

GENERALIZATION OF HILBERT AND HARDY–HILBERT INTEGRAL INEQUALITIES

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*Dedicated to the memory
of prof. Mladen Alić*

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Abstract. The main objective of this paper is a study of some new generalizations of Hilbert's and Hardy-Hilbert's type inequalities. We establish general form of multiple Hilbert-type inequality and we also introduce multiple inequality of Hardy-Hilbert type. Further, the best possible constants are obtained for some general cases.

1. Introduction

Hilbert's and Hardy-Hilbert's type inequalities are some significant weight inequalities which play an important role in analysis and its applications. First, let us recall the well known Hilbert's integral inequality (1) and its equivalent form which we call Hardy-Hilbert's inequality (2).

THEOREM A. *The following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^\infty f^2(x) dx \cdot \int_0^\infty g^2(x) dx \right)^{\frac{1}{2}}, \quad (1)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx dy \right)^2 < \pi^2 \int_0^\infty f^2(x) dx, \quad (2)$$

where π and π^2 are the best constants.

Although classical, they are field of interest of numerous mathematicians and were generalized in many different ways. For more details see [11].

Very recently, Brnetić and Pečarić ([8], [9]) gave some further generalizations of Hilbert's and Hardy-Hilbert's inequalities. So we shall state their result that will take our attention.

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THEOREM B. If $n \geq 2$ is an integer and $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$, then

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n x_i)^\lambda} dx_1 dx_2 \cdots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \left(\Gamma(1 - p_i A_{i+1}) \Gamma(\lambda - n + 1 + p_i A_{i+1}) \right)^{\frac{1}{p_i}} \times \\ & \quad \times \left(\int_0^\infty x^{n-1-\lambda+p_i(A_i-A_{i+1})} f_i^{p_i}(x) dx \right)^{\frac{1}{p_i}} \end{aligned}$$

for any $\lambda > n - 2$ and $A_i \in \left(\frac{n - \lambda - 1}{p_{i-1}}, \frac{1}{p_{i-1}} \right)$, where Γ is the gamma-function and the indices i in A_i are taken modulo n .

Further, Yang and Rassias obtained in [2] the following result

THEOREM C. If $n \geq 2$ is an integer and $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$, then

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n x_i)^\lambda} dx_1 dx_2 \cdots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right) \left(\int_0^\infty x^{n-1-\lambda} f_i^{p_i}(x) dx \right)^{\frac{1}{p_i}} \end{aligned}$$

for any $\lambda > n - \min_{1 \leq i \leq n} \{p_i\}$, where Γ is gamma-function.

Above results include many interesting Hilbert-type inequalities as the special cases which can be found in [8] and [2].

On the other hand, these multiple Hilbert-type inequalities are not comparable, since it is impossible to obtain one result from another. Our aim is to obtain more general result from which both results will follow as special cases. Also, we introduce the general form of multiple Hardy-Hilbert inequality and obtain the best possible constants in some general cases.

The techniques that will be used in the proofs are mainly based on classical real analysis, especially on the well known Hölder's inequality and on Fubini's theorem.

2. Multiple Hilbert's and Hardy-Hilbert's inequalities

This section is dedicated to the most general form of multiple Hilbert's and Hardy-Hilbert's inequalities. Since we use Fubini's theorem, all the measures should be σ -finite. Further, throughout this paper, we assume that all the functions are non-negative and not identically equal to zero. Also we suppose that all the integrals in the paper converges, so we shall omit such types of conditions. Under these assumptions we present our general result.

THEOREM 1. Let $n \geq 2$ be an integer and $\sum_{i=1}^n \frac{1}{p_i} = 1, p_i > 1, i = 1, \dots, n,$

$\prod_{i,j=1}^n \phi_{ij}(x_j) = 1.$ Then the following inequalities hold and are equivalent

$$\begin{aligned} & \int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n) \\ & \leq \prod_{i=1}^n \left(\int_{\Omega} F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{1}{p_i}} \end{aligned} \tag{3}$$

and

$$\begin{aligned} & \int_{\Omega} h(x_n) \left(\int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1}) \right)^q d\mu_n(x_n) \\ & \leq \prod_{i=1}^{n-1} \left(\int_{\Omega} F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{q}{p_i}}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} F_i(x_i) &= \int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{j=1, j \neq i}^n \phi_{ij}^{p_i}(x_j) d\mu_j(x_j), \\ h(x_n) &= \phi_{nn}^{-q}(x_n) F_n^{1-q}(x_n) \quad \text{and} \quad \frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}. \end{aligned}$$

If $p_i > 0, i \in \{1, 2, \dots, n\}$ and $p_k < 0, k \neq i,$ then the reverse inequality in (3) is valid. Also, if $p_i > 0, i \in \{1, 2, \dots, n-1\}$ and $p_k < 0, k \neq i,$ then the reverse inequality in (4) is valid. Further, the inequality (4) holds also when $p_n > 0$ and $p_k < 0, k \neq n.$

Proof. First we prove the inequality (3). By applying Hölder’s inequality we have

$$\begin{aligned} & \int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n) \\ &= \int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^n (f_i(x_i) \prod_{j=1}^n \phi_{ij}(x_j)) d\mu_1(x_1) \dots d\mu_n(x_n) \\ &\leq \prod_{i=1}^n \left(\int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) f_i^{p_i}(x_i) \prod_{j=1}^n \phi_{ij}^{p_i}(x_j) d\mu_1(x_1) \dots d\mu_n(x_n) \right)^{\frac{1}{p_i}} \end{aligned}$$

and we obtain the inequality (3).

Now we prove the inequality (4). We denote

$$I(x_n) = \int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1}).$$

By putting

$$f_n(x_n) = h(x_n) \cdot (I(x_n))^{q-1}$$

in the inequality (3) we obtain

$$\begin{aligned} \int_{\Omega} h(x_n) (I(x_n))^q d\mu_n(x_n) &\leq \prod_{i=1}^{n-1} \left(\int_{\Omega} F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{1}{p_i}} \times \\ &\times \left(\int_{\Omega} h^{p_n}(x_n) (I(x_n))^{p_n(q-1)} F_n(x_n) \phi_{nn}^{p_n}(x_n) d\mu_n(x_n) \right)^{1-\frac{1}{q}}. \end{aligned} \tag{5}$$

Now we put

$$h(x_n) = \phi_{nn}^{-q}(x_n) F_n^{1-q}(x_n),$$

in the inequality (5). Then, by using $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and denoting by I the integral of the left-hand side of the inequality (5), we can easily obtain

$$I \leq \prod_{i=1}^{n-1} \left(\int_{\Omega} F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{1}{p_i}} \times I^{1-\frac{1}{q}},$$

which gives the inequality (4). Analogously, we obtain the reverse inequalities, by using the reverse Hölder’s inequality.

It remains to prove that the inequalities (3) and (4) are equivalent. We only have to check that the inequality (3) follows from the inequality (4). For this purpose let’s suppose that the inequality (4) is valid. We start with the following identity

$$\begin{aligned} &\int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n) \\ &= \int_{\Omega} \phi_{nn}^{-1}(x_n) F_n^{-\frac{1}{p_n}}(x_n) \left(\int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1}) \right) \times \\ &\quad \times F_n^{\frac{1}{p_n}}(x_n) f_n(x_n) \phi_{nn}(x_n) d\mu_n(x_n). \end{aligned}$$

Now, by applying Hölder’s inequality with conjugate parameters q and p_n , we have

$$\begin{aligned} &\int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n) \\ &\leq \left(\int_{\Omega} h(x_n) \left(\int_{\Omega} \cdots \int_{\Omega} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1}) \right)^q d\mu_n(x_n) \right)^{\frac{1}{q}} \times \\ &\quad \times \left(\int_{\Omega} F_n(x_n) f_n^{p_n}(x_n) \phi_{nn}^{p_n}(x_n) d\mu_n(x_n) \right)^{\frac{1}{p_n}}, \end{aligned}$$

and it is obvious that the result follows by using the inequality (4). We obtain the equivalence of the reverse inequalities in a similar way, by using the reverse Hölder’s inequality. \square

REMARK 1. Equality in the previous theorem is possible if and only if it is equality in Hölder’s inequality. It means that the functions

$$(f_i(x_i) \prod_{j=1}^n \phi_{ij}(x_j))^{p_i}$$

are proportional. Hence, we obtain that the equality in Theorem 1 holds if and only if $f_i(x_i) = C_i \phi_{ii}(x_i)^{\frac{p_i}{1-p_i}}$, $i = 1, 2, \dots, n$, where C_i are arbitrary constants. It is possible only if the functions

$$\frac{\prod_{j=1, j \neq i}^n \phi_{ij}^{\frac{p_j}{1-p_j}}(x_j)}{\prod_{j=1, j \neq i}^n \phi_{ij}^{p_j}(x_j)}, \quad i = 1, 2, \dots, n$$

are adequate constants and

$$\int_{\Omega} F_i(x_i) \phi_{ii}^{\frac{p_i}{1-p_i}}(x_i) d\mu_i(x_i) < \infty, \quad i = 1, 2, \dots, n.$$

Otherwise, the inequalities in Theorem 1 are strict.

Let’s mention that the special case of Theorem 1, for $n = 2$, was proved in [14].

3. Applications to homogeneous functions

In this section we apply our general results to the homogeneous function

$$K(x_1, \dots, x_n) = \frac{1}{(\sum_{i=1}^n x_i)^\lambda},$$

of degree $-\lambda$, in n variables, and for some special choice of the functions ϕ_{ij} , $1 \leq i, j \leq n$. Further, let $\mu_1 = \dots = \mu_n$ be the Lebesgue measure and $\Omega = (0, \infty)$. Also, let $\phi_{ij}(x_j) = x_j^{A_{ij}}$. Then, the condition

$$\prod_{i,j=1}^n \phi_{ij}(x_j) = 1 \tag{6}$$

leads to

$$\prod_{i=1}^n \prod_{j=1}^n x_j^{A_{ij}} = \prod_{j=1}^n x_j^{\sum_{i=1}^n A_{ij}} = 1.$$

We set $\sum_{i=1}^n A_{ij} = 0, j = 1, 2, \dots, n$, so that the condition (6) is satisfied. We also define $\alpha_i := \sum_{j=1}^n A_{ij}, i = 1, 2, \dots, n$. For our homogeneous function $K(x_1, \dots, x_n) = \frac{1}{(\sum_{i=1}^n x_i)^\lambda}$, we have

$$F_i(x_i) = \int_0^\infty \dots \int_0^\infty \frac{\prod_{j=1, j \neq i}^n x_j^{p_i A_{ij}}}{(\sum_{k=1}^n x_k)^\lambda} dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n.$$

Then, by using the substitution $u_k = \frac{x_k}{x_i}, k = 1, \dots, n, k \neq i$, we have

$$\begin{aligned} F_i(x_i) &= x_i^{n-1-\lambda+p_i\alpha_i-p_iA_{ii}} \times \\ &\quad \times \int_0^\infty \dots \int_0^\infty \frac{\prod_{j=1, j \neq i}^n u_j^{p_i A_{ij}}}{(1 + \sum_{k=1, k \neq i}^n u_k)^\lambda} du_1 \dots du_{i-1} du_{i+1} \dots du_n \\ &= \frac{1}{\Gamma(\lambda)} x_i^{n-1-\lambda+p_i\alpha_i-p_iA_{ii}} \Gamma(\lambda - n + 1 - p_i\alpha_i + p_iA_{ii}) \prod_{j=1, j \neq i}^n \Gamma(p_i A_{ij} + 1), \end{aligned}$$

where we used the well known formula for gamma function (see, for instance, [2], Lemma 5.1.):

$$\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{k-1} u_i^{a_i-1} du_1 \dots du_{k-1}}{(1 + \sum_{i=1}^{k-1} u_i)^{\sum_{i=1}^k a_i}} = \frac{\prod_{i=1}^k \Gamma(a_i)}{\Gamma(\sum_{i=1}^k a_i)}.$$

Also, we have

$$h(x_n) = \frac{1}{\Gamma(\lambda)^{1-q}} x_n^{(1-q)(n-1-\lambda)-q\alpha_n} \left(\Gamma(\lambda - n + 1 - p_n\alpha_n + p_nA_{nn}) \prod_{j=1}^{n-1} \Gamma(p_nA_{nj} + 1) \right)^{1-q},$$

Then, by using Theorem 1, we obtain the following result

THEOREM 2. *Let $n \geq 2$ be an integer, $\sum_{i=1}^n \frac{1}{p_i} = 1, p_i > 1, i = 1, \dots, n$,*

$\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and $\sum_{i=1}^n A_{ij} = 0$. Then the following inequalities hold and are equivalent

$$\begin{aligned} &\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^\lambda} dx_1 \dots dx_n \\ &< K \prod_{i=1}^n \left(\int_0^\infty x_i^{n-1-\lambda+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}} \end{aligned} \tag{7}$$

and

$$\int_0^\infty x_n^{(1-q)(n-1-\lambda)-q\alpha_n} \left(\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^\lambda} dx_1 \dots dx_{n-1} \right)^q dx_n < K^q \prod_{i=1}^{n-1} \left(\int_0^\infty x^{n-1-\lambda+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{q}{p_i}}, \tag{8}$$

where

$$K = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\lambda - n + 1 - p_i\alpha_i + p_iA_{ii})^{\frac{1}{p_i}} \prod_{i,j=1, i \neq j}^n \Gamma(p_iA_{ij} + 1)^{\frac{1}{p_i}},$$

$$A_{ij} > -\frac{1}{p_i}, i \neq j \text{ and } A_{ii} - \alpha_i > \frac{n - \lambda - 1}{p_i}.$$

If $p_i > 0, i \in \{1, 2, \dots, n\}$, and $p_k < 0, k \neq i$, then the reverse inequality in (7) is valid for $A_{ij} > -\frac{1}{p_i}, j \neq i, A_{ii} - \alpha_i > \frac{n - \lambda - 1}{p_i}, A_{kj} < -\frac{1}{p_k}, k \neq j$ and $A_{kk} - \alpha_k < \frac{n - \lambda - 1}{p_k}$.

Also, if $p_i > 0, i \in \{1, 2, \dots, n - 1\}$, and $p_k < 0, k \neq i$, then the reverse inequality in (8) is valid for $A_{ij} > -\frac{1}{p_i}, j \neq i, A_{ii} - \alpha_i > \frac{n - \lambda - 1}{p_i}, A_{kj} < -\frac{1}{p_k}, k \neq j$ and $A_{kk} - \alpha_k < \frac{n - \lambda - 1}{p_k}$.

Further, the inequality (8) holds also when $p_n > 0$ and $p_k < 0, k \neq n$, for $A_{nj} > -\frac{1}{p_n}, j \neq n, A_{nn} - \alpha_n > \frac{n - \lambda - 1}{p_n}, A_{kj} < -\frac{1}{p_k}, k \neq j$ and $A_{kk} - \alpha_k < \frac{n - \lambda - 1}{p_k}$.

REMARK 2. Note that the inequalities in Theorem 2 are strict (under the assumption that the functions f_i are not identically equal to zero). That follows from Remark 1.

REMARK 3. It is very interesting to elaborate why multiple Hilbert-type inequality (7) generalizes both inequalities given in Theorem B and Theorem C.

If we put $A_{ii} = (n - \lambda) \frac{p_i - 1}{p_i^2}$ and $A_{ij} = (\lambda - n) \frac{1}{p_i p_j}, i \neq j$, then the condition $\sum_{i=1}^n A_{ij} = 0$ is satisfied (also $\sum_{j=1}^n A_{ij} = 0$), and we obtain the result of Theorem C as a special case of our result.

On the other hand, if we put $A_{ii} = A_i, A_{i,i+1} = -A_{i+1}, A_{ij} = 0$, where $|i - j| > 1$ and the indices are taken modulo n , then we obtain the result of Theorem B as a special case (obviously, the condition $\sum_{i=1}^n A_{ij} = 0$ is satisfied).

REMARK 4. If we put $n = 2$ and $A_{11} = A_1, A_{22} = A_2, A_{12} = -A_2$ and $A_{21} = -A_1$ in Theorem 2 we obtain the result proved in [9]. If $\lambda > 0, \frac{1}{p} + \frac{1}{q} = 1$,

$p > 1$, then the following inequalities hold and are equivalent

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < K \left(\int_0^\infty x^{1-\lambda+q(A_1-A_2)} f^q(x) dx \right)^{\frac{1}{q}} \left(\int_0^\infty x^{1-\lambda+p(A_2-A_1)} g^p(x) dx \right)^{\frac{1}{p}}, \tag{9}$$

and

$$\int_0^\infty y^{(\lambda-1)(q-1)+q(A_1-A_2)} \left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \right)^q dy < K^q \left(\int_0^\infty x^{1-\lambda+q(A_1-A_2)} f^q(x) dx \right) \tag{10}$$

where $K = B(1 - qA_2, \lambda - 1 + qA_2)^{\frac{1}{q}} B(1 - pA_1, \lambda - 1 + pA_1)^{\frac{1}{p}}$, $A_1 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$, $A_2 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right)$ and B is the beta function.

Now we consider some special choices of the parameters A_{ij} to obtain the best possible constants. More precisely, let the parameters A_{ij} satisfy constraint

$$\lambda - n + 1 - p_i \alpha_i + p_i A_{ii} = p_i A_{ki} + 1, \quad k \neq i, \quad i = 1, 2, \dots, n. \tag{11}$$

It is easy to see that the constant K from the Theorem 2 may be written in the form

$$K^* = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\tilde{A}_i + 1), \tag{12}$$

where $\tilde{A}_i = p_n A_{ni}$, $1 \leq i \leq n - 1$ and $\tilde{A}_n = p_1 A_{1n}$. Further, the inequalities (7) and (8) take form

$$\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^\lambda} dx_1 \dots dx_n < K^* \prod_{i=1}^n \left(\int_0^\infty x^{-1-p_i \tilde{A}_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}}, \tag{13}$$

and

$$\int_0^\infty x_n^{(q-1)(1+p_n \tilde{A}_n)} \left(\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^\lambda} dx_1 \dots dx_{n-1} \right)^q dx_n < (K^*)^q \prod_{i=1}^{n-1} \left(\int_0^\infty x^{-1-p_i \tilde{A}_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{q}{p_i}}. \tag{14}$$

In the following theorem we show that, if the parameters A_{ij} satisfy condition (11), then one obtains the best possible constant.

THEOREM 3. *If the parameters A_{ij} , $1 \leq i, j \leq n$ satisfy condition (11) and $\sum_{i=1}^n A_{ij} = 0$, $j = 1, 2, \dots, n$, then the constants K and K^q from Theorem 2 are the best possible.*

Proof. Let us define the functions

$$\tilde{f}_i(x_i) = \begin{cases} 0 & x \in (0, 1) \\ x_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} & x \in [1, \infty), \end{cases}$$

where $0 < \varepsilon < \min_{1 \leq i \leq n} \{p_i + p_i \tilde{A}_i\}$. If we put these functions into the inequality (7) i.e

(13), then the right-hand side of the inequality becomes $\frac{K^*}{\varepsilon}$, since

$$\varepsilon \prod_{i=1}^n \left(\int_0^\infty x^{-1-p_i \tilde{A}_i} \tilde{f}_i^{p_i}(x) dx \right)^{\frac{1}{p_i}} = 1. \tag{15}$$

Further, let J denote the left-hand side of the inequality (13) multiplied by ε , for above choice of the functions \tilde{f}_i . By using the substitution $u_i = \frac{x_i}{x_n}$, $i = 1, 2, \dots, n - 1$ in J , we find

$$J = \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left(\int_{\frac{1}{x_n}}^\infty \dots \int_{\frac{1}{x_n}}^\infty \frac{\prod_{i=1}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=1}^{n-1} u_i)^\lambda} du_1 \dots du_{n-1} \right) dx_n. \tag{16}$$

It is easy to see that the following inequality holds

$$\begin{aligned} J \geq \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left(\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=1}^{n-1} u_i)^\lambda} du_1 \dots du_{n-1} \right) dx_n \\ - \varepsilon \int_1^\infty x_n^{-1} \sum_{j=1}^{n-1} I_j(x_n) dx_n, \end{aligned} \tag{17}$$

where for $j = 1, 2, \dots, n - 1$, $I_j(x_n)$ is defined by

$$I_j(x_n) := \int_{D_j} \frac{\prod_{i=1}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=1}^{n-1} u_i)^\lambda} du_1 \dots du_{n-1},$$

satisfying $D_j = \{(u_1, u_2, \dots, u_{n-1}); 0 < u_j \leq \frac{1}{x_n}, 0 < u_k < \infty, k \neq j\}$. Without loss of generality, we only need to estimate the integral $I_1(x_n)$. For $n=2$ we have

$$I_1(x_n) = \int_0^{\frac{1}{x_n}} \frac{u_1^{\tilde{A}_1 - \frac{\varepsilon}{p_1}}}{(1 + u_1)^\lambda} du_1 \leq \int_0^{\frac{1}{x_n}} u_1^{\tilde{A}_1 - \frac{\varepsilon}{p_1}} du_1 = \left(1 - \frac{\varepsilon}{p_1} + \tilde{A}_1\right)^{-1} x_n^{\frac{\varepsilon}{p_1} - \tilde{A}_1 - 1},$$

and for $n > 2$ we find

$$\begin{aligned} I_1(x_n) &\leq \left(\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=2}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=2}^{n-1} u_i)^\lambda} du_2 \dots du_{n-1} \right) \cdot \int_0^{\frac{1}{x_n}} u_1^{\tilde{A}_1 - \frac{\varepsilon}{p_1}} du_1 \\ &\leq \left(1 - \frac{\varepsilon}{p_1} + \tilde{A}_1\right)^{-1} x_n^{\frac{\varepsilon}{p_1} - \tilde{A}_1 - 1} \int_0^\infty \dots \int_0^\infty \frac{\prod_{i=2}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=2}^{n-1} u_i)^{\lambda - 1 + \frac{\varepsilon}{p_1} - \tilde{A}_1 - \varepsilon}} du_2 \dots du_{n-1} \\ &= \left(1 - \frac{\varepsilon}{p_1} + \tilde{A}_1\right)^{-1} x_n^{\frac{\varepsilon}{p_1} - \tilde{A}_1 - 1} \frac{1}{\Gamma(\lambda - 1 + \frac{\varepsilon}{p_1} - \tilde{A}_1 - \varepsilon)} \prod_{i=2}^n \Gamma(\tilde{A}_i - \frac{\varepsilon}{p_i} + 1). \end{aligned}$$

Hence, we have $I_j(x_n) \leq x_n^{\frac{\varepsilon}{p_i} - \tilde{A}_i - 1} O_j(1)$, for $\varepsilon \rightarrow 0^+$, $j = 1, 2, \dots, n - 1$, and consequently

$$\int_1^\infty x_n^{-1} \sum_{j=1}^{n-1} I_j(x_n) dx_n \leq O(1). \tag{18}$$

Since for $\varepsilon \rightarrow 0^+$ we have

$$\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=1}^{n-1} u_i)^\lambda} du_1 \dots du_{n-1} = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\tilde{A}_i + 1) + o(1)$$

we conclude, by using (17) and (18), that

$$J \geq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\tilde{A}_i + 1) \quad \text{when } \varepsilon \rightarrow 0^+. \tag{19}$$

Now let us suppose that the constant factor $K^* = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\tilde{A}_i + 1)$ is not the best possible. Then, there exists a positive constant C , smaller than K^* such that the inequality (13) is still valid if we replace K^* by C . In particular, for the above choice of the functions \tilde{f}_i we obtain, by using (15) and (19), that $K^* \leq C$ when $\varepsilon \rightarrow 0^+$. This contradiction shows that the constant factor K^* in (13) is the best possible.

We also prove that K^* is the best possible in the reverse inequality in (13). By using the same notation as before, we obtain from (16), the following inequality

$$\begin{aligned} J &\leq \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left(\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{(1 + \sum_{i=1}^{n-1} u_i)^{\lambda - \varepsilon}} du_1 \dots du_{n-1} \right) dx_n \\ &= \frac{1}{\Gamma(\lambda - \varepsilon)} \prod_{i=1}^n \Gamma(\tilde{A}_i - \frac{\varepsilon}{p_i} + 1). \end{aligned} \tag{20}$$

Now, let us suppose that there exists the constant D , greater than K^* , such that the reverse inequality in (13) is still valid if we replace K^* by D . In particular, for above choice of functions \tilde{f}_i we obtain, by using (15) and (20), that $K^* \geq D$ when $\varepsilon \rightarrow 0^+$. This contradiction shows that the constant factor K^* is the best possible.

Further, since the equivalence keeps the best possible constant, the proof is complete. \square

Now, if we put $\tilde{A}_i = \frac{\lambda}{r_i} - 1$, where $i = 1, 2, \dots, n$ and $\sum_{i=1}^n \frac{1}{r_i} = 1$, in the inequalities (13) and (14), we obtain the following

COROLLARY 1. *Let $n \geq 2$ be an integer, $\lambda > 0$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{r_i} = 1$, $r_i > 1$, $i = 1, \dots, n$ and $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. Then the following inequalities hold and are*

equivalent

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^\lambda} dx_1 \dots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right) \left(\int_0^\infty x_i^{p_i(1-\frac{\lambda}{r_i})-1} f_i^{p_i}(x_i) dx_i\right)^{\frac{1}{p_i}}, \tag{21}$$

and

$$\int_0^\infty x_n^{\frac{q}{n}\lambda-1} \left(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^\lambda} dx_1 \dots dx_{n-1}\right)^q dx_n < \left[\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right)\right]^q \prod_{i=1}^{n-1} \left(\int_0^\infty x_i^{p_i(1-\frac{\lambda}{r_i})-1} f_i^{p_i}(x_i) dx_i\right)^{\frac{q}{p_i}}. \tag{22}$$

If $p_i > 0, i \in \{1, 2, \dots, n\}$, and $p_k < 0, k \neq i$, then the reverse inequality in (21) is valid. Also, if $p_i > 0, i \in \{1, 2, \dots, n - 1\}$, and $p_k < 0, k \neq i$, then the reverse inequality in (22) is valid. Further, the inequality (22) holds also when $p_n > 0$ and $p_k < 0, k \neq n$. Also, the constant factors involved in the right-hand sides of the inequalities (21) and (22) and their's reverses are the best possible.

REMARK 5. It is interesting to consider the case $n = 2$ for the choice of the parameters A_{ij} as in Remark 2. The constraint (11) leads to the condition $pA_1 + qA_2 = 2 - \lambda$, and in that case the constant factors in the inequalities (9) and (10) are the best possible. That result was proved in [15].

4. Further results

Now we present some special cases of multiple Hardy’s inequalities. If we put

$$K(x_1, x_2, \dots, x_n) = \begin{cases} g(x_n), & x_1, x_2, \dots, x_{n-1} \leq x_n \\ 0, & \text{otherwise} \end{cases}$$

in Theorem 1, where $\Omega = [a, b], a < b$, we obtain following result

THEOREM 4. Let $n \geq 2$ be an integer, $\sum_{i=1}^n \frac{1}{p_i} = 1, p_i > 1, i = 1, \dots, n, \frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$. Then the following inequalities hold and are equivalent

$$\int_{[a,b] \times [a,x_n]^{n-1}} g(x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n) < \prod_{i=1}^{n-1} \left(\int_{[a,b]} \left(\int_{[x_i,b] \times [a,x_n]^{n-2}} g(x_n) \prod_{j=1, j \neq i}^n \phi_{ij}^{p_i}(x_j) d\mu_j(x_j)\right) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i)\right)^{\frac{1}{p_i}} \times$$

$$\times \left(\int_{[a,b]} \left(\int_{[a,x_n]^{n-1}} \prod_{j=1}^{n-1} \phi_{nj}^{p_n}(x_j) d\mu_j(x_j) \right) g(x_n) f_n^{p_n}(x_n) \phi_{nn}^{p_n}(x_n) d\mu_n(x_n) \right)^{\frac{1}{p_n}} \quad (23)$$

and

$$\begin{aligned} & \int_{[a,b]} h(x_n) g^q(x_n) \left(\int_{[a,x_n]^{n-1}} \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1}) \right)^q d\mu_n(x_n) \\ & < \prod_{i=1}^{n-1} \left(\int_{[a,b]} \left(\int_{[x_i,b] \times [a,x_n]^{n-2}} g(x_n) \prod_{j=1, j \neq i}^n \phi_{ij}^{p_i}(x_j) d\mu_j(x_j) \right) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{q}{p_i}}, \end{aligned} \quad (24)$$

where

$$h(x_n) = \phi_{nn}^{-q}(x_n) g^{1-q}(x_n) \left[\int_{[a,x_n]^{n-1}} \prod_{j=1}^{n-1} \phi_{nj}^{p_n}(x_j) d\mu_j(x_j) \right]^{1-q}.$$

If $p_i > 0$, $i \in \{1, 2, \dots, n\}$, and $p_k < 0$, $k \neq i$, then the reverse inequality in (23) is valid. Also, if $p_i > 0$, $i \in \{1, 2, \dots, n-1\}$, and $p_k < 0$, $k \neq i$, then the reverse inequality in (24) is valid. Further, the inequality (24) holds also when $p_n > 0$ and $p_k < 0$, $k \neq n$.

We also obtain the result dual to Theorem 3 by putting

$$K(x_1, x_2, \dots, x_n) = \begin{cases} g(x_n), & x_1, x_2, \dots, x_{n-1} \geq x_n \\ 0, & \text{otherwise} \end{cases}$$

in Theorem 1.

THEOREM 5. Let $n \geq 2$ be an integer, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, \dots, n$, $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$. Then the following inequalities hold and are equivalent

$$\begin{aligned} & \int_{[a,b] \times [x_n,b]^{n-1}} g(x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \dots d\mu_n(x_n) \\ & < \prod_{i=1}^{n-1} \left(\int_{[a,b]} \left(\int_{[a,x_i] \times [x_n,b]^{n-2}} g(x_n) \prod_{j=1, j \neq i}^n \phi_{ij}^{p_i}(x_j) d\mu_j(x_j) \right) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{1}{p_i}} \times \\ & \times \left(\int_{[a,b]} \left(\int_{[x_n,b]^{n-1}} \prod_{j=1}^{n-1} \phi_{nj}^{p_n}(x_j) d\mu_j(x_j) \right) g(x_n) f_n^{p_n}(x_n) \phi_{nn}^{p_n}(x_n) d\mu_n(x_n) \right)^{\frac{1}{p_n}} \end{aligned} \quad (25)$$

and

$$\int_{[a,b]} h(x_n) g^q(x_n) \left(\int_{[x_n,b]^{n-1}} \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \dots d\mu_{n-1}(x_{n-1}) \right)^q d\mu_n(x_n)$$

$$\left\langle \prod_{i=1}^{n-1} \left(\int_{[a,b]} \left(\int_{[a,x_i] \times [x_n,b]^{n-2}} g(x_n) \prod_{j=1, j \neq i}^n \phi_{ij}^{p_i}(x_j) d\mu_j(x_j) \right) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{q}{p_i}} \right\rangle \quad (26)$$

where

$$h(x_n) = \phi_{nn}^{-q}(x_n) g^{1-q}(x_n) \left[\int_{[x_n,b]^{n-1}} \prod_{j=1}^{n-1} \phi_{nj}^{p_n}(x_j) d\mu_j(x_j) \right]^{1-q}.$$

If $p_i > 0$, $i \in \{1, 2, \dots, n\}$, and $p_k < 0$, $k \neq i$, then the reverse inequality in (25) is valid. Also, if $p_i > 0$, $i \in \{1, 2, \dots, n-1\}$, and $p_k < 0$, $k \neq i$, then the reverse inequality in (26) is valid. Further, the inequality (26) holds also when $p_n > 0$ and $p_k < 0$, $k \neq n$.

REMARK 6. Note that the special cases of Theorems 3 and 4, for $n = 2$, were proved in [14].

REMARK 7. The inequalities in this paper are related to general weighted Hardy-type inequalities as described in the book [16] and the references given there. The general problems in this connection are fairly completely solved in the one-dimensional case but this is far from being true in the multidimensional case. Hence the results in this paper could be helpful to begin to understand how such weight characterizations of multidimensional Hardy type inequalities could look like in the general multidimensional case.

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