (\gamma^{\frac{q}{2}} [c(q)]^q + \delta^{\frac{q}{2}})^{\frac{1}{q}}$ .

We note that  $q_0$  is such that, for each  $q \in [2, q_0[$ , it results  $1 - k(q) > 0$ .

### 3. Proof of the theorems

Theorem 1.1 will be proved by the aid of the Leray-Schauder fixed point theorem. For this let us fix  $v \in W^{1,2q}(\Omega)$ ,  $q < \tilde{q} = \min\{q_0, r\}$ . Then  $f(x, v, Dv)$  belongs to  $L^q(\Omega)$ . In fact,  $f_2 \in L^q(\Omega)$  for all  $q \leq r$  and by condition (1.8)

$$|f(x, v, Dv)| \leq f_1(|v|)[f_2(x) + |Dv|^2] \leq f_1(\max |v|)[f_2(x) + |Dv|^2].$$

If we consider for fixed  $\tau \in [0, 1]$  the problem

$$\begin{cases} \mathcal{A}(x, v, Dv, D^2u) = \tau f(x, v, Dv) & \text{a.e. in } \Omega \\ u \in W_l^q(\Omega) \end{cases} \quad (3.1)$$

by means of Theorem 2.1, for each  $q \in (2, \tilde{q})$  problem (3.1) admits a unique solution. So we have defined an operator

$$T : W^{1,2q}(\Omega) \times [0, 1] \rightarrow W_l^q(\Omega) \hookrightarrow W^{1,2q}(\Omega)$$

where the image  $u = T(v, \tau)$  is the unique solution of (3.1). It is easily seen that each fixed point of the mapping  $T(\cdot, 1)$  is a solution of (1.7). The condition  $\mathcal{A}(x, z, p, 0) = 0$  as required above and uniqueness of the solution of problem (3.1) show that  $T(v, 0) = 0$  for all  $v \in W^{1,2q}(\Omega)$ . Taking into account Theorem 2.1, the continuity and the compactness of the mapping  $T$  follows as in [10], [12]. To assert the existence of a fixed point of  $T(\cdot, 1)$  it remains to prove the a priori estimate

$$\|u\|_{W^{1,2q}(\Omega)} \leq C \quad (3.28)$$

with a constant  $C$  independent of  $u$  and  $\tau$  for each solution  $u$  of the problem

$$\begin{cases} \mathcal{A}(x, u, Du, D^2u) = \tau f(x, u, Du) & \text{a.e. in } \Omega \\ u \in W_l^q(\Omega). \end{cases} \quad (3.3)$$

In a first step we prove a priori bound for  $\|u\|_{L^\infty(\Omega)}$ .

LEMMA 3.1. *Suppose that conditions (A) and (1.9) are fulfilled. Then*

$$\|u\|_{L^\infty(\Omega)} \leq G = M + R \left\{ \text{diam } \Omega + \sup_{\partial\Omega} \frac{1}{\sigma(x)} \right\} \quad (3.30)$$

for each solution  $u$  of problem (3.3), where  $R$  is such that  $\int_{B_R(0)} h(p) dp = \int_{\Omega} g(x) dx$  and  $B_R(0)$  is a ball with center at the origin and radius  $R$ .

*Proof.* If we set

$$a^{ij}(x) = \int_0^1 \frac{\partial \mathcal{A}}{\partial \xi_{ij}}(x, u, Du, sD^2u) ds \in L^\infty(\Omega),$$

since  $\mathcal{A}(x, z, p, 0) = 0$ , the function  $u$ , solution of problem (3.3), solves the problem

$$\begin{cases} a^{ij}(x)D_{ij}u = \tau f(x, u, Du) & \text{a.e. in } \Omega \\ u \in W_l^q(\Omega). \end{cases}$$

Hence the estimate will follow from condition (1.9) and Theorem 2.6.1 in [14].

In the second step of the proof of the existence theorem we obtain a priori estimate for the  $L^{2q}(\Omega)$  norm of the gradient  $Du$ . For this goal an approach due to Amann and Crandall [1] will be used.

**THEOREM 3.1.** *Suppose conditions (A), (1.3), (1.4), (1.5), (1.6), (1.8), (1.9) be fulfilled. Then there exists a constant  $C = C(\alpha, \gamma, \delta, q, \partial\Omega, f_1, f_2, M, R, \text{diam } \Omega, \sup_{\partial\Omega} \frac{1}{\sigma(x)})$  such that, for each  $q \in (2, \tilde{q})$ ,*

$$\|Du\|_{L^{2q}(\Omega)} \leq C \tag{3.31}$$

for each solution  $u$  of problem (3.3).

*Proof.* Let  $u$  be a solution of problem (3.3). The equation in (3.3) is equivalent to the one

$$\mathcal{A}(x, u, Du, D^2u) + b(x)|Du|^2 - f_2(x)u(x) = F(x)$$

where

$$b(x) = -\tau \frac{f(x, u, Du)}{f_2(x) + |Du|^2}; \quad F(x) = \tau \frac{f(x, u, Du)f_2(x)}{f_2(x) + |Du|^2} - f_2(x)u(x).$$

From condition (1.8) it follows that  $b \in L^\infty(\Omega)$  and  $F \in L^q(\Omega)$  ( $q \leq r$ ). In fact,

$$|b(x)| \leq f_1(\|u\|_{L^\infty(\Omega)})$$

and

$$|F(x)| \leq \tau f_1(\|u\|_{L^\infty(\Omega)})f_2(x) + f_2(x)|u(x)| \leq f_2(x)[f_1(\|u\|_{L^\infty(\Omega)}) + \|u\|_{L^\infty(\Omega)}]. \tag{3.32}$$

For the fixed solution  $u$  of problem (3.3), we consider the problem with  $\rho \in [0, 1]$

$$\begin{cases} \mathcal{A}(x, u, Du, D^2v) + b(x)|Dv|^2 - f_2(x)v(x) = \rho F(x) & \text{a.e. in } \Omega \\ v \in W_1^q(\Omega). \end{cases} \tag{3.7}$$

Let us note that the function  $v = 0$  solves (3.7) if  $\rho = 0$ , and  $u$  solves it if  $\rho = 1$ . Consider solutions  $v_1$  and  $v_2$  of problem (3.7) corresponding to the respective values  $\rho_1 < \rho_2$  of the parameter  $\rho$ . In order to prove estimate (3.31), the first step is to obtain an uniqueness result for problem (3.7). Then, in the second step, we estimate  $\|Dv_2\|_{L^{2q}(\Omega)}$  in terms of  $\|Dv_1\|_{L^{2q}(\Omega)}$ , assuming  $\rho_2 - \rho_1 > 0$  small enough.

**LEMMA 3.2.** *Let  $v_1, v_2 \in W_1^q(\Omega)$  be solutions to problem (3.7) with  $\rho_1 < \rho_2$ . Then*

$$\|v_1 - v_2\|_{L^\infty(\Omega)} \leq (\rho_2 - \rho_1)[f_1(\|u\|_{L^\infty(\Omega)}) + \|u\|_{L^\infty(\Omega)}]. \tag{3.34}$$

*Proof.* The functions  $v_1, v_2$  satisfy the problem

$$\begin{cases} \mathcal{A}(x, u, Du, D^2v_1) - \mathcal{A}(x, u, Du, D^2v_2) + b(x)[|Dv_1|^2 - |Dv_2|^2] \\ \quad - f_2(x)[v_1(x) - v_2(x)] = F(x)(\rho_1 - \rho_2) & \text{a.e. in } \Omega \\ v_1 - v_2 \in W_1^q(\Omega). \end{cases} \tag{3.9}$$

It is possible to verify that

$$\mathcal{A}(x, u, Du, D^2v_1) - \mathcal{A}(x, u, Du, D^2v_2) = \sum_{i,j=1}^2 A_{ij}(x)D_{ij}(v_1 - v_2)$$

where  $A_{ij}(x) = \int_0^1 \frac{\partial \mathcal{A}}{\partial \xi_{ij}}(x, u, Du, s(D^2v_1 - D^2v_2) + D^2v_2) ds$  and similarly

$$b(x) [ |Dv_1|^2 - |Dv_2|^2 ] = \sum_{i=1}^2 b_i(x) D_i(v_1 - v_2)$$

where  $b_i(x) = 2b(x) \int_0^1 [sD_i(v_1 - v_2) + D_i v_2] ds$ . If we set  $w = v_1 - v_2$ , problem (3.9) is equivalent to

$$\begin{cases} \mathcal{L}w = \sum_{i,j=1}^2 A_{ij}(x) D_{ij}w + \sum_{i=1}^2 b_i(x) D_i w - f_2(x)w(x) = F(x)(\rho_1 - \rho_2) & \text{a.e. in } \Omega \\ w \in W_l^q(\Omega). \end{cases}$$

By (3.32), since  $\rho_1 - \rho_2$  is negative, we get

$$\mathcal{L}w \geq (\rho_1 - \rho_2) f_2(x) [f_1(\|u\|_{L^\infty(\Omega)}) + \|u\|_{L^\infty(\Omega)}].$$

Setting  $W = (\rho_2 - \rho_1) [f_1(\|u\|_{L^\infty(\Omega)}) + \|u\|_{L^\infty(\Omega)}]$  it results  $\mathcal{L}W = -f_2(x)W$  and  $\mathcal{L}w \geq \mathcal{L}W$ . Then the function  $w \in W_l^q(\Omega)$  satisfies the problem

$$\begin{cases} \mathcal{L}(w - W) \geq 0 & \text{a.e. in } \Omega \\ \frac{\partial(w - W)}{\partial l} + \sigma(x)(w - W) \leq 0 & \text{on } \partial\Omega. \end{cases}$$

By means of Lemma 2.1.3 and Remark to the end of Chapter 2 in [14], we obtain  $w \leq W$  a.e. in  $\Omega$ . If we replace  $w$  with  $-w$  and consider the same problem, we derive an estimate for  $w$  from below  $w \geq -W$  a.e. in  $\Omega$ . Then

$$\|w\|_{L^\infty(\Omega)} \leq W = (\rho_2 - \rho_1) [f_1(\|u\|_{L^\infty(\Omega)}) + \|u\|_{L^\infty(\Omega)}].$$

Thus the statement is proved.

**COROLLARY 3.1.** *If problem (3.7) has a solution  $v$  for some  $\rho \in [0, 1]$ , then it is a unique solution.*

*Proof.* It follows from (3.34) putting  $\rho_1 = \rho_2$ .

Now we are in position to prove a gradient estimate.

Let us consider the equation

$$\Delta w = \Delta w - \alpha [ \mathcal{A}(x, u, Du, D^2v_1) - \mathcal{A}(x, u, Du, D^2v_2) ] + \alpha H(x)$$

where

$$H(x) = F(x)(\rho_1 - \rho_2) - b(x) [ |Dv_1|^2 - |Dv_2|^2 ] + f_2(x)w(x).$$

By means of condition (A), the Minkowsky inequality and Lemma 2.3 we have

$$\begin{aligned} \left( \int_{\Omega} |\Delta w|^q dx \right)^{\frac{1}{q}} &\leq \left( \int_{\Omega} \left( \left| \Delta w - \alpha \left[ \mathcal{A}(x, u, Du, D^2 v_1) - \mathcal{A}(x, u, Du, D^2 v_2) \right] \right|^2 \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}} \\ &+ \alpha \left( \int_{\Omega} |H(x)|^q dx \right)^{\frac{1}{q}} \leq 2^{\frac{q-2}{2q}} (\gamma^{\frac{q}{2}} [c(q)]^q + \delta^{\frac{q}{2}})^{\frac{1}{q}} \left( \int_{\Omega} |\Delta w|^q dx \right)^{\frac{1}{q}} \\ &+ \alpha \left( \int_{\Omega} |H(x)|^q dx \right)^{\frac{1}{q}} = k(q) \left( \int_{\Omega} |\Delta w|^q dx \right)^{\frac{1}{q}} + \alpha \left( \int_{\Omega} |H(x)|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Bearing in mind Theorem 2.1,  $k(q) < 1$  for all  $q \in (2, \tilde{q})$ , then it results

$$\left( \int_{\Omega} |\Delta w|^q dx \right)^{\frac{1}{q}} \leq \frac{\alpha}{1 - k(q)} \left( \int_{\Omega} |H(x)|^q dx \right)^{\frac{1}{q}}$$

and therefore, from Lemma 2.3,

$$\|w\|_{W_1^q(\Omega)} \leq \frac{\alpha c(q)}{1 - k(q)} \|H(x)\|_{L^q(\Omega)}.$$

From this, taking into account estimate (3.30), it follows

$$\begin{aligned} \|w\|_{W_1^q(\Omega)} &\leq C_1 \left[ (\rho_2 - \rho_1) \|F(x)\|_{L^q(\Omega)} + \|b(x) [ |Dv_1|^2 - |Dv_2|^2 ]\|_{L^q(\Omega)} + \|f_2 w\|_{L^q(\Omega)} \right] \\ &\leq C_1 \left[ 2(\rho_2 - \rho_1) \|f_2\|_{L^q(\Omega)} [f_1(G) + G] + \|b(x)\|_{L^\infty(\Omega)} \| |Dv_1|^2 - |Dv_2|^2 \|_{L^q(\Omega)} \right] \\ &\leq C_2 (\|f_2\|_{L^q(\Omega)}, \|b(x)\|_{L^\infty(\Omega)}, G) [1 + \|Dv_1\|_{L^{2q}(\Omega)}^2 + \|Dw\|_{L^{2q}(\Omega)}^2]. \end{aligned} \quad (3.10)$$

We can estimate the  $L^{2q}$ -norm of  $Dw$  by the help of the Gagliardo-Nirenberg inequality [9] and by estimates (3.30) and (3.34) it follows

$$\|Dw\|_{L^{2q}(\Omega)}^2 \leq k \|w\|_{L^\infty(\Omega)} \|w\|_{W_1^q(\Omega)} \leq k \|w\|_{W_1^q(\Omega)} (\rho_2 - \rho_1) [f_1(G) + G]. \quad (3.37)$$

Bearing in mind bound (3.10),

$$\|w\|_{W_1^q(\Omega)} \leq C_2 \left[ 1 + \|Dv_1\|_{L^{2q}(\Omega)}^2 + k(\rho_2 - \rho_1) \|w\|_{W_1^q(\Omega)} [f_1(G) + G] \right]$$

and hence

$$\|w\|_{W_1^q(\Omega)} \leq C_3 [1 + \|Dv_1\|_{L^{2q}(\Omega)}^2]$$

assuming  $(\rho_2 - \rho_1) \leq \eta$  to be so small that  $C_2 k(\rho_2 - \rho_1) [f_1(G) + G] < 1$ .

If we consider (3.37), by the last inequality we derive

$$\|Dw\|_{L^{2q}(\Omega)}^2 \leq k(\rho_2 - \rho_1) [f_1(G) + G] C_3 [1 + \|Dv_1\|_{L^{2q}(\Omega)}^2]$$

whence

$$\|Dv_2\|_{L^{2q}(\Omega)}^2 \leq C \|Dv_1\|_{L^{2q}(\Omega)}^2 + C \|Dw\|_{L^{2q}(\Omega)}^2 \leq C_4 + C_5 \|Dv_1\|_{L^{2q}(\Omega)}^2.$$

Setting  $\rho_1 = 0, \rho_2 = \eta$  and bearing in mind uniqueness result (Corollary 3.1), we get

$$\|Dv_\eta\|_{L^{2q}(\Omega)}^2 \leq C_4 \quad (3.38)$$

whenever there exists the solution  $v_\eta$  of problem (3.7) with  $\rho = \eta$ .

Thus if we put  $\rho_1 = k\eta$  and  $\rho_2 = (k+1)\eta$  with  $k = 1, \dots, m-1$  and repeat the above procedure  $m$  times, we derive the desired estimate

$$\|Du\|_{L^{2q}(\Omega)}^2 \leq C_6.$$

Applying once again Leray-Schauder fixed point theorem, bearing in mind estimates (3.34), (3.38), we derive solvability of problem (3.7) with  $\rho = \eta$ . This completes the proof of Theorem 3.1.

Finally, from bounds (3.30) and (3.31) we derive estimate (3.28) and therefore the existence result, i.e. Theorem 1.1.

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