Mathematical
Inequalities
& Applications
Volume 8, Number 4 (2005), 559–564

GEOMETRIC MEANS AND HADAMARD PRODUCTS

BAO QI FENG AND ANDREW TONGE

(*communicated by T. Ando*)

Abstract. Ando [1] proved that for m commuting positive definite matrices, the m-fold Hadamard product of their geometric mean is bounded above by their Hadamard product. We obtain a natural extension to the non-commutative case.

In what follows, if M is a positive definite $n \times n$ matrix and $\alpha \in \mathbb{R}$, then M^α will denote its unique positive α^{th} *power*. In 1979, Ando [1] developed a robust definition of the geometric mean of two positive definite $n \times n$ matrices M_1 and M_2 that had been introduced by Pusz and Woronowicz [5]:

$$G(M_1, M_2) := M_2^{1/2} \left(M_2^{-1/2} M_1 M_2^{-1/2} \right)^{1/2} M_2^{1/2}.$$

The geometric mean is symmetric in M_1, M_2, monotone in each variable and satisfies the arithmetic-geometric mean inequality

$$G(M_1, M_2) \leqslant A(M_1, M_2) := \frac{1}{2}(M_1 + M_2).$$

Given $0 < \alpha < 1$, the geometric mean is generalized to the α-mean $G^{(\alpha)}(M_1, M_2)$:

$$G^{(\alpha)}(M_1, M_2) := M_2^{1/2} (M_2^{-1/2} M_1 M_2^{-1/2})^\alpha M_2^{1/2}.$$

The usual geometric mean is just $G^{(1/2)}(M_1, M_2)$. In stead of the arithmetic-geometric mean inequality the Young inequality holds for α-mean:

$$G^{(\alpha)}(M_1, M_2) \leqslant A^{(\alpha)}(M_1, M_2) := \alpha M_1 + (1 - \alpha) M_2.$$

Ando [1] also researched the interaction of this geometric mean with the Hadamard (or Schur) product. If $M = (m_{ij})$, $N = (n_{ij})$ are matrices of the same size, their *Hadamard product* $M \circ N$ is the matrix of entry-wise products:

$$M \circ N := (m_{ij} n_{ij}).$$

Mathematics subject classification (2000): 15A24, 15A45, 15A48, 15A69, 47A63, 47A64.
Key words and phrases: arithmetic mean, geometric mean, α-power mean, Hadamard product, tensor product.

© ELEMENT, Zagreb
Paper MIA-08-51

Ando [1, Theorem 13] proved that for positive definite $n \times n$ matrices M, N we have

$$G(M,N) \circ G(M,N) \leqslant M \circ N, \tag{A_1}$$

which, in the commutative case, reduces to

$$(MN)^{\frac{1}{2}} \circ (MN)^{\frac{1}{2}} \leqslant M \circ N.$$

Using a different method, he succeeded in generalizing his inequality to the case of several commuting positive definite $n \times n$ matrices [1, Theorem 12]:

$$\prod_{1}^{m} \circ \left(\prod_{i=1}^{m} M_i \right)^{\frac{1}{m}} \leqslant \prod_{i=1}^{m} \circ M_i. \tag{A_2}$$

However, at the time, the notion of geometric mean for several non-commuting matrices was not available; therefore, inequality (A_1) was not developed beyond the case of two matrices.

In 1994, M. Sagae and K. Tanabe [6] successfully developed an approach to the geometric mean — in fact, the weighted geometric mean — of several positive definite matrices. The main purpose of this paper is to blend the ideas of Ando and the concepts of Sagae and Tanabe to extend inequality (A_1) to the case of several positive definite $n \times n$ matrices. As a result, we are able to provide a simpler proof of inequality (A_2).

The definition $G_w(M_1, M_2, \ldots, M_m)$ of the geometric mean of an m-tuple of positive definite matrices $(M_1, M_2, \ldots, M_m)$ $(m > 2)$ was given by Sagae and Tanabe in 1994.

When $m = 2$, define

$$G_w(M_1, M_2) := G^{(w_1)}(M_1, M_2).$$

Suppose that the definition for the case $m - 1$ has been well established. Now given an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1, define

$$G_w(M_1, M_2, \ldots, M_m) := G^{\left(\sum_{j=1}^{m-1} w_j\right)} \left(G_{\tilde{w}}(M_1, M_2, \ldots, M_{m-1}), M_m\right)$$

where

$$\tilde{w} = \left(w_1 / \sum_{j=1}^{m-1} w_j, w_2 / \sum_{j=1}^{m-1} w_j, \ldots, w_{m-1} / \sum_{j=1}^{m-1} w_j \right).$$

In order to work effectively with the general geometric mean of Sagae and Tanabe, it is convenient to introduce the $(\alpha_1, \ldots, \alpha_k)$-*power mean* for $(k+1)$-tuple positive definite matrices. Suppose that M_i $(i \geqslant 1)$ are positive definite $n \times n$ matrices and α_i $(i \geqslant 1)$ are real scalars. Starting with the two matrix basis, we can continue recursively to define

$$G^{(\alpha_1, \ldots, \alpha_k)}(M_1, \ldots, M_{k+1}) = G^{(\alpha_k)}(G^{(\alpha_1, \ldots, \alpha_{k-1})}(M_1, \ldots, M_k), M_{k+1})$$

for $k \geqslant 2$.

When $\alpha_i = 1 - \left(w_{i+1} \;/\; \sum_{j=1}^{i+1} w_j\right)$ for $i = 1, \ldots, m-1$, we have

$$G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_1, \ldots, M_m) = G_w(M_1, \ldots, M_m).$$

This general definition of geometric mean has many good properties, but in the case of equal weights it is not symmetric for $k > 2$ (see [3]). For us, it will be significant that the weighted geometric mean satisfies an arithmetic-geometric mean inequality.

THEOREM ([6], Theorem 1) *Let w be an m-tuple of positive numbers $(w_1, w_2, ..., w_m)$ summing to 1 and let $M_i (1 \leqslant i \leqslant m)$ be positive definite $n \times n$ matrices. Then*

$$G_w(M_1, \ldots, M_k) \leqslant A_w(M_1, \ldots, M_k) := w_1 M_1 + \cdots + w_k M_k.$$

The inequality is strict unless $M_1 = \cdots = M_k$.

From inequality (A_1) it is natural to conjecture that when M_i $(i = 1, 2, 3)$ are positive definite matrices (possibly non-commuting), the inequality

$$G_w(M_1, M_2, M_3) \circ G_w(M_1, M_2, M_3) \circ G_w(M_1, M_2, M_3) \leqslant M_1 \circ M_2 \circ M_3$$

holds true. The answer is positive, and in fact the order of the matrices is unimportant. This is to some extent surprising, since Ando's proof of inequality (A_1) depends on the symmetry of the geometric mean of two matrices. The main result in this paper is:

THEOREM 1. *Let w be an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1 and let M_i $(1 \leqslant i \leqslant m)$ be positive definite $n \times n$ matrices. If $(i_1, i_2, \ldots, i_m)$, $(j_1, j_2, \ldots, j_m)$, $\ldots$, $(k_1, k_2, \ldots, k_m)$ are arbitrary m-permutations of $\{1, 2, \ldots, m\}$, then*

$$G_w(M_{i_1}, \ldots, M_{i_m}) \circ G_w(M_{j_1}, \ldots, M_{j_m}) \circ \cdots \circ G_w(M_{k_1}, \ldots, M_{k_m}) \leqslant \prod_{i=1}^{m} \circ M_i.$$

In order to obtain the proof of Theorem 1, we need to develop some properties of tensor products of matrices. If $M = (m_{ij})$ is an $k \times l$ matrix and $N = (n_{ij})$ is an $s \times t$ matrix, then their *tensor (*or *Kronecker) product* is the $ks \times lt$ matrix

$$M \otimes N := \begin{bmatrix} m_{11}N & \cdots & m_{1l}N \\ \vdots & \ldots & \vdots \\ m_{k1}N & \cdots & m_{kl}N \end{bmatrix}.$$

The tensor product of finitely many matrices can be defined by induction.

The basic properties of the tensor product can be found in [2, p. 15] and [1, p. 224]. We need two more properties that we were unable to find in the literature.

PROPOSITION 1. *(i) Let M_i $(1 \leqslant i \leqslant k)$ be $m \times m$ matrices and let N_i $(1 \leqslant i \leqslant k)$ be $n \times n$ matrices. Then*

$$\prod_{i=1}^{k} (M_i \otimes N_i) = \left(\prod_{i=1}^{k} M_i\right) \otimes \left(\prod_{i=1}^{k} N_i\right).$$

(ii) *Let* M_i *be positive definite* $n_i \times n_i$ *matrices* $(1 \leqslant i \leqslant k)$. *Then, for any real number* α,

$$\left(\prod_{i=1}^{k} \otimes M_i\right)^{\alpha} = \prod_{i=1}^{k} \otimes M_i^{\alpha}.$$

Proof. (i) This follows easily by induction from the observation that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

which can be verified immediately from the definition of the tensor product of matrices.

(ii) Again, this follows by induction from the case of the product of two matrices. Since M_1 and M_2 are positive definite, we can write (see [4, Theorem 2.5.4]) $M_1 = U_1^* S U_1$ and $M_2 = U_2^* T U_2$, using appropriate unitary matrices U_1 and U_2, and positive definite diagonal matrices S and T. By (i),

$$M_1 \otimes M_2 = (U_1^* \otimes U_2^*)(S \otimes T)(U_1 \otimes U_2) = (U_1 \otimes U_2)^*(S \otimes T)(U_1 \otimes U_2)$$

and, since $U_1 \otimes U_2$ is unitary, the functional calculus (see [1, p. 212]) allows us to write

$$\begin{aligned}(M_1 \otimes M_2)^{\alpha} &= (U_1 \otimes U_2)^*(S \otimes T)^{\alpha}(U_1 \otimes U_2)\\ &= (U_1^* \otimes U_2^*)(S^{\alpha} \otimes T^{\alpha})(U_1 \otimes U_2) = (U_1^* S U_1)^{\alpha} \otimes (U_2^* T U_2)^{\alpha} = M_1^{\alpha} \otimes M_2^{\alpha}. \quad \square\end{aligned}$$

In [1], Ando pointed out a fundamental commutativity relation between the geometric mean and tensor product of two positive definite matrices, namely

$$G(M_1 \otimes M_2, N_1 \otimes N_2) = G(M_1, N_1) \otimes G(M_2, N_2). \tag{1}$$

In fact, this can readily be extended to the α*-power mean*. The analog of identity (1) is

$$G^{(\alpha)}(M_1 \otimes M_2, N_1 \otimes N_2) = G^{(\alpha)}(M_1, N_1) \otimes G^{(\alpha)}(M_2, N_2), \tag{2}$$

and this follows easily from Proposition 1.

A multi-stage induction argument leads to a simple, but powerful extension of identity (2):

PROPOSITION 2. *Let* M_{ij} $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant k)$ *be positive definite* $n \times n$ *matrices, and let* w *be an* m*-tuple of positive numbers* $(w_1, w_2, \ldots, w_m)$ *summing to* 1. *Then*

$$G_w\left(\prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj}\right) = \prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}).$$

Proof. With the help of Proposition 1, a preliminary induction shows that

$$G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_1 \otimes N_1, \ldots, M_m \otimes N_m) = G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_1, \ldots, M_m) \otimes G^{(\alpha_1,\ldots,\alpha_{m-1})}(N_1, \ldots, N_m),$$

and then a second induction gives

$$G^{(\alpha_1,\ldots,\alpha_{m-1})}\left(\prod_{j=1}^{k}\otimes M_{1j},\ldots,\prod_{j=1}^{k}\otimes M_{mj}\right)=\prod_{j=1}^{k}\otimes G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_{1j},\ldots,M_{mj}).$$

Setting $\alpha_i = 1-\left(w_{i+1}\ /\ \sum_{j=1}^{i+1} w_j\right)$ for $i=1,\ldots,m-1$ yields the statement of Proposition 2. We omit the simple details. $\square$

In many situations, properties of tensor products transfer to Hadamard products. This is thanks to an important connection between the two products (e.g. see [1, Lemma 4]): there is a positive linear map Φ_k from n^k-dimensional space to n-dimensional space of matrices such that, for all $n\times n$ matrices M_i $(1\leqslant i\leqslant k)$,

$$\Phi_k\left(\prod_{i=1}^{k}\otimes M_i\right)=\prod_{i=1}^{k}\circ M_i. \tag{3}$$

With this, we can quickly identify the key to our proof of Theorem 1:

PROPOSITION 3. *Let M_{ij} $(1\leqslant i\leqslant m,\ 1\leqslant j\leqslant k)$ be positive definite $n\times n$ matrices, and let w be an m-tuple of positive numbers $(w_1,w_2,\ldots,w_m)$ summing to* 1. *Then*

$$\prod_{j=1}^{k}\otimes\ G_w(M_{1j},\ldots,M_{mj})\leqslant\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\otimes M_{ij}$$

and

$$\prod_{j=1}^{k}\circ\ G_w(M_{1j},\ldots,M_{mj})\leqslant\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\circ M_{ij}.$$

Proof. By Sagae and Tanabe's arithmetic-geometric mean inequality and Proposition 2,

$$\prod_{j=1}^{k}\otimes\ G_w(M_{1j},\ldots,M_{mj})=G_w\left(\prod_{j=1}^{k}\otimes\ M_{1j},\ldots,\prod_{j=1}^{k}\otimes\ M_{mj}\right)\leqslant\sum_{i=1}^{m}w_i\prod_{j=1}^{k}\otimes\ M_{ij}.$$

The Hadamard product inequality follows by (3) from the tensor product inequality. $\square$

Proof of Theorem 1. In Proposition 3, let

$$\begin{aligned}(M_{11},\ldots,M_{m1})&=(M_{i_1},\ldots,M_{i_m}),\\(M_{12},\ldots,M_{m2})&=(M_{j_1},\ldots,M_{j_m}),\\&\vdots\\(M_{1m},M_{2m},\ldots,M_{mm})&=(M_{k_1},\ldots,M_{k_m}).\end{aligned}$$

Then, since the Hadamard product is commutative,

$$\begin{aligned} G_w(M_{i_1},\dots,M_{i_m}) \circ \cdots \circ G_w(M_{k_1},\dots,M_{k_m}) &= \prod_{j=1}^{m} \circ\, G_w(M_{1j},\dots,M_{mj}) \\ &\leqslant \sum_{i=1}^{m} w_i \prod_{j=1}^{m} \circ M_{ij} \\ &= \sum_{i=1}^{m} w_i \prod_{j=1}^{m} \circ M_j = \prod_{i=1}^{m} \circ M_i. \quad \square \end{aligned}$$

It is clear that inequality (A_2) is an immediate corollary of Theorem 1. Let $(i_1, i_2, \dots, i_m)$, $(j_1, j_2, \dots, j_m)$, $\dots$, $(k_1, k_2, \dots, k_m)$ be arbitrary m-permutations of $\{1, 2, \dots, m\}$. If M_i $(1 \leqslant i \leqslant m)$ commute and $w_1 = w_2 = \cdots = w_m = 1/m$, then

$$\begin{aligned} G_w(M_{i_1},\dots,M_{i_m}) &= G_w(M_{j_1},\dots,M_{j_m}) \\ &= \cdots \\ &= G_w(M_{k_1},\dots,M_{k_m}) = \left(\prod_{i=1}^{m} M_i\right)^{\frac{1}{m}}. \end{aligned}$$

Acknowledgements We thank the referee for his helpful suggestions.

REFERENCES

[1] T. ANDO, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl., **26**, (1979), 203–241, MR **80f**: 15023.

[2] R. BHATIA, *Matrix analysis*, Springer–Verlag New York, 1997, MR **98i**: 15003.

[3] B. Q. FENG, A. M. TONGE, *Matrix versions of some refinements of the arithmetic-geometric mean inequality*, submitted.

[4] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press 1990, MR **91i**:15027.

[5] W. PUSZ, S. L. WORONOWICZ, *Functional calculus for sesquilinear forms and the purification map*, Reports on Mathematical Physics, **8**, (1975), 159–170, MR **54**: # 8316.

[6] M. SAGAE, K. TANABE, *Upper and lower bounds for the arithmetic-geometric-harmonic means of positive definite matrices*, Linear and Multilinear Algebra, **37**, (1994), 279–282, MR **95m**: 15030.

(Received July 18, 2003)

Bao Qi Feng
Department of Mathematical Sciences
Kent State University, Tuscarawas Campus
New Philadelphia
OH 44663-9403
USA
e-mail: bfeng@tusc.kent.edu

Andrew Tonge
Department of Mathematical Sciences
Kent State University
Kent
OH 44240
USA
e-mail: tonge@math.kent.edu