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WEIGHTED INEQUALITIES FOR POSITIVE OPERATORS

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(*communicated by L.-E. Persson*)

Abstract. A technique arising from Schur's Lemma and its converse is shown to generate weighted Lebesgue norm inequalities for a wide class of linear and non-linear positive operators. In many cases the best constants for these inequalities are determined as well. A sharp converse to Schur's Lemma is proved via a minimax principle for a class of positive operators on Banach Function Spaces. This shows that all such inequalities can be generated by this technique and establishes a structure theorem for weight pairs.

Examples involving Hardy and Stieltjes operators are given as well as several Opial-type inequalities. As an illustration of the structure theorem a new proof is given of necessity in the well-known weight characterization for the Hardy operator.

1. Introduction

Let (X,μ) and (Y,ν) be σ-finite measure spaces and L^+_μ and L^+_ν denote the collections of non-negative measurable functions on (X,μ) and (Y,ν) respectively. Suppose that for $i=1,\dots,n$ the maps $T_i : L^+_\nu \to L^+_\mu$ have *formal adjoints* $T_i^* : L^+_\mu \to L^+_\nu$, that is,

$$\int_X (T_i f)\varphi\, d\mu = \int_Y f\,(T_i^*\varphi)\, d\nu, \quad \text{for all } f \in L^+_\nu \text{ and } \varphi \in L^+_\mu .$$

Define the map $T : L^+_\nu \to L^+_\mu$ by

$$(Tf)^q = \prod_{i=1}^n (T_i f^{r_i})^{q_i/r_i} \tag{1.1}$$

where $q = q_1 + q_2 + \cdots + q_n$. Our purpose is to give a method for generating weighted inequalities of the form

$$\left(\int_X (Tf)^q u\, d\mu\right)^{1/q} \leqslant C\left(\int_Y f^p v\, d\nu\right)^{1/p}, \quad f \in L^+_\nu, \tag{1.2}$$

for indices satisfying $r_i < q \leqslant p < \infty$ for all i. If $q < p$ and $r_i \leqslant q_i$ for all i then the method generates essentially all such inequalities and always produces the best

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constant. If $q = p$ and $r_i \leqslant q_i$ for all i then the method generates essentially all such inequalities and produces constants arbitrarily close to optimal.

The methods of the paper can be traced back to Frobenius and Schur but are naturally more closely connected with recent work. Schur's lemma [14] gave a method for proving that a matrix with non-negative entries was bounded as a map on ℓ^2. Many generalizations and applications of the result and its converse [7] followed. See, for example [1, 3, 4, 5, 9, 10, 11, 13, 16, 18, 19]. In [9, 16], the method was used on positive, linear operators that need not be integral operators. In [5] it was extended to non-linear, positive operators of the form $f \mapsto f^{\alpha}Tf$ with T a positive integral operator. The operators in (1.1) include both these cases and much more.

The minimax principle introduced in [6] evolved in [5, 19] and elsewhere. Here we give a substantial extension of the principle to include operators in Banach Function Spaces. In this general setting we are able to improve the clever iteration given in [7] to establish a sharp converse to Schur's Lemma for operators of the form (1.1). This sharp converse was proved for matrix operators on Lebesgue sequence spaces in [3, 11] and for positive operators with formal adjoints on Lebesgue spaces in [16].

The plan of the paper emphasizes our focus on weighted inequalities. In the next section we state and prove our method for generating weighted inequalities of the form (1.2). We also state the converse results which show that the method generates all such inequalities. The minimax principle is given in Section 3 and in Section 4 we apply it in the Lebesgue space case to give our general Schur's Lemma, the sharp converse, and the proofs deferred from Section 2.

From a wealth of possible examples we select a collection that illustrates the versatility of the method. These are given in Section 5. The first two examples are weighted norm inequalities with best constant for the Hardy operator and the Stieltjes transformation. The next two show how using product operators like (1.1) can restore homogeneity in inequalities with nonhomogeneous constraints. Example 5.5 gives the best constant in a known weighted Opial inequality and Example 5.6 is an unusual variant of Hardy's inequality which does not seem to be accessible by other methods.

Section 5 concludes with a new proof of a well-known result. See, for example, [12] and the references given there. The idea is to use the sharp converse of Schur's Lemma as a structure theorem for weight functions, a notion that promises to provide a new technique in weighted norm inequalities.

The remainder of this introduction is devoted to notation and definitions used in the paper. For a σ-finite measure space (Y, λ) we let L^+_λ be the collection of λ-measurable functions $\varphi : Y \to [0, \infty]$. Arithmetic on $[0, \infty]$ is used throughout with the conventions that $0(\infty) = (\infty)0 = 0$, $0/0 = 0$, $\infty/\infty = 0$, and $\infty^0 = 1$. With these conventions the expression $f \leqslant \infty g$, for $f, g \in L^+_\lambda$, means that f vanishes wherever g does.

As usual we write $f_n \uparrow f$, for $f, f_1, f_2, \cdots \in L^+_\lambda$ to mean that the sequence $f_1, f_2, \ldots$ is non-increasing and converges pointwise λ-almost everywhere to f. Similarly for $f_n \downarrow f$. We use a prime to denote the harmonic conjugate of an index so that $1/p + 1/p' = 1$ whenever $1 \leqslant p \leqslant \infty$.

We say that an operator $T : L^+_\nu \to L^+_\mu$ is $[0, \infty]$-*linear* if $T(af + g) = aTf + Tg$ for all $a \in [0, \infty]$; *preserves order* if $Tf \leqslant Tg$ whenever $f \leqslant g$; is *order continuous*

if it preserves order and $Tf_n \uparrow Tf$ whenever $f_n \uparrow f$; is *strongly order continuous* if it is order continuous and $Tf_n \downarrow Tf$ whenever both $f_n \downarrow f$ and $Tf_1 < \infty$ λ-almost everywhere.

Note that the Arithmetic-Geometric Mean inequality (AGM) remains valid on $[0,\infty]$: If $a_1,\dots,a_n \in [0,\infty]$ and $\theta_1,\dots,\theta_n \in (0,1]$ satisfy $\theta_1+\cdots+\theta_n = 1$ then

$$\prod_{i=1}^{n} a_i^{\theta_i} \leqslant \sum_{i=1}^{n} \theta_i a_i.$$

For definitions and properties of Banach Function Norms we refer [2]. One simple consequence of the definition [2, I.1.1] is that for any Banach Function Norm over a σ-finite measure space there exists a positive function with finite norm. Another fact that we will need is [2, Proposition I.3.6]. If a Banach Function Norm is absolutely continuous then it has a dominated convergence property.

If $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ are Banach Function Norms on L^+_μ and L^+_ν respectively and $J : L^+_\nu \to L^+_\mu$ the inequality

$$\|Jf\|_\mu \leqslant C\|f\|_\nu, \quad f \in L^+_\nu, \tag{1.3}$$

is equivalent to

$$\sup_{f\in L^+_\nu} \frac{\|Jf\|_\mu}{\|f\|_\nu} \leqslant C$$

and the least constant C for which they hold is called the *best constant* in (1.3). A function g satisfying $0 < \|g\|_\nu < \infty$ for which $\|Jg\|_\mu/\|g\|_\nu$ is the best constant in (1.3) is called an *extremal* for (1.3).

2. Weighted Norm Inequalities

Suppose that for $i = 1,\dots,n$ the maps $T_i : L^+_\nu \to L^+_\mu$ have formal adjoints $T_i^* : L^+_\mu \to L^+_\nu$ and T is defined by (1.1) with $q = q_1+\cdots+q_n$ and $r_i < q \leqslant p < \infty$. Fix a weight $u \in L^+_\mu$. For each positive $g \in L^+_\nu$ we define

$$\begin{aligned} v_g &= \sum_{i=1}^{n} (q_i/q) g^{r_i-p} T_i^*(u(Tg)^q/T_i(g^{r_i})), \\ C_g &= \left(\int_Y g^p v_g \, d\nu\right)^{(1/q)-(1/p)}. \end{aligned} \tag{2.1}$$

Note that by our convention, $C_g = 1$ when $p = q$ even if $\int_Y g^p v_g \, d\nu = \infty$.

THEOREM 2.1. *If* $0 < g < \infty$ ν*-almost everywhere and* $0 < Tg < \infty$ $u\mu$*-almost everywhere then*

$$\left(\int_X (Tf)^q u \, d\mu\right)^{1/q} \leqslant C_g \left(\int_Y f^p v_g \, d\nu\right)^{1/p}, \quad f \in L^+_\nu. \tag{2.2}$$

If $0 < \int_X (Tg)^q u\, d\mu < \infty$ *then* C_g *is the best constant in* (2.2) *and* g *is an extremal for* (2.2).

When $q < p$ this method generates all weighted norm inequalities for operators T provided $q_i \geqslant r_i$ for $i = 1, \ldots, n$ as we see in our first converse to Theorem 2.1. The condition (2.3) imposed in Theorem 2.2 and the restriction $0 < v < \infty$ imposed in Theorems 2.2 and 2.3 are not essential and can be easily removed by reducing (1.2) to an equivalent inequality. See Theorem 4.4.

THEOREM 2.2. *Suppose that* $q < p$ *and* $q_i \geqslant r_i$ *for* $i = 1, \ldots, n$, *and that the best constant in* (1.2) *is* $C < \infty$. *If* $0 < v < \infty$ *and* T *satisfies,*

$$\text{if } Tf = T(\chi_E f) \text{ for all } f \in L_\nu^+ \text{ then } \nu(Y \setminus E) = 0 \tag{2.3}$$

then there exists a $g \in L_\nu^+$ *satisfying* $0 < g < \infty$ *and* $0 < \int_X (Tg)^q u\, d\mu < \infty$ *such that* $v = v_g$ *and* $C = C_g$.

This result may be viewed as a structure theorem for weights. Once the operator and the indices are fixed, Theorem 2.2 states that every weight pair for which the inequality holds is of the form (u, v_g) for some positive g. See Theorem 5.7 for an example of how this idea may be used.

The classical Hardy inequality [8, Theorem 327] has no extremal function, showing that Theorem 2.2 does not extend to the case $p = q$. Thus, when $p = q$ there are inequalities of the form (1.2) that are not generated by Theorem 2.1 for any choice of the function g. However, even when $p = q$ the method of Theorem 2.1 can generate inequalities as close as desired to any given inequality of the form (1.2).

THEOREM 2.3. *Suppose* $q \leqslant p$, $q_i \geqslant r_i$ *for* $i = 1, \ldots, n$, C *is the best constant in* (1.2) *and* $C < A < \infty$. *If* $0 < v < \infty$ *then there exists a* $g \in L_\nu^+$ *satisfying* $0 < g < \infty$ *and* $0 < \int_X (Tg)^q u\, d\mu < \infty$ *such that* $v_g \leqslant A^p v$ *and* $C_g \leqslant 1$ *so that*

$$\left(\int_X (Tf)^q u\, d\mu\right)^{1/q} \leqslant C_g \left(\int_Y f^p v_g\, d\nu\right)^{1/p} \leqslant A \left(\int_Y f^p v\, d\nu\right)^{1/p}$$

holds for all $f \in L_\nu^+$.

We prove Theorem 2.1 below but the rest of the proofs are deferred until Section 4 because they depend on the minimax principle of Section 3. To begin, however, we must take a closer look at operators with formal adjoints.

The most popular positive operators are the integral operators with non-negative kernels and it is immediate that they have formal adjoints. Although the identity operator is not in general an integral operator, it clearly has a formal adjoint as well. More generally, if a positive operator $J : L_\nu^+ \to L_\mu^+$ arises as the restriction to the positive functions of an order continuous linear operator on the space of real-valued functions then, according to [9, p. 141], it necessarily has a formal adjoint.

In the next lemma we see that an operator with a formal adjoint inherits many of the properties of integration. We review the standard proofs here with an eye to arithmetic in $[0, \infty]$.

LEMMA 2.4. *Suppose $J : L_\nu^+ \to L_\mu^+$ has a formal adjoint. Then the formal adjoint is unique, J is $[0,\infty]$-linear and strongly order continuous. Also, if $1 < q < \infty$ and $f, g \in L_\nu^+$ then*

$$J(fg) \leqslant (J(f^q))^{1/q}(J(g^{q'}))^{1/q'}.$$

Proof. If J^* and $J^\star$ are both formal adjoints of J then for all E with $\nu E < \infty$ and all $\varphi \in L_\mu^+$ we have

$$\int_E J^*\varphi \, d\nu = \int_X (J\chi_E)\varphi \, d\mu = \int_E J^\star\varphi \, d\nu$$

and hence $J^*\varphi = J^\star\varphi$ ν-almost everywhere. This shows that the formal adjoint is unique.

Let J^* be the formal adjoint of J. If $\mu E < \infty$, $a \in [0,\infty]$ and $f, g \in L_\nu^+$ then

$$\begin{aligned}\int_E J(af+g)\, d\mu &= \int_Y (af+g)J^*\chi_E \, d\nu = a\int_Y fJ^*\chi_E \, d\nu + \int_Y gJ^*\chi_E \, d\nu \\ &= a\int_E Jf \, d\mu + \int_E Jg \, d\mu = \int_E aJf + Jg \, d\mu\end{aligned}$$

so we have $J(af+g) = aJf + Jg$ ν-almost everywhere. This proves the $[0,\infty]$-linearity of J. If $f \leqslant g$ and $\mu E < \infty$ then

$$\int_E (Jf)\, d\mu = \int_Y fJ^*\chi_E \, d\nu \leqslant \int_Y gJ^*\chi_E \, d\nu = \int_E Jg \, d\mu$$

so $Jf \leqslant Jg$. Thus J is order-preserving and so if $\{f_n\}$ is a non-decreasing sequence in L_ν^+ then Jf_n is a non-decreasing sequence in L_μ^+. Let f and φ be the pointwise limits of these two sequences. To see that $\varphi = Jf$ we use the formal adjoint again. If $\mu E < \infty$ then by the Monotone Convergence Theorem applied twice we have

$$\int_E \varphi \, d\mu = \lim_{n\to\infty}\int_E Jf_n \, d\mu = \lim_{n\to\infty}\int_Y f_nJ^*\chi_E \, d\nu = \int_Y fJ^*\chi_E \, d\nu = \int_E Jf \, d\mu.$$

This implies that $\varphi = Jf$ μ-almost everywhere as desired and we have shown that J is order continuous. To prove strong order continuity we apply the Dominated Convergence Theorem. If $\{f_n\}$ is a non-increasing sequence in L_ν^+ then Jf_n is a non-increasing sequence in L_μ^+. Once again, let f and φ be the pointwise limits. If Jf_1 is finite μ-almost everywhere and $\mu E < \infty$ then the sets $E_m = \{x \in E : Jf_1(x) \leqslant m\}$ increase with m to E, except for a set of λ-measure zero. Moreover

$$\int_Y f_1J^*\chi_{E_m} \, d\nu = \int_{E_m} Jf_1 \, d\mu < \infty.$$

Thus, by the Dominated Convergence Theorem applied twice we have

$$\begin{aligned}\int_{E_m} \varphi \, d\mu &= \lim_{n\to\infty}\int_{E_m} Jf_n \, d\mu = \lim_{n\to\infty}\int_Y f_nJ^*\chi_{E_m} \, d\nu \\ &= \int_Y fJ^*\chi_{E_m} \, d\nu = \int_{E_m} Jf \, d\mu.\end{aligned}$$

As $m \to \infty$ the Monotone Convergence Theorem shows that

$$\int_E \varphi \, d\mu = \int_E Jf \, d\mu.$$

This implies that $\varphi = Jf$ μ-almost everywhere and we have established the strong order continuity of J.

To prove the analogue of Hölder's inequality we first dispense with the case where the right hand side is zero. Since $fg \leqslant \infty f^q$ the $[0,\infty]$-linearity of J implies $J(fg) \leqslant \infty J(f^q)$ so $J(fg)$ vanishes wherever $J(f^q)$ does. Similarly $J(fg)$ vanishes wherever $J(g^{q'})$ does. It remains to prove the inequality where both $J(f^q)$ and $J(g^{q'})$ are positive and finite. Using the homogeneity of J we may assume that both are 1. Now by the AGM we have

$$J(fg) \leqslant J((1/q)f^q + (1/q')g^{q'}) = (1/q)J(f^q) + (1/q')J(g^{q'}) = 1.$$

This completes the proof.

The key argument for the proof of Theorem 2.1 is isolated in the next lemma so that it may be re-used more readily. Define the maps R_i, $i = 1, \ldots, n$, by

$$R_i g(x) = \begin{cases} \infty, & \text{if } q_i < r_i \text{ and } T_i g^{r_i}(x) = \infty \\ \prod_{j=1}^n (T_j g^{r_j})^{q_j/r_j - \delta_{ij}}, & \text{otherwise.} \end{cases}$$

This definition is complicated by difficulties with the rules for exponents when extended real values are involved. However, when $0 < T_i g^{r_i} < \infty$ the definition reduces to

$$R_i g = (Tg)^q / T_i g^{r_i}.$$

LEMMA 2.5. *If $f, g \in L_v^+$ then*

$$\int_X (T(fg))^q u \, d\mu \leqslant \int_Y f^q \left(\sum_{i=1}^n (q_i/q) g^{r_i} T_i^*(u R_i g) \right) \tag{2.4}$$

with equality when $f \equiv 1$.

Proof. We can apply the Hölder inequality from Lemma 2.4 with q replaced by q/r_i to get

$$T_i((fg)^{r_i}) = T_i(f^{r_i} g_i^{r_i^2/q} g^{r_i(q-r_i)/q}) \leqslant (T_i(f^q g^{r_i}))^{r_i/q} (T_i(g^{r_i}))^{(q-r_i)/q}$$

for each i. If for some x, $0 < Tg(x) < \infty$ then $0 < (T_i g^{r_i})(x) < \infty$ for all i so

$$\begin{aligned} T(fg)(x)^q &= \prod_{i=1}^n T_i((fg)^{r_i})(x)^{q_i/r_i} \\ &\leqslant \prod_{i=1}^n T_i(f^q g^{r_i})(x)^{q_i/q} T_i(g^{r_i})(x)^{(q_i/r_i)-(q_i/q)} \\ &= Tg(x)^q \prod_{i=1}^n [T_i(f^q g^{r_i})(x)/T_i(g^{r_i})(x)]^{q_i/q} \\ &\leqslant Tg(x)^q \sum_{i=1}^n (q_i/q) T_i(f^q g^{r_i})(x)/T_i(g^{r_i})(x). \end{aligned}$$

The last inequality is an application of the AGM. We have shown that

$$T(fg)(x)^q \leqslant \sum_{i=1}^{n} (q_i/q) T_i(f^q g^{r_i})(x) R_i g(x) \tag{2.5}$$

whenever $0 < Tg(x) < \infty$. If $Tg(x) = 0$ then because $fg \leqslant \infty g$ we have $T(fg)(x) \leqslant \infty Tg(x) = 0$. Thus (2.5) holds when $Tg(x) = 0$. If $Tg(x) = \infty$ then $T_j(g^{r_j})(x) > 0$ for all j and $T_i(g^{r_i})(x) = \infty$ for some i. It follows from the definition of the R_j's that $R_j g(x) = \infty$ for $i \neq j$ and $R_i g(x) = \infty$ as well unless $q_i = r_i$. If $n > 1$ we can choose $j \neq i$ to get $R_j g(x) = \infty$ and if $n = 1$ then $q_1 = q > r_1$ by assumption so we can choose $j = 1$ to get $R_j g(x) = \infty$. For this j, if $T_j(f^q g^{r_j})(x) > 0$ then (2.5) holds with infinite right hand side and if $T_j(f^q g^{r_j})(x) = 0$ then (2.5) holds with zero left hand side because $(fg)^{r_j} \leqslant \infty f^q g^{r_j}$ implies $T_j((fg)^{r_j})(x) \leqslant \infty T_j(f^q g^{r_j})(x) = 0$ and hence $T(fg)(x) = 0$. We conclude that (2.5) holds for all x.

It is easy to check that the inequality (2.5) reduces to equality when $f \equiv 1$. This property is retained when we integrate (2.5) to get

$$\begin{aligned}
\int_X (T(fg))^q u \, d\mu &\leqslant \sum_{i=1}^{n} (q_i/q) \int_X T_i(f^q g^{r_i}) u R_i g \, d\mu \\
&= \sum_{i=1}^{n} (q_i/q) \int_Y f^q g^{r_i} T_i^*(u R_i g) \, d\nu \\
&= \int_Y f^q \left(\sum_{i=1}^{n} (q_i/q) g^{r_i} T_i^*(u R_i g) \right) d\nu.
\end{aligned}$$

This completes the proof.

Proof of Theorem 2.1. The hypothesis that $0 < Tg < \infty$ $u\mu$-almost everywhere together with the definitions of v_g and $R_i g$ yield

$$v_g = \sum_{i=1}^{n} (q_i/q) g^{r_i - p} T_i^*(u R_i g).$$

Since $0 < g < \infty$, $(f/g)g = f$ so we can replace f by f/g in (2.4) to get

$$\int_X (Tf)^q u \, d\mu \leqslant \int_Y f^q g^{p-q} v_g \, d\nu.$$

If $p = q$ then $C_g = 1$ so we just take qth roots to get (2.2). If $p > q$ then we apply Hölder's inequality with indices p/q and $p/(p-q)$ to get

$$\left(\int_X (Tf)^q u \, d\mu \right)^{1/q} \leqslant C_g \left(\int_Y f^p v_g \, d\nu \right)^{1/p}.$$

This proves the first statement of Theorem 2.1. For the second statement we use the fact that (2.4) is equality when $f \equiv 1$ and take qth roots to get

$$\left(\int_X (Tg)^q u \, d\mu \right)^{1/q} = \left(\int_Y g^p v_g \, d\nu \right)^{1/q} = C_g \left(\int_Y g^p v_g \, d\nu \right)^{1/p}.$$

Thus, if $0 < \int_X (Tg)^q u\, d\mu < \infty$ then C_g is the best constant and g is an extremal for (2.2). This completes the proof.

Looked at in the right way, Lemma 2.5 enables us to reduce inequalities involving T, a map between two different function spaces, to inequalities involving a map on a single function space. If $0 < v < \infty$ the new map $S : L_v^+ \to L_v^+$ is defined by

$$(Sg)^p = \frac{1}{v} \sum_{i=1}^{n} (q_i/q) g^{r_i} T_i^*(uR_i g). \tag{2.6}$$

With $f \equiv 1$, Lemma 2.5 shows that

$$\int_X (Tg)^q u\, d\mu = \int_Y (Sg)^p v\, d\nu \tag{2.7}$$

so (1.2) becomes

$$\left(\int_Y (Sf)^p v\, d\nu \right)^{1/q} \leqslant C \left(\int_Y f^p v\, d\nu \right)^{1/p}$$

or equivalently,

$$\sup_{f \in L_v^+} \frac{\|Sf\|_{L^p_{v\nu}}}{\|f\|^{q/p}_{L^p_{v\nu}}} \leqslant C^{q/p}.$$

This is the sort of inequality we address in Section 3. The condition (3.1) imposed on the operator S is motivated by the following consequence of Lemma 2.5. Using (2.7) to write (2.4) in terms of S we have

$$\int_Y S(fg)^p v\, d\nu \leqslant \int_Y f^q (Sg)^p v\, d\nu$$

which may be written as

$$\|S(fg)\|_{L^p_{v\nu}} \leqslant \|f^{q/p} Sg\|_{L^p_{v\nu}}. \tag{2.8}$$

3. A Minimax Principle

Let (Y, λ) be a σ-finite measure space and let $L = L_\lambda^+$ be the collection of non-negative, extended real valued, λ-measurable functions on Y.

THEOREM 3.1. *Suppose that $\|\cdot\|$ is a Banach Function Norm on L, $S : L \to L$, and $0 < \alpha \leqslant 1$. If*

$$\|S(fg)\| \leqslant \|f^\alpha Sg\|, \quad f, g \in L, \tag{3.1}$$

then

$$\sup_{f \in L} \frac{\|Sf\|}{\|f\|^\alpha} \leqslant \inf_{0<g} \operatorname{ess\,sup}_{y \in Y}{}_\lambda \frac{Sg(y)}{g(y)} \|g\|^{1-\alpha} \leqslant \inf_{0<g,\ \|g\| \leqslant 1} \operatorname{ess\,sup}_{y \in Y}{}_\lambda \frac{Sg(y)}{g(y)} \tag{3.2}$$

with equality if S is order continuous.

Proof. First observe that if $f, g \in L$ then $\|f^\alpha g^{1-\alpha}\| \leqslant \|f\|^\alpha \|g\|^{1-\alpha}$: The homogeneity of $\|\cdot\|$ reduces the observation to the case $\|f\| = \|g\| = 1$ where we use the AGM to get

$$\|f^\alpha g^{1-\alpha}\| \leqslant \|\alpha f + (1-\alpha) g\| \leqslant \alpha \|f\| + (1-\alpha)\|g\| = 1$$

as required. With this in hand we address (3.2).

If $f, g \in L$ with g positive then by (3.1)

$$\|Sf\| \leqslant \|(f/g)^\alpha Sg\| \leqslant \operatorname{ess\,sup}_{y\in Y}{}_\lambda \frac{Sg(y)}{g(y)} \|f^\alpha g^{1-\alpha}\| \leqslant \operatorname{ess\,sup}_{y\in Y}{}_\lambda \frac{Sg(y)}{g(y)} \|f\|^\alpha \|g\|^{1-\alpha}.$$

If $\|f\| = 0$ then $\|Sf\| = 0$ so by our convention $\|Sf\|/\|f\|^\alpha$ is zero. Otherwise we divide by $\|f\|^\alpha$, take the supremum over f, and take the infimum over g to get

$$\sup_{f\in L} \frac{\|Sf\|}{\|f\|^\alpha} \leqslant \inf_{0<g\in L} \operatorname{ess\,sup}_{y\in Y}{}_\lambda \frac{Sg(y)}{g(y)} \|g\|^{1-\alpha}.$$

The second inequality in (3.2) is trivial.

If S is order continuous let

$$C = \sup_{f\in L} \frac{\|Sf\|}{\|f\|^\alpha}.$$

We need to consider only the case $C < \infty$. Fix a finite $A > C$ and choose a positive $g_0 \in L$ such that $\|g_0\| \leqslant 1 - A^{-1}C$. Such a g_0 exists because λ is σ-finite and $\|\cdot\|$ is a Banach Function Norm. For $n = 0, 1, \ldots$ define

$$g_{n+1} = g_0 + A^{-1} S g_n.$$

Clearly $g_0 \leqslant g_1$ and if $g_{n-1} \leqslant g_n$ then $Sg_{n-1} \leqslant Sg_n$ so $g_n \leqslant g_{n+1}$. By induction the sequence $g_0, g_1, \ldots$ is non-decreasing. Let g be its pointwise limit and note that $0 < g_0 \leqslant g$. The order continuity of S implies that

$$g = g_0 + A^{-1} S g. \tag{3.3}$$

Now $\|g_0\| \leqslant 1 - A^{-1}C \leqslant 1$ and if $\|g_n\| \leqslant 1$ then

$$\|g_{n+1}\| \leqslant \|g_0\| + A^{-1}\|Sg_n\| \leqslant 1 - A^{-1}C + A^{-1}C\|g_n\|^\alpha \leqslant 1.$$

By induction, $\|g_n\| \leqslant 1$ for all n and the Fatou property of $\|\cdot\|$ yields $\|g\| \leqslant 1$. This and (3.3) imply

$$\operatorname{ess\,sup}_{y\in Y}{}_\lambda \frac{Sg(y)}{g(y)} \leqslant A.$$

Since the argument holds for any $A > C$ we have

$$\inf_{0<g\in L, \|g\|\leqslant 1} \operatorname{ess\,sup}_{y\in Y}{}_\lambda \frac{Sg(y)}{g(y)} \leqslant C.$$

This inequality completes the cycle and ensures equality in (3.2).

As an immediate consequence we have a version of Schur's Lemma for operators satisfying (3.1).

COROLLARY 3.2. *Suppose that* $\|\cdot\|$ *is a Banach Function Norm on* L, $S : L \to L$, $0 < \alpha \leqslant 1$ *and* (3.1) *holds. If there exists a positive* $g \in L$ *satisfying* $Sg \leqslant Cg$ *for some* $C > 0$ *then*

$$\|Sf\| \leqslant \left(C\|g\|^{1-\alpha}\right)\|f\|^{\alpha}, \quad f \in L.$$

It is natural to ask if the supremum and infimum are achieved in (3.2). The answer is yes when S has a positive (formal) eigenvector.

COROLLARY 3.3. *Suppose that* $\|\cdot\|$ *is a Banach Function Norm on* L, $S : L \to L$, $0 < \alpha \leqslant 1$ *and* (3.1) *holds. If there exists a positive* $g \in L$ *satisfying* $\|g\| < \infty$ *and* $Sg = Cg$ *for some* $C > 0$ *then*

$$\sup_{f \in L} \frac{\|Sf\|}{\|f\|^{\alpha}} = C\|g\|^{1-\alpha}.$$

Proof. Since g is positive (3.2) yields

$$C\|g\|^{1-\alpha} = \frac{\|Sg\|}{\|g\|^{\alpha}} \leqslant \sup_{f \in L} \frac{\|Sf\|}{\|f\|^{\alpha}} \leqslant \operatorname{ess\,sup}_{\lambda\, y \in Y} \frac{Sg(y)}{g(y)}\|g\|^{1-\alpha} = C\|g\|^{1-\alpha}$$

provided $g \not\equiv \infty$. If $g \equiv \infty$ then $\|g\| < \infty$ implies that $\|\cdot\| \equiv 0$ and the conclusion is trivially valid.

Our next result shows that when S and $\|\cdot\|$ are well-behaved and $\alpha < 1$ then such an eigenvector always exists. To ensure that the eigenvector we generate below is positive we need an additional assumption: That S does not achieve its norm on a proper ideal. If E is a measurable subset of Y we let

$$L(E) = \chi_E L = \{f \in L : f(y) = 0 \text{ for } y \notin E\}$$

be the ideal of functions supported on E. If E does not have full measure in Y then we say that $L(E)$ is a proper ideal. We will assume that

$$\sup_{f \in L(E)} \frac{\|Sf\|}{\|f\|^{\alpha}} < \sup_{f \in L} \frac{\|Sf\|}{\|f\|^{\alpha}} \quad \text{whenever } \lambda(Y \setminus E) > 0. \tag{3.4}$$

THEOREM 3.4. *Suppose that* $\|\cdot\|$ *is an absolutely continuous Banach Function Norm on* L, $S : L \to L$ *is strongly order continuous,* $0 < \alpha < 1$ *and* (3.1) *holds. If*

$$\sup_{f \in L} \frac{\|Sf\|}{\|f\|^{\alpha}} = C < \infty$$

then there exists a $g \in L$ *such that* $\|g\| = 1$ *and* $Sg = Cg$. *If, in addition,* (3.4) *holds then* g *is positive.*

Proof. If $C = 0$ then $S \equiv 0$ and the theorem holds trivially. Otherwise fix a positive $g_0 \in L$ with $\|g_0\| = 1$. For each positive integer k let $D_k > 1$ be the solution to

$$(1/k) + D_k^{\alpha} = D_k$$

and note that D_k decreases to 1 as k increases to ∞. Set

$$g_0^{(k)} = (1/k)g_0 \quad \text{and} \quad g_{n+1}^{(k)} = g_0^{(k)} + C^{-1}Sg_n^{(k)}, \quad n = 0, 1, \ldots.$$

As in the proof of Theorem 3.1, we find that the sequence $g_0^{(k)}, g_1^{(k)}, \ldots$ is non-decreasing. Let $g^{(k)}$ be its pointwise limit. The order continuity of S implies

$$g^{(k)} = (1/k)g_0 + C^{-1}Sg^{(k)}. \tag{3.5}$$

Now $\|g_0^{(k)}\| = 1/k \leqslant D_k$ and if $\|g_n^{(k)}\| \leqslant D_k$ then

$$\|g_{n+1}^{(k)}\| \leqslant (1/k)\|g_0\| + C^{-1}\|Sg_n^{(k)}\| \leqslant (1/k) + D_k^{\alpha} = D_k.$$

By induction $\|g_n^{(k)}\| \leqslant D_k$ for all n and the Fatou property of $\|\cdot\|$ yields $\|g^{(k)}\| \leqslant D_k$.

Since $Sg^{(k)} \leqslant Cg^{(k)}$, Corollary 3.2 shows that

$$C = \sup_{f \in L} \frac{\|Sf\|}{\|f\|^{\alpha}} \leqslant C\|g^{(k)}\|^{1-\alpha}.$$

It follows that $\|g^{(k)}\| \geqslant 1$ and we have

$$1 \leqslant \|g^{(k)}\| \leqslant D_k. \tag{3.6}$$

Now we are ready to vary k. It is easy to verify that $g^{(1)}, g^{(2)}, \ldots$ is a non-increasing sequence. Let g be its pointwise limit. Since $Sg^{(1)} \leqslant Cg^{(1)} \leqslant Cg_0$ we see that $Sg^{(1)}$ is finite λ-almost everywhere. The strong order continuity of S applied to (3.5) gives

$$g = C^{-1}Sg$$

and the dominated convergence property of $\|\cdot\|$ ([2, Proposition I.3.6]) applied to (3.6) yields

$$1 \leqslant \|g\| \leqslant 1.$$

This completes the proof of the first statement.

To show that g is positive λ-almost everywhere we set $E = \{y \in Y : g(y) > 0\}$ so that $g \in L(E)$. Then

$$\sup_{f \in L} \frac{\|Sf\|}{\|f\|^{\alpha}} = C = \frac{\|Sg\|}{\|g\|^{\alpha}} \leqslant \sup_{f \in L(E)} \frac{\|Sf\|}{\|f\|^{\alpha}}$$

and in view of our hypothesis (3.4) we have $\lambda(E) = 0$ as required.

If we work with ideals in L instead of L itself then we can better understand why g is required to be positive in the infimum of Theorem 3.1. For $g \in L$ we set

$$E_g = \{y \in Y : g(y) > 0\}.$$

The ideal $L(E_g)$ consists of those functions that vanish wherever g does. Note that for any E the ideal $L(E)$ is an order ideal as well as a multiplicative ideal. That is, if $f \leqslant g \in L(E)$ then $f \in L(E)$ and also if $f \in L$ and $g \in L(E)$ then $fg \in L(E)$. We have defined $L(E_g)$ to be the order ideal generated by g rather than the multiplicative

ideal generated by g which may be smaller. One easily checks that $gL \subset L(E_g)$ and it is worth noting that if g takes the value ∞ on a set of positive measure then the inclusion is proper.

If $E \subset Y$ and $S : L \to L$ satisfies (3.1) then $S_E : L(E) \to L(E)$ defined by

$$S_E(f) = \chi_E Sf$$

satisfies $\|S_E f\| = \|Sf\|$ for all $f \in L(E)$ and

$$\|S_E(fg)\| \leqslant \|f^\alpha S_E g\|, \quad f, g \in L(E).$$

It is natural to identify the ideal $L(E)$ with the cone $L_\lambda^+(E)$ of non-negative functions on E and by making this identification we can apply the results of this section to the operator S_E. The outcome of this process is recorded below. Since it includes all the results of this section as special cases, the next theorem also serves as a summary.

Recall that λ is a σ-finite measure on Y and $L = L_\lambda^+(Y)$ is the collection of λ-measurable functions on Y with values in $[0, \infty]$. For $g \in L$, $E_g = \{y \in Y : g(y) > 0\}$ and for $E \subset Y$, $L(E) = \chi_E L$.

THEOREM 3.5. *Suppose that $\|\cdot\|$ is a Banach function norm on L, $S : L \to L$, $0 < \alpha \leqslant 1$, and*

$$\|S(fg)\| \leqslant \|f^\alpha Sg\|, \quad f, g \in L.$$

1. *If E subset$Y$ is λ-measurable then*

$$\sup_{f \in L(E)} \frac{\|Sf\|}{\|f\|^\alpha} \leqslant \inf_{E_g = E} \operatorname{ess\,sup}_{y \in E}{}_\lambda \frac{Sg(y)}{g(y)} \|g\|^{1-\alpha} \leqslant \inf_{E_g = E,\ \|g\| \leqslant 1} \operatorname{ess\,sup}_{y \in E}{}_\lambda \frac{Sg(y)}{g(y)}$$

with equality if S is order continuous.

2. *If $Sg \leqslant Cg$ then*

$$\sup_{f \in L(E_g)} \frac{\|Sf\|}{\|f\|^\alpha} \leqslant C\|g\|^{1-\alpha}.$$

3. *If $\|g\| < \infty$ and $Sg = Cg$ then*

$$\sup_{f \in L(E_g)} \frac{\|Sf\|}{\|f\|^\alpha} = C\|g\|^{1-\alpha}.$$

4. *If $\|\cdot\|$ is absolutely continuous, S is strongly order continuous, $0 < \alpha < 1$, $E \subset Y$ is λ-measurable, and*

$$\sup_{f \in L(E)} \frac{\|Sf\|}{\|f\|^\alpha} = C < \infty$$

then there exists a $g \in L(E)$ such that $\|g\| = 1$ and $Sg = Cg$ on E. If for every $E_1 \subset E$, S satisfies

$$\sup_{f \in L(E_1)} \frac{\|Sf\|}{\|f\|^\alpha} < \sup_{f \in L(E)} \frac{\|Sf\|}{\|f\|^\alpha} \quad \text{whenever } \lambda(E \setminus E_1) > 0 \tag{3.7}$$

then $g > 0$ on E.

4. Back to the Lebesgue Case

The general results of Section 3 include the situation introduced in Section 2. Our first result is an analogue of Theorem 3.5 in this case. With this, the proofs of Theorems 2.2 and 2.3 will be easy.

Recall that L^+_μ and L^+_ν are the collections of functions from the σ-finite measure spaces (X,μ) and (Y,ν), respectively, taking values in $[0,\infty]$. The maps $T_i : L^+_\nu \to L^+_\mu$ and $T_i^* : L^+_\mu \to L^+_\nu$ are formal adjoints for $i = 1,\dots,n$, T is defined by

$$(Tf)^q = \prod_{i=1}^n (T_i f^{r_i})^{q_i/r_i},$$

and $r_i < q = q_1 + \cdots + q_n \leqslant p$ for all i. For $u \in L^+_\mu$ and $v \in L^+_\nu$ with $0 < v < \infty$, the operator S is given by

$$(Sg)^p = \frac{1}{v}\sum_{i=1}^n (q_i/q) g^{r_i} T_i^*(uR_i g)$$

where

$$R_i g(x) = \begin{cases} \infty, & \text{if } q_i < r_i \text{ and } T_i(g^{r_i})(x) = \infty \\ \prod_{j=1}^n T_j(g^{r_j})^{q_j/r_j - \delta_{ij}}, & \text{otherwise.} \end{cases}$$

For $g \in L^+_\nu$ we set

$$B_g = \left(\int_E g^p v\, d\nu\right)^{(1/q)-(1/p)}$$

and note that $B_g = 1$ when $p = q$ even if $g \notin L^p_{v\nu}$.

As a notational convenience of using arithmetic on $[0,\infty]$ we write $f \leqslant \infty g$ to mean that f vanishes wherever g does. Thus $f \leqslant \infty \chi_E$ means that $f = 0$ off the set E and $\infty g = \infty \chi_E$ means that $g = 0$ off E and $g > 0$ on E.

THEOREM 4.1.

1. *If $E \subset Y$ is ν-measurable then*

$$\sup_{f \leqslant \infty \chi_E} \frac{\left(\int_X (Tf)^q u\, d\mu\right)^{1/q}}{\left(\int_E f^p v\, d\nu\right)^{1/p}} \leqslant \inf_{\infty g = \infty \chi_E} B_g \left(\operatorname*{ess\,sup}_{y\in E}{}_\lambda \frac{Sg(y)}{g(y)}\right)^{p/q}$$
$$\leqslant \inf_{\infty g = \infty \chi_E,\ \int_E gv\, d\nu \leqslant 1} \left(\operatorname*{ess\,sup}_{y\in E}{}_\lambda \frac{Sg(y)}{g(y)}\right)^{p/q}$$

with equality if $q_i \geqslant r_i$ for $i = 1,\dots,n$.

2. *If $g \in L^+_\nu$ satisfies $Sg \leqslant Cg$ then*

$$\left(\int_X (Tf)^q u\, d\mu\right)^{1/q} \leqslant C^{p/q} B_g \left(\int_Y f^p v\, d\nu\right)^{1/p}, \quad f \leqslant \infty g. \tag{4.1}$$

3. *If $g \in L^p_{v\nu}$ and $Sg = Cg$ then the constant in (4.1) is best possible.*

4. *If* $q < p$, $q_i \geqslant r_i$ *for* $i = 1, \dots, n$, $E \subset Y$ *and*

$$\left(\int_X (Tf)^q u\, d\mu\right)^{1/q} \leqslant C^{p/q} \left(\int_Y f^p v\, d\nu\right)^{1/p}, \quad f \leqslant \infty \chi_E,$$

then there exists a $g \leqslant \infty \chi_E$ *such that* $B_g = 1$ *and* $Sg = Cg$ *on* E. *If* T *satisfies* (2.3) *then* $\infty g = \infty \chi_E$.

Proof. It is a simple matter to check that if the theorem holds for the indices $p, q, r_1, \dots, r_n, q_1, \dots, q_n$ and m is any positive real number then it also holds for the indices $mp, mq, mr_1, \dots, mr_n, mq_1, \dots, mq_n$. Therefore, we may assume without loss of generality that $p \geqslant 1$. This ensures that $\|\cdot\|_{L^p_{\nu\nu}}$ is a Banach Function Norm. Note that $\|\cdot\|_{L^p_{\nu\nu}}$ has the dominated convergence property and so is absolutely continuous.

The measure $\lambda = \nu\nu$ is σ-finite because ν is σ-finite and $\nu < \infty$ ν-almost everywhere. Take $\|\cdot\| = \|\cdot\|_{L^p_{\nu\nu}}$ and $\alpha = q/p$ and apply Theorem 3.5 to the operator S above. In this special case the condition (3.1) is just (2.8) which was established previously. The conclusions of Theorem 3.5 are readily reformulated to yield Theorem 4.1 by using (2.7) to express the results in terms of T. Only two things remain. To show that S is strongly order continuous when $q_i \geqslant r_i$ for $i = 1, \dots, n$ and to show that if T satisfies (2.3) then S satisfies (3.7). These are established in the next two lemmas.

LEMMA 4.2. *If* $q_i \geqslant r_i$ *for* $i = 1, \dots, n$ *then* S *is strongly order continuous.*

Proof. By Lemma 2.4, T_j is strongly order continuous (SOC) and it is easy to check that

$$g \mapsto T_j(g^{r_j})^{q_j/r_j - \delta_{ij}}$$

is also SOC because the exponent $q_j/r_j - \delta_{ij}$ is non-negative for all i and j. A standard argument shows that sums and products of SOC operators are again SOC. Thus R_i is SOC for each i and to complete the proof it is enough to show that

$$g \mapsto T_i^*(uR_i g)$$

is SOC. Of course, T_i^* has formal adjoint T_i so Lemma 2.4 shows that T_i^* is SOC. If $g_n \uparrow g$ then $uR_i g_n \uparrow uR_i g$ so $T_i^*(uR_i g_n) \uparrow T_i^*(uR_i g)$.

For non-increasing sequences a bit more analysis is required. If $g_n \downarrow g$ and $T_i^*(uR_i g_1) < \infty$ then we let $E = \{x : u(x) R_i g_1(x) < \infty\}$. For any f

$$\infty \chi_{X \setminus E} f = 0 \leqslant uR_i g_1$$

so

$$\infty T_i^*(\chi_{X\setminus E} f) \leqslant T_i^*(uR_i g_1) < \infty.$$

It follows that $T_i^*(\chi_{X\setminus E} f) = 0$ and, since T_i^* is additive, that $T_i^* f = T_i^*(\chi_E f)$.

Now $\chi_E uR_i g_1 < \infty$ and $\chi_E uR_i$ is SOC so $\chi_E uR_i g_n \downarrow \chi_E uR_i g$. Since T_i^* is SOC and $T_i^*(\chi_E uR_i g_1) = T_i^*(uR_i g_1) < \infty$ we have

$$T_i^*(uR_i g_n) = T_i^*(\chi_E uR_i g_n) \downarrow T_i^*(\chi_E uR_i g) = T_i^*(uR_i g).$$

This shows that $g \to T_i^*(uR_i g)$ is SOC and completes the proof.

LEMMA 4.3. *If* $q < p$, $q_i \geqslant r_i$ *for* $i = 1, \dots, n$ *and* T *satisfies* (2.3) *then* S *satisfies* (3.7).

Proof. The assumption $q_i \geqslant r_i$ for $i = 1, \dots, n$ gives T a q-superadditivity property: If f_0 and f_1 have disjoint supports then $(f_0 + f_1)^{r_i} = f_0^{r_i} + f_1^{r_i}$ for all i so

$$\begin{aligned}
(T(f_0 + f_i))^q &= \prod_{i=1}^{n} (T_i(f_0^{r_i} + f_1^{r_i}))^{q_i/r_i} \\
&= \prod_{i=1}^{n} (T_i(f_0^{r_i}) + T_i(f_1^{r_i}))^{q_i/r_i} \\
&\geqslant \prod_{i=1}^{n} ((T_i(f_0^{r_i}))^{q_i/r_i} + (T_i(f_1^{r_i}))^{q_i/r_i}) \\
&\geqslant \prod_{i=1}^{n} (T_i(f_0^{r_i}))^{q_i/r_i} + \prod_{i=1}^{n} (T_i(f_1^{r_i}))^{q_i/r_i} \\
&= (Tf_0)^q + (Tf_1)^q.
\end{aligned}$$

Consequently

$$\|Tf_0\|_q^q + \|Tf_1\|_q^q \leqslant \|T(f_0 + f_1)\|_q^q. \tag{4.2}$$

With this in hand we suppose that T satisfies (2.3), fix $E_1 \subset E$, and suppose that $E_0 = E \setminus E_1$ has positive λ-measure. Define

$$M = \sup_{f \leqslant \infty \chi_E} \frac{\|Tf\|_q}{\|f\|_p}, \quad M_0 = \sup_{f \leqslant \infty \chi_{E_0}} \frac{\|Tf\|_q}{\|f\|_p}, \quad M_1 = \sup_{f \leqslant \infty \chi_{E_1}} \frac{\|Tf\|_q}{\|f\|_p}.$$

In view of (2.7) our object is to show that $M_1 < M$. Since $\lambda = v\nu$ and E_0 has positive λ-measure, it also has positive ν-measure so (2.3) shows that $M_0 > 0$. To complete the proof it will suffice to establish

$$M_0^s + M_1^s \leqslant M^s \tag{4.3}$$

where $s = pq/(p-q)$. If $M_1 = 0$ then (4.3) holds trivially. If $M_1 > 0$ and m_0 and m_1 satisfy $0 < m_0 < M_0$ and $0 < m_1 < M_1$ then there exist functions $f_0 \leqslant \infty \chi_{E_0}$ and $f_1 \leqslant \infty \chi_{E_1}$ such that $m_0 \|f_0\|_p \leqslant \|Tf_0\|_q$ and $m_1 \|f_1\|_p \leqslant \|Tf_1\|_q$. The homogeneity of T ensures that we can scale f_0 and f_1 so that they also satisfy $\|f_0\|_p^p = m_0^s$ and $\|f_1\|_p^p = m_1^s$. Using (4.2) and the definition of M we get

$$\begin{aligned}
m_0^s + m_1^s &= m_0^q \|f_0\|_p^q + m_1^q \|f_1\|_p^q \\
&\leqslant \|Tf_0\|_q^q + \|Tf_1\|_q^q \\
&\leqslant \|T(f_0 + f_1)\|_q^q \\
&\leqslant M^q \|f_0 + f_1\|_p^q \\
&= M^q (\|f_0\|_p^p + \|f_1\|_p^p)^{q/p} \\
&= M^q (m_0^s + m_1^s)^{q/p}.
\end{aligned}$$

Thus, $(m_0^s + m_1^s)^{(1-q/p)} \leqslant M^q$ whenever $m_0 < M_0$ and $m_1 < M_1$ and so we have (4.3). This completes the proof.

Proof of Theorem 2.2. By replacing v in (1.2) by $C^p v$ we can reduce Theorem 2.2 to the case $C = 1$. Let $E = Y$ and take g to be the function satisfying $Sg = g$, $B_g = 1$ and $g > 0$ whose existence is guaranteed by Theorem 4.1(4). In view of (2.7) we have

$$\int_X (Tg)^q u\, d\nu = \int_Y (Sg)^p v\, d\nu = \int_Y g^p v\, d\nu = 1$$

and because $v > 0$ it is clear that $g < \infty$ ν-almost everywhere. Therefore, $v_g = g^{-p} v (Sg)^p = v$ and it follows that $C_g = B_g = 1$.

Proof of Theorem 2.3. If $S \equiv 0$ then by (2.7) $uT \equiv 0$ as well and the theorem is trivial. Otherwise, with $E = Y$ in Theorem 4.1 (1), we have

$$C = \inf_{g>0,\ \int_Y gv\, d\nu \leqslant 1} \left(\operatorname*{ess\,sup}_{y \in Y}{}_{\lambda} \frac{Sg(y)}{g(y)} \right)^{p/q}.$$

Since $C < A < \infty$ there exists $g_0 > 0$ with $\int_Y g_0 v\, d\nu \leqslant 1$ such that $Sg_0 < A^{q/p} g_0$ ν-almost everywhere. With $g = A^{-1} g_0$ this becomes $Sg < Ag$ and we have $v_g = g^{-p} v (Sg)^p \leqslant A^p v$. Now

$$C_g = \left(\int_Y g^p v_g\, d\nu \right)^{(1/q)-(1/p)} \leqslant \left(\int_Y A^{-p} g_0^p A^p v\, d\nu \right)^{(1/q)-(1/p)} \leqslant 1.$$

The condition (2.3) of Theorem 2.2 and the restriction $0 < v < \infty$ of Theorems 2.2 and 2.3 do not reduce the generality of these results. This is because an inequality of the form (1.1) which is not trivially false is equivalent to another of the same form for which (2.3) holds and $0 < v < \infty$. This is presented in the next theorem.

THEOREM 4.4. *Let T be an operator of the form* (1.1). *Suppose $0 < C < \infty$ and let $Y_0 = \{y \in Y : v(y) = 0\}$. If there exists an $f_0 \in L^p_{v\nu}$ such that $uTf_0 \neq uT(f_0 \chi_{Y \setminus Y_0})$ on a set of positive*

μ-measure then

$$\left(\int_X (Tf)^q u\, d\mu \right)^{1/q} \leqslant C \left(\int_Y f^p v\, d\nu \right)^{1/p}, \quad f \in L^+_\nu(Y), \tag{4.4}$$

fails. Otherwise, (4.4) *holds if and only if*

$$\left(\int_{X_1} (Tf)^q u\, d\mu \right)^{1/q} \leqslant C \left(\int_{Y_1} f^p v\, d\nu \right)^{1/p}, \quad f \in L^+_\nu(Y_1), \tag{4.5}$$

where

$$X_1 = \{x \in X : u(x) T(\chi_{Y \setminus Y_0})(x) > 0\}$$

and

$$Y_1 = \{y \in Y : 0 < v(y) < \infty; \textstyle\sum_{i=1}^n T_i^*(\chi_{X_1})(y) > 0\}.$$

Moreover, if (4.5) *holds,* $E \subset Y_1$ *and* $uTf = uT(f\chi_E)$ *for all* $f \in L^+_v(Y_1)$ *then* $v(Y_1 \setminus E) = 0$.

Proof. Suppose $f \in L^p_{vv}$ satisfies $uTf \neq uT(f\chi_{Y\setminus Y_0})$ on a set of positive μ-measure. Then for some i both $T_i(f^{r_i}) > T_i(f^{r_i}\chi_{Y\setminus Y_0})$ and $T_j(f^{r_j}) > 0$ for $i \neq j$ hold on a set of positive $u\mu$-measure. The $[0,\infty]$-linearity of T_i yields $T_i(\infty f^{r_i}\chi_{Y_0}) = \infty$ on this set. Therefore

$$\int_X T(\infty f\chi_{Y_0} + f)^q u\,d\mu \geqslant \int_X T_i(\infty f^{r_i}\chi_{Y_0})^{q_i/r_i} \prod_{j\neq i} T_j(f^{r_j})^{q_j/r_j} u\,d\mu = \infty.$$

However, v vanishes on Y_0 so

$$\int_Y (\infty f\chi_{Y_0} + f)^p v\,dv = \int_Y f^p v\,dv < \infty.$$

We conclude that (4.4) fails for the function $\infty f\chi_{Y_0} + f$. This proves the first statement of the theorem.

If (4.4) holds and $f \in L^+_v(Y_1)$ then

$$\left(\int_{X_1} (Tf)^q u\,d\mu\right)^{1/q} \leqslant \left(\int_X (Tf)^q u\,d\mu\right)^{1/q} \leqslant C\left(\int_Y f^p v\,dv\right)^{1/p} = C\left(\int_{Y_1} f^p v\,dv\right)^{1/p}$$

so (4.5) holds. On the other hand, suppose (4.5) holds and fix $f \in L^+_v(Y)$. If the right hand side of (4.4) is infinite there is nothing to prove so we may assume that $f \in L^p_{vv}$ and, in particular, that $f = 0$ vv-almost everywhere on $\{y \in Y : v(y) = \infty\}$. By hypothesis, we also have $uTf \equiv uT(f\chi_{Y\setminus Y_0})$ so

$$uTf \equiv uT(f\chi_{\{y\in Y: 0<v(y)<\infty\}}). \tag{4.6}$$

For each i, $T_i^*\chi_{X_1} = \chi_{Y_2}T_i^*\chi_{X_1}$ where

$$Y_2 = \{y \in Y : \textstyle\sum_{i=1}^n T_i^*(\chi_{X_1})(y) > 0\}.$$

Therefore

$$\int_{X_1} T_i(f^{r_i})\,d\mu = \int_Y f^{r_i}T_i^*\chi_{X_1}\,dv = \int_Y f^{r_i}\chi_{Y_2}T_i^*\chi_{X_1}\,dv = \int_{X_1} T_i(f^{r_i}\chi_{Y_2})\,d\mu$$

and since $T_i(f^{r_i}\chi_{Y_2}) \leqslant T_i(f^{r_i})$ it follows that $T_i(f^{r_i}) = T_i(f^{r_i}\chi_{Y_2})$ μ-almost everywhere on X_1. Hence $Tf = T(f\chi_{Y_2})$ μ-almost everywhere on X_1. Combining this with (4.6) yields

$$uTf \equiv uT(f\chi_{Y_1}).$$

With this we can prove (4.4). Observe that

$$uTf \equiv uT(f\chi_{Y\setminus Y_0}) \leqslant \infty uT(\chi_{Y\setminus Y_0})$$

so $uTf = 0$ off X_1. Now

$$\begin{aligned}\left(\int_X (Tf)^q u\,d\mu\right)^{1/q} &= \left(\int_{X_1} (Tf)^q u\,d\mu\right)^{1/q} = \left(\int_{X_1} T(f\chi_{Y_1})^q u\,d\mu\right)^{1/q}\\ &\leqslant C\left(\int_{Y_1} f^p v\,dv\right)^{1/p} \leqslant C\left(\int_Y f^p v\,dv\right)^{1/p}.\end{aligned}$$

This is (4.4).

Finally, suppose that (4.5) holds and $E \subset Y_1$ satisfies $uTf = uT(f\chi_E)$ on X_1 for all $f \in L_\nu^+(Y_1)$. Since $\nu\nu$ is a σ-finite measure on $Y \setminus Y_0$ we can choose a positive function $f \in L_{\nu\nu}^p(Y_1)$. Then $T\chi_{Y\setminus Y_0} = T\chi_{Y_1} \leqslant \infty Tf$ so $Tf > 0$ on X_1. Also, using (4.5) we see that $\int_{X_1}(Tf)^q u\,d\mu < \infty$. We have

$$\infty \int_{X_1} T_i(\chi_{Y_1\setminus E})^{q_i/r_i} \prod_{j\neq i} T_j(f^{r_j})^{q_j/r_j} u\,d\mu$$
$$\leqslant \int_{X_1} T(\infty\chi_{Y_1\setminus E} + f)^q u\,d\mu = \int_{X_1}(Tf)^q u\,d\mu < \infty.$$

Since Tf is positive on X_1 this implies $T_i(\chi_{Y_1\setminus E}) = 0$ and hence $T_i\chi_{Y_1} = T_i(\chi_E)$ for each i. Now

$$\int_{Y_1} T_i^*\chi_{X_1}\,d\nu = \int_{X_1} T_i\chi_{Y_1}\,d\mu = \int_{X_1} T_i\chi_E\,d\mu = \int_E T_i^*\chi_{X_1}\,d\nu.$$

Thus $T_i^*(\chi_{X_1}) = 0$ ν-almost everywhere off E and so the definition of Y_1 yields $\nu(Y_1 \setminus E) = 0$.

5. Examples and Applications

Our first example illustrates the simplicity of generating inequalities using Theorem 2.1 by exhibiting a weighted Hardy inequality with best constant.

EXAMPLE 5.1. If $1 < q \leqslant p < \infty$ and $\alpha > 0$ then

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^q e^{-\alpha x} x\,dx\right)^{1/q} \leqslant \alpha^{(1/p)-(2/q)} \left(\int_0^\infty f(y)^p e^{-\alpha y}\,dy\right)^{1/p},$$

for all $f \geqslant 0$. The constant is best possible.

Proof. Let $n = 1$, $T_1 f(x) = \frac{1}{x}\int_0^x f$, $u(x) = xe^{-\alpha x}$, $r_1 = 1$, $q_1 = q$, and apply Theorem 2.1 with $g(y) \equiv 1$. Using the formulas (2.1), we readily calculate

$$v_g = e^{-\alpha y}/\alpha \quad \text{and} \quad C_g = \alpha^{(2/p)-(2/q)}$$

and then (2.2) simplifies to the above inequality.

Next we show that the Stieltjes transformation has norm 1 as a map from L^2 to a certain weighted L^2.

EXAMPLE 5.2.

$$\int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y}\,dy\right)^2 \frac{2}{\log(x)^2 + \pi^2}\,dx \leqslant \int_0^\infty f(y)^2\,dy, \quad f \geqslant 0.$$

The constant 1 is best possible.

Proof. Let $n = 1$, $T_1 f(x) = \int_0^\infty f(y)/(x+y)\,dy$, $u(x) = 2/(\log(x)^2 + \pi^2)$, $r_1 = 1$, $q_1 = 2$, $p = q = 2$ and apply Theorem 2.1 with $g(y) = \log(y)/(y-1)$. Since $p = q$ we have $C_g = 1$ and checking that $v_g \equiv 1$ in (2.1) completes the proof.

In the next example we look at a weighted Hardy inequality with a nonhomogeneous boundary condition. In the next three examples, $AC(I)$ denotes the collection of absolutely continuous functions on the interval I.

EXAMPLE 5.3. Suppose u and w are weights with w positive and set $U(y) = \int_y^1 u$. Then

$$\int_0^1 |h|u \leqslant C \int_0^1 |h'|^2/w, \quad h \in AC[0,1], h(0) = 0, h(1) = 1,$$

with $C = \frac{1}{2}\int_0^1 Uw + \frac{1}{2}\left(\int_0^1 U^2 w \int_0^1 w\right)^{1/2}$. If the constant C is finite then it is best possible.

Proof. We make the substitution $h(x) = \int_0^x f / \int_0^1 f$ to see that the desired inequality is equivalent to

$$\int_0^1 \left(\int_0^x f\right)\left(\int_0^1 f\right) u(x)\,dx \leqslant C \int_0^1 f^2/w, \quad f \geqslant 0.$$

For this we apply Theorem 2.1 with $n = 2$, $T_1 f(x) = \int_0^x f$, $T_2 f(x) = \int_0^1 f$, $r_1 = r_2 = q_1 = q_2 = 1$, $p = q = 2$ and, of course, $u = u$. Set $g = Uw + bw$ where $b^2 = \int_0^1 U^2 w / \int_0^1 w$. Since $p = q$ we have $C_g = 1$ and a calculation yields $wv_g = C$.

If C is finite it is easy to check that $\int_0^1 (T_1 g)(T_2 g)u < \infty$ to conclude that the constant is best possible.

Next is an unweighted variant of Opial's inequality with nonhomogeneous boundary conditions.

EXAMPLE 5.4. Suppose $p \geqslant 3$. Then

$$\int_0^1 |h||1-h||h'| \leqslant \frac{1}{6}\int_0^1 |h'|^p, \quad h \in AC[0,1],\ h(0) = 0,\ h(1) = 1.$$

The constant is best possible.

Proof. The substitution $h(x) = \int_0^x f / \int_0^1 f$ shows that the desired inequality is equivalent to

$$\int_0^1 \left(\int_0^x f\right)\left(\int_x^1 f\right) f(x) \left(\int_0^1 f\right)^{p-3} dx \leqslant \frac{1}{6}\int_0^1 f(y)^p\,dy, \quad f \geqslant 0.$$

To generate this from Theorem 2.1 we put $n = 4$, $T_1 f(x) = \int_0^x f$, $T_2 f(x) = \int_x^1 f$, $T_3 f(x) = f(x)$, $u(x) \equiv 1$, and $T_4 f(x) = \int_0^1 f$. Take the r_i's and q_i's all to be 1

except for $q_4 = p - 3$, and set $g(y) \equiv 1$. Straightforward calculations yield $v_g = 1/6$ and $C_g = 1$ to prove the result. (If $p = 3$ the operator T_4 does not appear so we are applying Theorem 2.1 with $n = 3$. The calculations are the same.)

In the previous examples the functions g satisfied $\int (Tg)^q u < \infty$ so they were extremals and the constants were automatically best possible. Next we give an example where g is not an extremal. As we see in the proof, a sequence of functions tending to g provides our substitute for an extremal. The following weighted Opial-type inequality appeared without best constant in [15].

EXAMPLE 5.5. Let $u(x) = 1/x$. Then

$$\int_0^\infty |hh'|u \leqslant 2\int_0^\infty |h'|^2, \quad h' \in AC[0,\infty), h(0) = 0.$$

The constant is best possible.

Proof. The desired inequality is equivalent to

$$\int_0^\infty \left(\int_0^x f(y)\,dy\right) f(x)\,\frac{dx}{x} \leqslant 2\int_0^\infty f(y)^2\,dy, \quad f \geqslant 0.$$

Let $n = 2$, $T_1 f(x) = \int_0^x f(y)\,dy$, $T_2 f(x) = f(x)$, $u(x) = 1/x$, $r_1 = r_2 = q_1 = q_2 = 1$, $p = q = 2$ and set $g(y) = y^{-1/2}$. Since $p = q$ we have $C_g = 1$ and integration yields $v_g = 2$. The inequality therefore holds by Theorem 2.1 but without a guarantee that the best constant is 2. To see that the inequality does not hold for any constant less than 2 we set $f_k(y) = y^{-1/2}\chi_{(1/k,k)}(y)$ for $k = 1, 2, \ldots$. The best constant can be no less than

$$\lim_{k\to\infty} \frac{\int_0^\infty \left(\int_0^x f_k(y)\,dy\right) f_k(x)\,dx/x}{\int_0^\infty f_k(y)^2\,dy} = 2.$$

We can generate inequalities with given weights u and v provided we can solve the equation $v_g = Av$ for some function g and constant A. Often this can be reduced to a differential equation and solved explicitly. Once the solution is found there is no need to exhibit the solution procedure, we merely apply Theorem 2.1 to the appropriate function g. The next example is of this sort. Both the weights u and v are constant in this unweighted Hardy type inequality. (Although the example below is not strictly a Hardy type inequality in the sense of [12], we refer to it as such because of the Hardy averaging operators it involves.)

EXAMPLE 5.6.

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)\left(\frac{1}{x}\int_0^x f^2\right) dx \leqslant \frac{9}{2}\int_0^\infty f^3, \quad f \geqslant 0.$$

The constant is best possible.

Proof. Apply Theorem 2.1 with both $T_1 f(x) = T_2 f(x) = \frac{1}{x}\int_0^x f$ but let $q_1 = r_1 = 1$, $q_2 = r_2 = 2$ and $p = q = 3$. Let $g(y) = y^{-1/3}$ and check that $C_g = 1$ and

$v_g(y) \equiv 9/2$. As in the previous example, some extra argument is needed to show that the constant is best possible. We omit the details.

We conclude with an application of Theorem 2.2 viewed as a structure theorem. Although the result is well-known the method is new. See [17] or the references in [12].

THEOREM 5.7. *If* $1 < q < p$ *and*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q u(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p v\right)^{1/p}, \quad f \geqslant 0,$$

for some finite C *then*

$$\left(\int_0^\infty \left(\int_t^\infty u\right)^{r/p} \left(\int_0^t v^{1-p'}\right)^{r/p'} u(t)\,dt\right)^{1/r} \leqslant C.$$

Here $1/r = 1/q - 1/p$.

Proof. By Theorem 2.2 there exists a positive function g such that $v = v_g$ and $C_g \leqslant C < \infty$. Using the definition of v_g and reducing the region of integration we have

$$\begin{aligned}\int_0^t v^{1-p'} &= \int_0^t \left(\int_y^\infty u(x)\left(\int_0^x g\right)^{q-1} dx\right)^{1-p'} g(y)\,dy \\ &\leqslant \int_0^t \left(\int_t^\infty u(x)\left(\int_0^t g\right)^{q-1} dx\right)^{1-p'} g(y)\,dy \\ &= \left(\int_0^t g\right)^{q-p'q+p'} \left(\int_t^\infty u\right)^{1-p'}.\end{aligned}$$

This estimate and the hypothesis of the theorem yields

$$\begin{aligned}&\left(\int_0^\infty \left(\int_t^\infty u\right)^{r/p} \left(\int_0^t v^{1-p'}\right)^{r/p'} u(t)\,dt\right)^{1/r} \\ &\qquad \leqslant \left(\int_0^\infty \left(\int_0^t g\right)^q u(t)\,dt\right)^{1/r} \leqslant C^{q/r}\left(\int_0^\infty g^p v\right)^{q/(pr)} = C^{q/r} C_g^{q/p} \leqslant C.\end{aligned}$$

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