

## THE WEIGHTED EULER IDENTITY

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*(communicated by P. S. Bullen)*

*Abstract.* Some new weighted generalisations of Euler-type identities are given, by using weighted Montgomery identity.

### 1. Introduction

For every function  $f : [a, b] \rightarrow \mathbb{R}$  such that  $f^{(n-1)}$  is a continuous function of bounded variation on  $[a, b]$  for some  $n \geq 1$  and for every  $x \in [a, b]$ , the following two formulae have been proved (see [8]):

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + T_n(x) + P_n(x), \quad (1.1)$$

and

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + T_{n-1}(x) + R_n(x), \quad (1.2)$$

where

$$T_m(x) = \sum_{k=1}^m \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right]$$

with convention  $T_0(x) = 0$ , and

$$P_n(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left[ B_n^* \left( \frac{x-t}{b-a} \right) \right] df^{(n-1)}(t),$$

$$R_n(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left[ B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right] df^{(n-1)}(t).$$

Here  $B_k(x)$ ,  $k \geq 0$ , are the Bernoulli polynomials,  $B_k = B_k(0)$ ,  $k \geq 0$ , the Bernoulli numbers, and  $B_k^*(x)$ ,  $k \geq 0$ , are periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leq x < 1, \quad B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R}.$$

*Mathematics subject classification* (2000): 26D15, 26D20.

*Key words and phrases:* Bernoulli polynomials, weighted Montgomery identity.

From the properties of Bernoulli polynomials  $B_0^*(t) = 1$ ,  $B_1^*$  is a discontinuous function with the jump of  $-1$  at each integer, and  $B_k^*$ ,  $k \geq 2$ , is a continuous function (see [1]). The formulæ (1.1) and (1.2) are extensions of the Euler formula (see [9]).

Here, as in the rest of paper, we write  $\int_a^b \varphi(t) dg(t)$  to denote the Riemann-Stieltjes integral with respect to a function  $g : [a, b] \rightarrow \mathbb{R}$  of bounded variation, and  $\int_a^b \varphi(t) dt$  for the Riemann integral.

The aim of this paper is to give one new weighted generalization of the Euler identity, which can be obtained by using the weighted Montgomery identity given by Pečarić in [10]. (For other weighted generalizations of the Euler identity see [4], [2]). This identity is used to obtain some Ostrowski type inequalities (weighted generalizations of the results from [8] and [2]), as well as the generalizations of the estimations of the difference of two weighted integral means (generalizations of the results from [3], [5], [6], [7], [11]).

## 2. Weighted Euler type identities

**THEOREM 1.** *Let's suppose  $f^{(n-1)}$  is a continuous function of bounded variation on  $[a, b]$  for some  $n \geq 1$ . If  $w : [a, b] \rightarrow [0, \infty)$  is some probability density function i.e. integrable function satisfying  $\int_a^b w(t) dt = 1$ , and  $W(t) = \int_a^t w(x) dx$  for  $t \in [a, b]$ ,  $W(t) = 0$  for  $t < a$  and  $W(t) = 1$  for  $t > b$ , the weighted Peano kernel is*

$$P_w(x, t) = \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1 & x < t \leq b. \end{cases} \quad (2.1)$$

Then the following identities hold

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t) dt + \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} \times \\ &\quad \times \left( B_k \left( \frac{x-a}{b-a} \right) - \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t) dt + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} \times \\ &\quad \times \left( B_k \left( \frac{x-a}{b-a} \right) - \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \end{aligned}$$

$$\begin{aligned}
 & - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right. \\
 & \left. - \int_a^b w(s) \left( B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right) ds \right) df^{(n-1)}(t)
 \end{aligned} \tag{2.3}$$

and for  $n > 1$

$$\begin{aligned}
 f(x) &= \int_a^b w(t) f(t) dt \\
 &+ \sum_{k=1}^n \frac{(b-a)^{k-2}}{(k-1)!} \left( \int_a^b P_w(x,t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
 &- \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left( \int_a^b P_w(x,s) B_{n-1}^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t)
 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
 f(x) &= \int_a^b w(t) f(t) dt \\
 &+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left( \int_a^b P_w(x,t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
 &- \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left( \int_a^b P_w(x,s) \left[ B_{n-1}^* \left( \frac{s-t}{b-a} \right) - B_{n-1} \left( \frac{s-a}{b-a} \right) \right] ds \right) df^{(n-1)}(t).
 \end{aligned} \tag{2.5}$$

*Proof.* Multiplying identity (1.1) by  $w(x)$  and then integrating we obtain

$$\begin{aligned}
 \int_a^b w(x) f(x) dx &= \left( \int_a^b w(x) dx \right) \frac{1}{b-a} \int_a^b f(t) dt + \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} \times \\
 &\times \left( \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
 &- \frac{(b-a)^{n-1}}{n!} \int_a^b \left( \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t).
 \end{aligned}$$

If we subtract this identity from (1.1) we obtain (2.2). Further

$$\begin{aligned}
 & B_k \left( \frac{x-a}{b-a} \right) - \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \\
 &= B_k \left( \frac{x-a}{b-a} \right) - W(t) B_k \left( \frac{t-a}{b-a} \right) \Big|_a^b + \frac{k}{b-a} \int_a^b W(t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt \\
 &= B_k \left( \frac{x-a}{b-a} \right) - B_k \left( \frac{b-a}{b-a} \right) + \frac{k}{b-a} \int_a^b W(t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{k}{b-a} \int_x^b B_{k-1} \left( \frac{t-a}{b-a} \right) dt + \frac{k}{b-a} \int_a^b W(t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt \\
&= \frac{k}{b-a} \int_a^b P_w(x, t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt.
\end{aligned}$$

Similarly we get for  $n > 1$

$$B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds = \frac{n}{b-a} \int_a^b P_w(x, s) B_{n-1}^* \left( \frac{s-t}{b-a} \right) ds$$

since  $B_n^*$ ,  $n > 1$  is a continuous function: for  $n = 1$  we have

$$\begin{aligned}
B_1^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_1^* \left( \frac{s-t}{b-a} \right) ds \\
= -P_w(x, t) + \frac{1}{b-a} \int_a^b P_w(x, s) B_0^* \left( \frac{s-t}{b-a} \right) ds.
\end{aligned}$$

The identity (2.4) follows. The proofs of the identities (2.5) and (2.3) are similar, (using the identity (1.2)).  $\square$

REMARK 1. We could also obtain identities (2.4) and (2.5) by applying identities (1.1) and (1.2) with  $f'(x)$  :

$$\begin{aligned}
f'(x) &= \frac{f(b) - f(a)}{b-a} + \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k)}(b) - f^{(k)}(a) \right] \\
&\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left[ B_n^* \left( \frac{x-t}{b-a} \right) \right] df^{(n)}(t) \\
&= \sum_{k=0}^n \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k)}(b) - f^{(k)}(a) \right] \\
&\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left[ B_n^* \left( \frac{x-t}{b-a} \right) \right] df^{(n)}(t)
\end{aligned}$$

and

$$\begin{aligned}
f'(x) &= \frac{f(b) - f(a)}{b-a} + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k)}(b) - f^{(k)}(a) \right] \\
&\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left[ B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right] df^{(n)}(t) \\
&= \sum_{k=0}^{n-1} \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k)}(b) - f^{(k)}(a) \right] \\
&\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left[ B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right] df^{(n)}(t)
\end{aligned}$$

Now by putting these two formulae in the weighted Montgomery identity (see [10])

$$f(x) = \int_a^b w(t)f(t) dt + \int_a^b P_w(x,t)f'(t) dt$$

where the weighted Peano kernel is given by (2.1), we obtain

$$\begin{aligned} f(x) &= \int_a^b w(t)f(t) dt \\ &+ \sum_{k=0}^n \frac{(b-a)^{k-1}}{k!} \left( \int_a^b P_w(x,t) B_k \left( \frac{t-a}{b-a} \right) dt \right) [f^{(k)}(b) - f^{(k)}(a)] \\ &- \frac{(b-a)^{n-1}}{n!} \int_a^b P_w(x,t) \left( \int_a^b \left[ B_n^* \left( \frac{t-s}{b-a} \right) \right] df^{(n)}(s) \right) dt \end{aligned}$$

and

$$\begin{aligned} f(x) &= \int_a^b w(t)f(t) dt \\ &+ \sum_{k=0}^{n-1} \frac{(b-a)^{k-1}}{k!} \left( \int_a^b P_w(x,t) B_k \left( \frac{t-a}{b-a} \right) dt \right) [f^{(k)}(b) - f^{(k)}(a)] \\ &- \frac{(b-a)^{n-1}}{n!} \int_a^b P_w(x,t) \left( \int_a^b \left[ B_n^* \left( \frac{t-s}{b-a} \right) - B_n \left( \frac{t-a}{b-a} \right) \right] df^{(n)}(s) \right) dt. \end{aligned}$$

Finally, if we interchange the order of integration and replace  $n$  with  $n-1$  we get (2.4) and (2.5). These identities are valid for  $n-1 \geq 1$ , i.e.  $n > 1$ .

REMARK 2. In the special case, if we take  $w(t) = \frac{1}{b-a}$ ,  $t \in [a, b]$  we have

$$\begin{aligned} &\left( B_k \left( \frac{x-a}{b-a} \right) - \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \right) \\ &= \left( B_k \left( \frac{x-a}{b-a} \right) - \frac{1}{b-a} \int_a^b B_k \left( \frac{t-a}{b-a} \right) dt \right) \\ &= B_k \left( \frac{x-a}{b-a} \right) - \frac{1}{(b-a)(k+1)} (B_{k+1}(1) - B_{k+1}(0)) \\ &= B_k \left( \frac{x-a}{b-a} \right) \end{aligned}$$

and similarly

$$\begin{aligned} &B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds = B_n^* \left( \frac{x-t}{b-a} \right), \\ &B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \\ &= B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \end{aligned}$$

Consequently, the identities (2.2) and (2.4) reduce to the Euler identity (1.1) and the identities (2.3) and (2.5) reduce to the identity (1.2). So we may regard them as weighted Euler identities.

**COROLLARY 1.** *Suppose that all assumptions of Theorem 1 hold. Additionally assume that  $w : [a, b] \rightarrow [0, \infty)$  is symmetric on  $[a, b]$ , i.e.  $w(t) = w(b - a - t)$ , for  $t \in [a, b]$ . Then*

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t) dt + \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\ &\quad - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \left( 2 \int_a^{\frac{a+b}{2}} w(t) B_{2k} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \\ &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t) \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t) dt + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\ &\quad + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \left( 2 \int_a^{\frac{a+b}{2}} w(t) B_{2k} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \\ &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right. \\ &\quad \left. - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \right) df^{(n-1)}(t). \end{aligned} \quad (2.7)$$

*Proof.* We have

$$\begin{aligned} &B_k \left( \frac{t-a}{b-a} \right) - \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \\ &= \begin{cases} B_k \left( \frac{t-a}{b-a} \right) - 2 \int_a^{\frac{a+b}{2}} w(t) B_k \left( \frac{t-a}{b-a} \right) dt, & \text{if } k \text{ is even,} \\ B_k \left( \frac{t-a}{b-a} \right), & \text{if } k \text{ is odd,} \end{cases} \end{aligned}$$

since  $B_n(1-x) = (-1)^n B_n(x)$ ,  $x \in [0, 1]$ . If we put this in (2.2) and (2.3) the proof follows.  $\square$

REMARK 3. Applying identity (2.6) with  $x = b$  we get

$$\begin{aligned}
 f(b) &= \int_a^b w(t)f(t) dt + \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} B_k(1) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
 &\quad - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \left( 2 \int_a^{\frac{a+b}{2}} w(t) B_{2k} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \\
 &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{b-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t)
 \end{aligned}$$

Since  $B_n(1) = (-1)^n B_n(0) = (-1)^n B_n$  for  $n \geq 0$  and  $B_{2n+1} = 0$  for  $n \geq 1$  (see [1]) and also for  $k = 1$ ,  $\frac{(b-a)^{k-1}}{k!} B_k(1) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] = \frac{1}{2} [f(b) - f(a)]$  the last identity reduces to

$$\begin{aligned}
 \frac{f(a)+f(b)}{2} &= \int_a^b w(t)f(t) dt + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \times \\
 &\quad \times \left( B_{2k} - 2 \int_a^{\frac{a+b}{2}} w(t) B_{2k} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \\
 &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{b-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t). \tag{2.8}
 \end{aligned}$$

We can regard this as the first Euler-Maclaurin formula (the generalized trapezoid identity). Similarly, applying identity (2.6) with  $x = \frac{a+b}{2}$  we get

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &= \int_a^b w(t)f(t) dt + \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{1}{2}\right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
 &\quad - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \left( 2 \int_a^{\frac{a+b}{2}} w(t) B_{2k} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \\
 &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{a+b-2t}{2(b-a)} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t)
 \end{aligned}$$

Since  $B_n(\frac{1}{2}) = -(1-2^{1-n}) B_n$  for  $n \geq 0$  (see [1]) the last identity reduces to

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &= \int_a^b w(t)f(t) dt - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \times \\
 &\quad \times \left( (1-2^{1-2k}) B_{2k} + 2 \int_a^{\frac{a+b}{2}} w(t) B_{2k} \left( \frac{t-a}{b-a} \right) dt \right) \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \\
 &\quad - \frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_n^* \left( \frac{a+b-2t}{2(b-a)} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) df^{(n-1)}(t) \tag{2.9}
 \end{aligned}$$

We can regard this as the second Euler-Maclaurin formula (the generalized midpoint identity).

### 3. Ostrowski type inequalities

For  $n \geq 1$  we write

$$\begin{aligned} T_{w,n}(x) &= \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} \left( B_k \left( \frac{x-a}{b-a} \right) - \int_a^b w(t) B_k \left( \frac{t-a}{b-a} \right) dt \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &= \sum_{k=1}^n \frac{(b-a)^{k-2}}{(k-1)!} \left( \int_a^b P_w(x,t) B_{k-1} \left( \frac{t-a}{b-a} \right) dt \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \end{aligned}$$

For all the results in this section we will use identities (2.4) and (2.5). But these results can also be obtained from identities (2.2) and (2.3).

**THEOREM 2.** Assume  $(p, q)$  is a pair of conjugate exponents, that is  $1 \leq p, q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $|f^{(n)}|^p : [a, b] \rightarrow \mathbb{R}$  be an  $R$ -integrable function for some  $n > 1$ . Then for  $x \in [a, b]$  the following inequalities hold

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t) dt - T_{w,n}(x) \right| \\ &\leq \frac{(b-a)^{n-2}}{(n-1)!} \left( \int_a^b \left| \int_a^b P_w(x,s) B_{n-1}^* \left( \frac{s-t}{b-a} \right) ds \right|^q dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t) dt - T_{w,n-1}(x) \right| \leq \frac{(b-a)^{n-2}}{(n-1)!} \times \\ &\times \left( \int_a^b \left| \int_a^b P_w(x,s) \left[ B_{n-1}^* \left( \frac{s-t}{b-a} \right) - B_{n-1} \left( \frac{s-a}{b-a} \right) \right] ds \right|^q dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p \end{aligned} \quad (3.2)$$

The constants on the right-hand sides of (3.1) and (3.2) are sharp for  $1 < p \leq \infty$  and the best possible for  $p = 1$ .

*Proof.* Let's denote  $C_1(t) = \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b P_w(x,s) B_{n-1}^* \left( \frac{s-t}{b-a} \right) ds$ . We use the identity (2.4) and apply the Hölder inequality to obtain

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t) dt - T_{w,n}(x) \right| \\ &= \left| \int_a^b C_1(t) f^{(n)}(t) dt \right| \leq \left( \int_a^b |C_1(t)|^q dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p \end{aligned}$$

For the proof of the sharpness of the constant  $\left(\int_a^b |C_1(t)|^q dt\right)^{\frac{1}{q}}$  we will find a function  $f$  for which the equality in (3.1) is obtained. For  $1 < p < \infty$  take  $f$  to be such that

$$f^{(n)}(t) = \operatorname{sgn} C_1(t) \times |C_1(t)|^{\frac{1}{p-1}}.$$

For  $p = \infty$  take

$$f^{(n)}(t) = \operatorname{sgn} C_1(t).$$

For  $p = 1$  we shall prove that

$$\left| \int_a^b C_1(t) f^{(n)}(t) dt \right| \leq \max_{t \in [a,b]} |C_1(t)| \left( \int_a^b |f^{(n)}(t)| dt \right) \tag{3.3}$$

is the best possible inequality. Suppose that  $|C_1(t)|$  attains its maximum at  $t_0 \in [a, b]$ . First we assume that  $C_1(t_0) > 0$ . For  $\varepsilon > 0$  define  $f_\varepsilon(t)$  by

$$f_\varepsilon(t) = \begin{cases} 0, & a \leq t \leq t_0, \\ \frac{1}{\varepsilon n!} (t - t_0)^n, & t_0 \leq t \leq t_0 + \varepsilon, \\ \frac{1}{n!} (t - t_0)^{n-1}, & t_0 + \varepsilon \leq t \leq b. \end{cases}$$

Then, for  $\varepsilon$  small enough

$$\left| \int_a^b C_1(t) f^{(n)}(t) dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} C_1(t) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C_1(t) dt.$$

Now, from inequality (3.3) we have

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C_1(t) dt \leq C_1(t_0) \int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon} dt = C_1(t_0).$$

Since,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C_1(t) dt = C_1(t_0)$$

the statement follows. In case  $C_1(t_0) < 0$ , we take

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!} (t - t_0 - \varepsilon)^{n-1}, & a \leq t \leq t_0, \\ -\frac{1}{\varepsilon n!} (t - t_0 - \varepsilon)^n, & t_0 \leq t \leq t_0 + \varepsilon, \\ 0, & t_0 + \varepsilon \leq t \leq b, \end{cases}$$

and the rest of proof is the same as above. For the inequality (3.2) the proof is similar.  $\square$

COROLLARY 2. *Suppose that all assumptions of Theorem 2 hold. Then the following inequality holds*

$$\begin{aligned} & \left| f(x) - \int_a^b w(t) f(t) dt - T_{w,n}(x) \right| \\ & \leq \frac{(b-a)^{n-1+\frac{1}{q}}}{(n-1)!} \left( \int_0^1 |B_{n-1}(s)| ds \right) \|f^{(n)}\|_p. \end{aligned} \quad (3.4)$$

*Proof.* Since  $0 \leq W(t) \leq 1$ ,  $t \in [a, b]$ , so  $|P_w(x, s)| \leq 1$  and  $B_{n-1}^*$  is a periodic function with period 1 and  $\int_0^1 |B_n^*(y+s)| ds = \int_0^1 |B_n^*(s)| ds = \int_0^1 |B_n(s)| ds$  for every  $y \in \mathbb{R}$ , we have

$$\begin{aligned} \left| \int_a^b P_w(x, s) B_{n-1}^* \left( \frac{s-t}{b-a} \right) ds \right| & \leq \int_a^b |P_w(x, s)| \left| B_{n-1}^* \left( \frac{s-t}{b-a} \right) \right| ds \\ & \leq \int_a^b \left| B_{n-1}^* \left( \frac{s-t}{b-a} \right) \right| ds = (b-a) \int_0^1 |B_{n-1}(s)| ds. \end{aligned}$$

So

$$\begin{aligned} \left( \int_a^b \left| \int_a^b P_w(x, s) B_{n-1}^* \left( \frac{s-t}{b-a} \right) ds \right|^q dt \right)^{\frac{1}{q}} & \leq (b-a) \left( \int_a^b \left| \int_0^1 |B_{n-1}(s)| ds \right|^q dt \right)^{\frac{1}{q}} \\ & = (b-a)^{1+\frac{1}{q}} \int_0^1 |B_{n-1}(s)| ds \end{aligned}$$

and by applying (3.1) the inequality is proved.  $\square$

REMARK 4. For  $n = 2$  inequality (3.4) reduces to

$$\left| f(x) - \int_a^b w(t) f(t) dt - T_{w,2}(x) \right| \leq \frac{1}{4} (b-a)^{1+\frac{1}{q}} \|f''\|_p;$$

for  $n = 3$  it reduces to

$$\left| f(x) - \int_a^b w(t) f(t) dt - T_{w,3}(x) \right| \leq \frac{\sqrt{3}}{34} (b-a)^{2+\frac{1}{q}} \|f'''\|_p;$$

and for  $n = 4$  to

$$\left| f(x) - \int_a^b w(t) f(t) dt - T_{w,4}(x) \right| \leq \frac{1}{192} (b-a)^{3+\frac{1}{q}} \|f^{(4)}\|_p.$$

REMARK 5. If we use the identities (2.8) and (2.9) for  $n = 2$  and  $w(t) = \frac{1}{b-a}$ ,  $t \in [a, b]$  and then apply the Hölder inequality with  $p = \infty$ ,  $q = 1$ , we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt - \frac{b-a}{12} [f'(b) - f'(a)] \right| \leq \|f''\|_\infty \times \frac{\sqrt{3}}{54} (b-a)^2$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{b-a}{24} [f'(b) - f'(a)] \right| \leq \|f''\|_\infty \times \frac{\sqrt{3}}{54} (b-a)^2.$$

By doing the same for  $n = 3$  we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt - \frac{b-a}{12} [f'(b) - f'(a)] \right| \leq \|f'''\|_\infty \times \frac{1}{192} (b-a)^3$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{b-a}{24} [f'(b) - f'(a)] \right| \leq \|f'''\|_\infty \times \frac{1}{192} (b-a)^3.$$

The last two inequalities were obtained in [2] (Corollary 1). In fact, inequality (3.2) from the Theorem 2 applied with  $n = 3$  is weighted generalization of Theorem 1 from [2] and applied with  $n = 2$  is weighted generalization of Corollary 1 from [8].

#### 4. Estimations of the difference of two weighted integral means

In this section we will use the following notations when  $n \geq 1$  and the function  $f : [a, b] \rightarrow \mathbb{R}$  is such that  $f^{(n-1)}$  is continuous and of bounded variation on  $[a, b]$ :

$$T_{w,n}^{[a,b]}(x) = \sum_{k=1}^n \frac{(b-a)^{k-1}}{k!} \left( B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right) dt \right) [f^{(k-1)}(b) - f^{(k-1)}(a)];$$

$$S_{w,n}^{[a,b]}(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_{n-1}^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_{n-1}^*\left(\frac{s-t}{b-a}\right) ds \right) df^{(n-1)}(t);$$

$$R_{w,n}^{[a,b]}(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left( B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right) - \int_a^b w(s) \left[ B_{n-1}^*\left(\frac{s-t}{b-a}\right) - B_{n-1}\left(\frac{s-a}{b-a}\right) \right] ds \right) df^{(n-1)}(t).$$

The following results are generalizations of the results from the recent papers [11], (in case  $[c, d] \subset [a, b]$ ), and [3], (in case  $[a, b] \cap [c, d] = [c, b]$ ). Other two possible cases, when  $[a, b] \cap [c, d] \neq \emptyset$  we simply get by change  $a \leftrightarrow c, b \leftrightarrow d$ .

**THEOREM 3.** *Let  $f : [a, b] \cup [c, d] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is a continuous function of bounded variation on  $[a, b]$  for some  $n \geq 1, w : [a, b] \rightarrow [0, \infty)$  and*

$u : [c, d] \rightarrow [0, \infty)$  some probability density functions. Then if  $[a, b] \cap [c, d] \neq \emptyset$  and  $x \in [a, b] \cap [c, d]$ , we have

$$\int_a^b w(t)f(t) dt - \int_c^d u(t)f(t) dt + T_{w,n}^{[a,b]}(x) - T_{u,n}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n^1(x,t) df^{(n-1)}(t) \quad (4.1)$$

and

$$\int_a^b w(t)f(t) dt - \int_c^d u(t)f(t) dt + T_{w,n-1}^{[a,b]}(x) - T_{u,n-1}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n^2(x,t) df^{(n-1)}(t), \quad (4.2)$$

where in case  $[c, d] \subset [a, b]$

$$K_n^1(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right), & t \in [a, c], \\ \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) \\ - \frac{(d-c)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{d-c} \right) - \int_c^d u(s) B_n^* \left( \frac{s-t}{d-c} \right) ds \right), & t \in [c, d], \\ \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right), & t \in \langle d, b \rangle, \end{cases}$$

and

$$K_n^2(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right. \\ \left. - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \right), & t \in [a, c], \\ \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right. \\ \left. - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \right) \\ - \frac{(d-c)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{d-c} \right) - B_n \left( \frac{x-c}{d-c} \right) \right. \\ \left. - \int_c^d u(s) \left[ B_n^* \left( \frac{s-t}{d-c} \right) - B_n \left( \frac{s-c}{d-c} \right) \right] ds \right), & t \in [c, d], \\ \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) \right. \\ \left. - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \right), & t \in \langle d, b \rangle, \end{cases}$$

and in case  $[a, b] \cap [c, d] = [c, b]$

$$K_n^1(x, t) = \begin{cases} \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right), & t \in [a, c], \\ \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - \int_a^b w(s) B_n^* \left( \frac{s-t}{b-a} \right) ds \right) \\ - \frac{(d-c)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{d-c} \right) - \int_c^d u(s) B_n^* \left( \frac{s-t}{d-c} \right) ds \right), & t \in [c, b], \\ - \frac{(d-c)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{d-c} \right) - \int_c^d u(s) B_n^* \left( \frac{s-t}{d-c} \right) ds \right), & t \in \langle b, d \rangle, \end{cases}$$

and

$$K_n^2(x, t) = \begin{cases} \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \right), & t \in [a, c], \\ \frac{(b-a)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{b-a} \right) - B_n \left( \frac{x-a}{b-a} \right) - \int_a^b w(s) \left[ B_n^* \left( \frac{s-t}{b-a} \right) - B_n \left( \frac{s-a}{b-a} \right) \right] ds \right) \\ - \frac{(d-c)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{d-c} \right) - B_n \left( \frac{x-c}{d-c} \right) - \int_c^d u(s) \left[ B_n^* \left( \frac{s-t}{d-c} \right) - B_n \left( \frac{s-c}{d-c} \right) \right] ds \right), & t \in [c, b], \\ - \frac{(d-c)^{n-1}}{n!} \left( B_n^* \left( \frac{x-t}{d-c} \right) - B_n \left( \frac{x-c}{d-c} \right) - \int_c^d u(s) \left[ B_n^* \left( \frac{s-t}{d-c} \right) - B_n \left( \frac{s-c}{d-c} \right) \right] ds \right), & t \in \langle b, d \rangle. \end{cases}$$

*Proof.* We subtract identities (2.2) for interval  $[a, b]$  and  $[c, d]$ , to get the first formula. By doing the same with identity (2.3), we get the second formula.  $\square$

**THEOREM 4.** Assume  $(p, q)$  is a pair of conjugate exponents and let  $|f^{(n)}|^p : [a, b] \rightarrow \mathbb{R}$  be an  $R$ -integrable function for some  $n \geq 1$ . Then we have

$$\begin{aligned} & \left| \int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt + T_{w,n}^{[a,b]}(x) - T_{u,n}^{[c,d]}(x) \right| \\ & \leq \left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n^1(x, t)|^q dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p \end{aligned} \tag{4.3}$$

and

$$\left| \int_a^b w(t)f(t) dt - \int_c^d u(t)f(t) dt + T_{w,n-1}^{[a,b]}(x) - T_{u,n-1}^{[c,d]}(x) \right| \leq \left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n^2(x,t)|^q dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p, \quad (4.4)$$

for every  $x \in [a, b] \cap [c, d]$ . The constants

$$\left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n^1(x,t)|^q dt \right)^{\frac{1}{q}} \text{ and } \left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n^2(x,t)|^q dt \right)^{\frac{1}{q}}$$

in the inequalities (4.3) and (4.4) are sharp for  $1 < p \leq \infty$  and the best possible for  $p = 1$ .

*Proof.* Use the identity (4.1) and apply the Hölder inequality to obtain

$$\left| \int_a^b w(t)f(t) dt - \int_c^d u(t)f(t) dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) \right| \leq \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n^1(x,t)| |f^{(n)}(t)| dt \leq \left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n^1(x,t)|^q dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p,$$

which proves the first inequality. Similarly we prove the second inequality. The proof for sharpness and the best possibility are similar as in Theorem 2.  $\square$

REMARK 6. Similar results to those in two last Theorems could be obtained using the identities (2.4) and (2.5) instead of (2.2) and (2.3).

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(Received September 23, 2003)

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