

SOME DETERMINANTAL INEQUALITIES

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(communicated by R. N. Mohapatra)

Abstract. In this paper we shall prove some new inequalities for Čebyšev functional and their applications to determinantal inequalities for n -tuples of continuous functions and vectors from unitary spaces.

1. Introduction

For two real n -tuples $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$ and $\bar{p} = (p_1, \dots, p_n)$ with $\sum_{i=1}^n p_i = 1$ the Čebyšev functional is:

$$T(\bar{a}, \bar{b}, \bar{p}) := \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \quad (1.1)$$

S. S. Dragomir [1] has proved:

LEMMA 1.1. *Let $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$ be two real n -tuples and $\bar{p} = (p_1, \dots, p_n)$ with $p_i \geq 0$, $i \in \{1, \dots, n\}$, $\sum_{i=1}^n p_i = 1$, in addition, we assume that for any $i, j \in \{1, \dots, n\}$ with $i < j$ and*

$$m(b_j - b_i) \leq a_j - a_i \leq M(b_j - b_i) \quad (1.2)$$

where m and M are real numbers. Then the following inequality

$$(m + M) T(\bar{a}, \bar{b}, \bar{p}) \geq T(\bar{a}, \bar{a}, \bar{p}) + mMT(\bar{b}, \bar{b}, \bar{p}) \quad (1.3)$$

is valid.

THEOREM 1.2. *Let assumptions of Lemma 1.1 be fulfilled and $0 < m \leq M$. Then*

$$T(\bar{a}, \bar{b}, \bar{p})^2 \geq \frac{4mM}{(m + M)^2} T(\bar{a}, \bar{a}, \bar{p}) T(\bar{b}, \bar{b}, \bar{p}) \quad (1.4)$$

holds. The constant 4 in (1.4) is best possible.

In this paper we shall give some related results to the above and other inequalities from [1].

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2. Results

First, we shall give a new proof of Theorem 1.2.

Proof of Theorem 1.2. Multiplying (1.3) with $T(\bar{b}, \bar{b}, \bar{p})$ we have

$$\begin{aligned} T(\bar{a}, \bar{a}, \bar{p}) T(\bar{b}, \bar{b}, \bar{p}) &\leq (m + M) T(\bar{a}, \bar{b}, \bar{p}) T(\bar{b}, \bar{b}, \bar{p}) - mM (T(\bar{b}, \bar{b}, \bar{p}))^2 \\ &= \frac{(m + M)^2}{4mM} (T(\bar{a}, \bar{b}, \bar{p}))^2 \\ &\quad - \left(\frac{m + M}{2\sqrt{mM}} T(\bar{a}, \bar{b}, \bar{p}) - \sqrt{mM} T(\bar{b}, \bar{b}, \bar{p}) \right)^2 \\ &\leq \frac{(m + M)^2}{4mM} (T(\bar{a}, \bar{b}, \bar{p}))^2 \quad \square \end{aligned}$$

THEOREM 2.1. *Let assumptions of Theorem 1.2 be fulfilled, then the following inequalities holds*

$$0 \leq (T(\bar{a}, \bar{a}, \bar{p}))^{\frac{1}{2}} (T(\bar{b}, \bar{b}, \bar{p}))^{\frac{1}{2}} - T(\bar{a}, \bar{b}, \bar{p}) \leq \frac{(M - m)^2}{4(M + m)} T(\bar{a}, \bar{b}, \bar{p}) \quad (2.1)$$

and

$$0 \leq \frac{T(\bar{a}, \bar{a}, \bar{p})}{T(\bar{a}, \bar{b}, \bar{p})} - \frac{T(\bar{a}, \bar{b}, \bar{p})}{T(\bar{b}, \bar{b}, \bar{p})} \leq (\sqrt{M} - \sqrt{m})^2 \quad (2.2)$$

Proof. The first inequality from (2.1) and (2.2) is well known and it is a simple consequence of Cauchy inequality. On the other side from (1.3) we have

$$\begin{aligned} 0 &\leq (T(\bar{a}, \bar{a}, \bar{p}))^{\frac{1}{2}} (T(\bar{b}, \bar{b}, \bar{p}))^{\frac{1}{2}} - T(\bar{a}, \bar{b}, \bar{p}) \\ &\leq (T(\bar{a}, \bar{a}, \bar{p}))^{\frac{1}{2}} (T(\bar{b}, \bar{b}, \bar{p}))^{\frac{1}{2}} - \frac{1}{M + m} T(\bar{a}, \bar{a}, \bar{p}) - \frac{mM}{M + m} T(\bar{b}, \bar{b}, \bar{p}) \\ &= \frac{(M - m)^2}{4(M + m)} T(\bar{b}, \bar{b}, \bar{p}) - \frac{1}{(M + m)} \left((T(\bar{a}, \bar{a}, \bar{p}))^{\frac{1}{2}} - \frac{M + m}{2} (T(\bar{b}, \bar{b}, \bar{p}))^{\frac{1}{2}} \right)^2 \\ &\leq \frac{(M - m)^2}{4(M + m)} T(\bar{b}, \bar{b}, \bar{p}) \end{aligned}$$

Divided by $T(\bar{a}, \bar{b}, \bar{p})$ the inequality (1.3) we have the second inequality in (2.2) :

$$\begin{aligned} 0 &\leq \frac{T(\bar{a}, \bar{a}, \bar{p})}{T(\bar{a}, \bar{b}, \bar{p})} - \frac{T(\bar{a}, \bar{b}, \bar{p})}{T(\bar{b}, \bar{b}, \bar{p})} \\ &\leq (M + m) - mM \frac{T(\bar{b}, \bar{b}, \bar{p})}{T(\bar{a}, \bar{b}, \bar{p})} - \frac{T(\bar{a}, \bar{b}, \bar{p})}{T(\bar{b}, \bar{b}, \bar{p})} \\ &= (\sqrt{M} - \sqrt{m})^2 - \left(\sqrt{mM} \left(\frac{T(\bar{b}, \bar{b}, \bar{p})}{T(\bar{a}, \bar{b}, \bar{p})} \right)^{\frac{1}{2}} - \left(\frac{T(\bar{a}, \bar{b}, \bar{p})}{T(\bar{b}, \bar{b}, \bar{p})} \right)^{\frac{1}{2}} \right)^2 \\ &\leq (\sqrt{M} - \sqrt{m})^2 \quad \square \end{aligned}$$

COROLLARY 2.2. *Let assumptions of Theorem 1.2 be fulfilled, then the following inequalities hold*

$$0 \leq (T(\bar{a}, \bar{a}, \bar{p}))^{\frac{1}{2}} (T(\bar{b}, \bar{b}, \bar{p}))^{\frac{1}{2}} - T(\bar{a}, \bar{b}, \bar{p}) \leq \frac{(M - m)^2}{4mM(M + m)} T(\bar{a}, \bar{a}, \bar{p}) \quad (2.3)$$

and

$$0 \leq \frac{T(\bar{b}, \bar{b}, \bar{p})}{T(\bar{a}, \bar{b}, \bar{p})} - \frac{T(\bar{a}, \bar{b}, \bar{p})}{T(\bar{a}, \bar{a}, \bar{p})} \leq \frac{(\sqrt{M} - \sqrt{m})^2}{mM} \quad (2.4)$$

Proof. Applying the inequality (1.2) we have

$$\frac{1}{M} (a_j - a_i) \leq (b_j - b_i) \leq \frac{1}{m} (a_j - a_i) \quad (2.5)$$

By Lemma 1.1 we have

$$(m + M) T(\bar{a}, \bar{b}, \bar{p}) \geq T(\bar{b}, \bar{b}, \bar{p}) + mM T(\bar{a}, \bar{a}, \bar{p}) \quad (2.6)$$

Proof of the inequalities (2.3) and (2.4) is similar the proof of Theorem 2.1 with using the inequality (2.6). \square

Similarly, by using Theorem 5 from [1] we can prove the following results related to that given in Corollary 6 from [1]:

THEOREM 2.3. *Let $f, g : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0$ for $x \in (a, b)$. Assume that*

$$0 < m = \inf_{x \in (a,b)} \left(\frac{f'(x)}{g'(x)} \right), \quad \sup_{x \in (a,b)} \left(\frac{f'(x)}{g'(x)} \right) < \infty \quad (2.7)$$

Then inequalities (2.1) and (2.2) with $a_i = f(x_i)$, $b_i = g(x_i)$ are also valid.

Similar, consequences of Lemma 7 from [1] is:

THEOREM 2.4. *Let $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$ and $\bar{p} = (p_1, \dots, p_n)$ be real n -tuples with $p_i \geq 0$, $i \in \{1, \dots, n\}$, $\sum_{i=1}^n p_i = 1$, $b_i \neq A_n(\bar{b}; \bar{p})$ for each $i \in \{1, \dots, n\}$ and*

$$0 < m \leq \frac{a_i - A_n(\bar{a}; \bar{p})}{b_i - A_n(\bar{b}; \bar{p})} \leq M < \infty \quad (2.8)$$

where $A_n(\bar{b}; \bar{p}) := \frac{\sum_{i=1}^n p_i b_i}{\sum_{i=1}^n p_i}$. *Then inequalities (2.1) and (2.2) are also valid.*

B. Mond, J. Pečarić and B. Tepeš in [2] has proved:

LEMMA 2.5. *Let $\bar{f} = (f_1, \dots, f_n)$, $\bar{g} = (g_1, \dots, g_n)$ be two n -tuples of continuous functions $f_1, \dots, f_n, g_1, \dots, g_n : [a, b] \rightarrow \mathbf{R}$ and $p : [a, b] \rightarrow (0, \infty)$ continuous function. Assume also that for $\bar{x} = (x_1, \dots, x_n)$, $a \leq x_1 < \dots < x_n \leq b$ we have inequality*

$$m \det \bar{g}(\bar{x}) \leq \det \bar{f}(\bar{x}) \leq M \det \bar{g}(\bar{x}) \quad (2.9)$$

where $\det \bar{f}(\bar{x}) = \det [f_i(x_j)]$ and $\det \bar{g}(\bar{x}) = \det [g_i(x_j)]$ are two determinants and m, M are given positive real numbers. Then one has the following inequality

$$(m + M) [\bar{f}, \bar{g}] \geq [\bar{f}, \bar{f}] + mM [\bar{g}, \bar{g}] \quad (2.10)$$

where $[\bar{f}, \bar{g}] = \det \left[\int_a^b p(x) f_i(x) g_j(x) \right]$.

LEMMA 2.6. If $\bar{f}(\bar{x})$, $\bar{g}(\bar{x})$, $\bar{p}(\bar{x})$ are the same as in Lemma 2.5, in addition, we assume that $0 < m \leq M$ and $\det [g_i(x)^{(j-1)}] \neq 0$. Assume also that for $a \leq x_1 < \dots < x_n \leq b$ one has inequality

$$0 < m = \inf_{x \in (a,b)} \frac{\det [f_i^{(j-1)}(x)]}{\det [g_i^{(j-1)}(x)]}, \quad \sup_{x \in (a,b)} \frac{\det [f_i^{(j-1)}(x)]}{\det [g_i^{(j-1)}(x)]} = M < \infty. \quad (2.11)$$

Then the inequality (2.10) is also valid.

LEMMA 2.7. Let x_1, \dots, x_r and y_1, \dots, y_r be vectors from unitary space X with finite dimension $\dim X = n$ and e_1, \dots, e_n be any orthonormal base in X . Assume also that $\det [(x_i, e_{j_k})]$ and $\det [(y_i, e_{j_k})]$ for $r < n$ and $1 \leq j_1 < j_2 < \dots < j_r \leq n$ satisfies inequality

$$m \det [(y_i, e_{j_k})] \leq \det [(x_i, e_{j_k})] \leq M \det [(x_i, e_{j_k})] \quad (2.12)$$

Then the following inequality is valid

$$(m + M) \det [(x_i, y_j)] \geq \Gamma(x_1, \dots, x_r) + mM \Gamma(y_1, \dots, y_r) \quad (2.13)$$

Now, we can give a new proof of the following result from [2]:

THEOREM 2.8. Let assumptions of Lemma 2.5 is fulfilled and $0 < m \leq M$. Then

$$[\bar{f}, \bar{f}] \cdot [\bar{g}, \bar{g}] \leq \frac{(m + M)^2}{4mM} [\bar{f}, \bar{g}]^2. \quad (2.14)$$

Proof. We are working as in the above proof of Theorem 1.2. From Lemma 2.5 multiplying (2.13) with $[\bar{g}, \bar{g}]$ we have

$$\begin{aligned} [\bar{f}, \bar{f}] \cdot [\bar{g}, \bar{g}] &\leq (m + M) [\bar{f}, \bar{g}] \cdot [\bar{g}, \bar{g}] - mM [\bar{g}, \bar{g}]^2 \\ &= \frac{(m + M)^2}{4mM} [\bar{f}, \bar{g}]^2 - \left(\frac{m + M}{2\sqrt{mM}} [\bar{f}, \bar{g}] - \sqrt{mM} [\bar{g}, \bar{g}] \right)^2 \\ &\leq \frac{(m + M)^2}{4mM} [\bar{f}, \bar{g}]^2 \quad \square \end{aligned}$$

Similarly to the proof of Theorem 2.1, using Lemma 2.5 instead of Lemma 1.1, we can prove:

THEOREM 2.9. *Let $f^-(\bar{x})$, $\bar{g}(\bar{x})$, $\bar{p}(\bar{x})$ be the same as in Lemma 2.5 and we assume that $0 < m \leq M$. Then the following inequalities hold*

$$0 \leq [f^-, f^-]^{\frac{1}{2}} [\bar{g}, \bar{g}]^{\frac{1}{2}} - [f^-, \bar{g}] \leq \frac{(M - m)^2}{4(m + M)} [\bar{g}, \bar{g}] \tag{2.15}$$

and

$$0 \leq \frac{[f^-, f^-]}{[f^-, \bar{g}]} - \frac{[f^-, \bar{g}]}{[\bar{g}, \bar{g}]} \leq (\sqrt{M} - \sqrt{m})^2 \tag{2.16}$$

COROLLARY 2.10. *If $f^-(\bar{x})$, $\bar{g}(\bar{x})$, $\bar{p}(\bar{x})$ are the same as in Lemma 2.5 and $0 < m \leq M$. Then, the following inequalities holds*

$$0 \leq [f^-, f^-]^{\frac{1}{2}} [\bar{g}, \bar{g}]^{\frac{1}{2}} - [f^-, \bar{g}] \leq \frac{(M - m)^2}{4mM(M + m)} [f^-, f^-], \tag{2.17}$$

and

$$0 \leq \frac{[\bar{g}, \bar{g}]}{[f^-, \bar{g}]} - \frac{[f^-, \bar{g}]}{[f^-, f^-]} \leq \frac{(\sqrt{M} - \sqrt{m})^2}{mM} \tag{2.18}$$

Proof. This proof is similar to the proof of Corollary 2.2. \square

THEOREM 2.11. *If $f^-(\bar{x})$, $\bar{g}(\bar{x})$, $\bar{p}(\bar{x})$ are the same as in Lemma 2.5. In addition, we assume that $0 < m \leq M$ and $\det [g_i^{(j-1)}(x)] \neq 0$. Assume also that for $a \leq x_1 < \dots < x_n \leq b$ one has inequality*

$$0 < m = \inf_{x \in (a,b)} \frac{\det [f_i^{(j-1)}(x)]}{\det [g_i^{(j-1)}(x)]}, \quad \sup_{x \in (a,b)} \frac{\det [f_i^{(j-1)}(x)]}{\det [g_i^{(j-1)}(x)]} = M < \infty. \tag{2.19}$$

Then the following inequalities hold

$$0 \leq [f^-, f^-]^{\frac{1}{2}} [\bar{g}, \bar{g}]^{\frac{1}{2}} - [f^-, \bar{g}] \leq \frac{(M - m)^2}{4(m + M)} [\bar{g}, \bar{g}] \tag{2.20}$$

and

$$0 \leq \frac{[f^-, f^-]}{[f^-, \bar{g}]} - \frac{[f^-, \bar{g}]}{[\bar{g}, \bar{g}]} \leq (\sqrt{M} - \sqrt{m})^2 \tag{2.21}$$

THEOREM 2.12. *Let x_1, \dots, x_r ; y_1, \dots, y_r ; e_1, \dots, e_n be vectors from unitary space X same as in Lemma 2.7 and $0 < m \leq M$. Then the following inequalities hold*

$$0 \leq (\Gamma(x_1, \dots, x_r))^{\frac{1}{2}} (\Gamma(y_1, \dots, y_r))^{\frac{1}{2}} - \det [(x_i, y_j)] \leq \frac{(M - m)^2}{4(M + m)} \Gamma(y_1, \dots, y_r) \tag{2.22}$$

and

$$0 \leq \frac{\Gamma(x_1, \dots, x_r)}{\det [(x_i, y_j)]} - \frac{\det [(x_i, y_j)]}{\Gamma(y_1, \dots, y_r)} \leq (\sqrt{M} - \sqrt{m})^2 \tag{2.23}$$

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