

A CERTAIN CLASS OF APPROXIMATION OPERATORS

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Abstract. In this paper we are studying the sequences of linear positive operators $(P_n^{\alpha,\beta})$ defined by means of the umbral calculus. We are proving that the sequence $(P_n^{\alpha,\beta})$ converges uniformly to the identity operator and we give some results for the rate of convergence for $P_n^{\alpha,\beta}f$.

1. Introduction

This section contains some basic facts needed in the subsequent analysis. The reader is referred to Gian-Carlo Rota and Steven Roman (1978) for the details about delta operators, as well as for historical remarks on the study of binomial sequence and Sheffer polynomials.

A sequence $(p_n)_{n \geq 0}$ of polynomials, where for all n, p_n is exactly of degree n , is called a polynomial sequence. Examples of such sequences are $(p_0(x) := 1)$:

1. $e_n(x) = x^n$ (the monomials);
2. $a_n(x) = x(x - na)^{n-1}$, (Abel);
3. $(x)^{[n,-1]} = x(x+1) \cdots (x+n-1)$, (upper-factorials);
4. $(x)^{[n,1]} = x(x-1) \cdots (x-n+1)$, (lower-factorials);
5. $p_n^{\alpha,\beta}(x) = x(x + \alpha + n\beta)^{[n-1,-\alpha]}$
 $= x(x + \alpha + n\beta)(x + 2\alpha + n\beta) \cdots (x + (n-1)\alpha + n\beta)$, (Gould).

A polynomial sequence $(a_n)_{n \geq 0}$ is called binomial, if for all x, y

$$a_n(x+y) = \sum_{k=0}^n \binom{n}{k} a_k(x) a_{n-k}(y), \quad n = 0, 1, 2, \dots$$

All above mentioned sequences are of binomial type. These binomial sequences occur in analysis and in combinatorics.

Further, let us denote by Π the (real) linear space of all polynomials with real coefficients. Let us put in evidence some operators $\Pi \rightarrow \Pi$. For instance, I is the

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identity, D is derivative, E^a is shift-operator, i.e. $(E^a f)(x) = f(x + a)$. We shall use the notation

$$\begin{aligned} \Pi^* &= \{U : \Pi \rightarrow \Pi; U \text{ linear}\} \\ \Pi_t^* &= \{U \in \Pi^*; E^a U = U E^a, \forall a\}. \end{aligned}$$

Let us remind that if $U_1, U_2 \in \Pi_t^*$, then $U_1 U_2 = U_2 U_1$.

An operator Q is called a delta operator if $Q \in \Pi_t^*$ and $Qe_1 = \text{const.} \neq 0$.

Let $X, X \in \Pi^*$, be defined as $(Xp)(x) = xp(x)$, $p \in \Pi$. If U is shift-invariant operator, i.e. $U \in \Pi_t^*$ then the linear operator U' defined by $U' = UX - XU$, is the so-called Pincherle derivative of U . Π_δ^* denotes the set of all delta operators.

Examples of delta operators:

- 1) $Q_1 = D(I - D)$;
- 2) $\Delta_a = E^a - I = E^{aD} - I, a \neq 0$;
- 3) $\nabla = I - E^{-1}$;
- 4) $A = DE^a$, (Abel operator);
- 5) $G = \frac{1}{\alpha}(I - E^{-\alpha})E^{-\beta} = \frac{1}{\alpha}\Delta_{-\alpha}E^{-\beta}, \alpha \neq 0$, (Gould operator).

At the same time is well-known that if Q is a delta operator, then $Q(\Pi_n) \subset \Pi_{n-1}$ where Π_n is the linear space of all polynomials of degree n .

It is known that

$$U \in \Pi_t^* \Rightarrow U' \in \Pi_t^*, Q \in \Pi_\delta^* \Rightarrow Q'^{-1} \text{ exists.}$$

Let $Q \in \Pi_\delta^*$. If $a_0(x) = 1, a_n(0) = 0, Qa_n = na_{n-1}, n \geq 1$, then (a_n) is a basic sequence associated to Q .

THEOREM 1.1. ([7], [8], [9])

- i) Every delta operator has an unique sequence of basic polynomials;
- ii) If (a_n) is a basic sequence for some delta operator Q , then it is binomial;
- iii) If (a_n) is a binomial sequence, then it is a basic sequence for some delta operator.

Now, a polynomial sequence (s_n) is called a Sheffer set or a set of Sheffer polynomials for the delta operator Q if $s_0(x) = \text{const.} \neq 0$ and $Qs_n = ns_{n-1}, n \neq 0$. A Sheffer set for the delta operator Q is related to the set of basic polynomials of Q by the following

THEOREM 1.2. ([7], [8], [9]) Let Q be a delta operator with basic polynomial set (a_n) . Then (s_n) is a Sheffer set relative to Q if and only if there exists an invertible shift-invariant operator S such that $s_n = S^{-1}a_n, n \geq 0$.

The defining property of polynomial sequence of binomial type has the following analog for Sheffer polynomials.

THEOREM 1.3. Let Q be a delta operator with basic polynomials (a_n) and let (s_n) be a Sheffer set relative to Q and to some invertible shift-invariant operator S . Then the following identity holds

$$s_n(x + y) = \sum_{k=0}^n \binom{n}{k} a_k(x) s_{n-k}(y).$$

This identity which characterizes the Sheffer polynomials, suggest us to consider the sequences of linear polynomial operators $L_n : C_{[0,1]} \rightarrow C_{[0,1]}$ defined by

$$(L_n(f))(x) = \frac{1}{s_n(1)} \sum_{k=0}^n \binom{n}{k} a_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

(see [3], [6]).

Now we consider the Gould operator

$$G = \frac{1}{\alpha}(I - E^{-\alpha})E^{-\beta}, \quad G : \Pi \rightarrow \Pi, \quad \alpha \neq 0.$$

If

$$q_n^\alpha(x) = x(x + \alpha)^{[n-1, -\alpha]}$$

is the basic set for the delta operator

$$Q = \frac{1}{\alpha}(I - E^{-\alpha})$$

then $s_n^{\alpha,\beta}(x) = (E^{n\beta} q_n^\alpha)(x)$ is a Sheffer set for the delta operator G . We have

$$(G s_n^{\alpha,\beta})(x) = (Q E^{-\beta} E^{n\beta} q_n^\alpha)(x) = n \left(E^{(n-1)\beta} q_{n-1}^\alpha \right) (x) = n s_{n-1}(x)$$

and

$$s_0^{\alpha,\beta}(x) = q_0^\alpha(x) = 1.$$

Let $P_n^{\alpha,\beta}$ be the linear positive operators defined by

$$(P_n^{\alpha,\beta} f)(x) = \sum_{k=0}^n I_{n,k}^{\alpha,\beta}(x) f\left(\frac{k}{n}\right), \quad f \in C[0, 1] \tag{1}$$

where

$$I_{n,k}^{\alpha,\beta}(x) = \binom{n}{k} \frac{p_k^{\alpha,\beta}(x) s_{n-k}^{\alpha,\beta}(1-x)}{s_n^{\alpha,\beta}(1)}$$

and

$$p_n^{\alpha,\beta}(x) = x(x + \alpha + n\beta)^{[n-1, -\alpha]}$$

is the basic set for G , $\alpha > 0, \beta > 0$.

REMARK Now we wish to notice that the Cheney-Sharma operator

$$(S_n^\beta f)(x) = \sum_{k=0}^n \binom{n}{k} \frac{x(x + k\beta)^{k-1} (1-x + (n-k)\beta)^{n-k}}{(1+n\beta)^n} f\left(\frac{k}{n}\right),$$

(see [2]), may be obtained as a limiting case of our operator (1). For $\alpha \rightarrow 0$ we have $Q \rightarrow D$ and $G \rightarrow DE^{-\beta}$ (the Abel operator). Next

$$p_n^{\alpha,\beta}(x) \rightarrow x(x + n\beta)^{n-1}, \quad s_n^{\alpha,\beta}(x) \rightarrow (x + n\beta)^n$$

and

$$P_n^{\alpha,\beta} \rightarrow S_n^\beta.$$

Now we remark that in the special case $\alpha \rightarrow 0, \beta = 0$ the operator (1) reduces, obviously, to the classic Bernstein operator, $P_n^{\alpha,\beta} \rightarrow B_n$ where

$$(B_n f)(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

2. Approximation properties

LEMMA 2.1. *The operators $P_n^{\alpha,\beta}$ defined by (1) verify*

$$\begin{aligned} (P_n^{\alpha,\beta} e_0)(x) &= e_0, & (P_n^{\alpha,\beta} e_1)(x) &= a_n x, \\ (P_n^{\alpha,\beta} e_2)(x) &= \frac{n-1}{n} d_n x^2 + \left(b_n + \frac{1}{n} c_n - \frac{n-1}{n} d_n \right) x, \end{aligned} \quad (2)$$

where

$$\begin{aligned} e_n(x) &= x^n, \quad n = 0, 1, 2, \dots \\ a_n &= \frac{1}{q_n^\alpha(1+n\beta)} \sum_{i=1}^n \binom{n-1}{i-1} q_{n-i}^{\alpha,\beta} p_i^{\alpha,\beta}(1), \\ b_n &= \frac{1}{q_n^\alpha(1+n\beta)} \sum_{i=2}^n \binom{n-2}{i-2} q_{n-i}^{\alpha,\beta} p_i^{\alpha,\beta}(1), \\ c_n &= \frac{1}{q_n^\alpha(1+n\beta)} \left[q_{n-1}^{\alpha,\beta} p_1^{\alpha,\beta}(1) + \sum_{i=2}^n \frac{n-i}{i-1} \binom{n-2}{i-2} q_{n-i}^{\alpha,\beta} p_i^{\alpha,\beta}(1) \right], \\ d_n &= \frac{1}{q_n^\alpha(1+n\beta)} \sum_{i=2}^n \binom{n-2}{i-2} q_{n-i}^{\alpha,\beta} (G'^{-2} p_{i-2}^{\alpha,\beta})(1), \end{aligned}$$

and G' is the Pincherle derivative of G .

Proof. Let us denote

$$S_m(x, y, n) = \sum_{k=0}^n \binom{n}{k} p_k^{\alpha,\beta}(x) s_{n-k}^{\alpha,\beta}(y) \left(\frac{k}{n}\right)^m$$

and let P the linear operator defined by $P = xG'^{-1}$. Using the identity

$$s_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k^{\alpha,\beta}(x) s_{n-k}^{\alpha,\beta}(y)$$

we obtain

$$S_m(x, y, n) = \frac{1}{n^m} \sum_{k=0}^n k! \binom{n}{k} S(m, k) P^k E^y s_{n-k}^{\alpha,\beta}(x)$$

where $S(m, k)$ are the Stirling numbers of second kind.

But

$$P E^y s_{n-1}(x) = \frac{x}{x+y} \sum_{i=1}^n \binom{n-1}{i-1} s_{n-i}^{\alpha,\beta}(0) p_i^{\alpha,\beta}(x+y),$$

$$\begin{aligned} P^2 E^y s_{n-2}(x) &= \frac{x}{x+y} \sum_{i=2}^n \binom{n-2}{i-2} s_{n-i}^{\alpha,\beta}(0) p_i^{\alpha,\beta}(x+y) \\ &\quad - xy \sum_{i=2}^n \binom{n-2}{i-2} s_{n-i}^{\alpha,\beta}(0) (G'^{-2} p_{i-2}^{\alpha,\beta})(x+y). \end{aligned}$$

Now we have

$$\begin{aligned}
 S_0(x, y, n) &= s_n^{\alpha, \beta}(x + y), \\
 S_1(x, y, n) &= \frac{x}{x + y} \sum_{i=1}^n \binom{n-1}{i-1} s_{n-i}^{\alpha, \beta}(0) p_i^{\alpha, \beta}(x + y), \\
 S_2(x, y, n) &= \frac{1}{n} \frac{x}{x + y} \sum_{i=1}^n \binom{n-1}{i-1} s_{n-i}^{\alpha, \beta}(0) p_i^{\alpha, \beta}(x + y) \\
 &\quad + \frac{n-1}{n} \frac{x}{x + y} \sum_{i=2}^n \binom{n-2}{i-2} s_{n-i}^{\alpha, \beta}(0) p_i^{\alpha, \beta}(x + y) \\
 &\quad - \frac{n-1}{n} xy \sum_{i=2}^n \binom{n-2}{i-2} p_{n-i}^{\alpha, \beta}(0) (G^{t-2} p_{i-2}^{\alpha, \beta})(x + y).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (P_n e_0)(x) &= e_0, \\
 (P_n e_1)(x) &= \frac{x}{s_n^{\alpha, \beta}(1)} \sum_{i=1}^n \binom{n-1}{i-1} s_{n-i}^{\alpha, \beta}(0) p_i^{\alpha, \beta}(1), \\
 (P_n e_2)(x) &= \frac{x}{n s_n^{\alpha, \beta}(1)} \sum_{i=1}^n \binom{n-1}{i-1} s_{n-i}^{\alpha, \beta}(0) p_i^{\alpha, \beta}(1) \\
 &\quad + \frac{(n-1)x}{n s_n^{\alpha, \beta}(1)} \sum_{i=2}^n \binom{n-2}{i-2} s_{n-i}^{\alpha, \beta}(0) p_i^{\alpha, \beta}(1) \\
 &\quad - \frac{(n-1)x(1-x)}{n s_n^{\alpha, \beta}(1)} \sum_{i=2}^n \binom{n-2}{i-2} s_{n-i}^{\alpha, \beta}(0) (G^{t-2} p_{i-2}^{\alpha, \beta})(1).
 \end{aligned}$$

We obtain (2) with

$$d_n \leq b_n \leq a_n \leq 1 \text{ and } 0 \leq c_n \leq 1.$$

THEOREM 2.2. *Let $P_n^{\alpha, \beta}$ be defined by (1). If $\alpha = \alpha(n) > 0$, $\beta = \beta(n) > 0$ with $\lim_{n \rightarrow \infty} n\alpha(n) = \lim_{n \rightarrow \infty} n\beta(n) = 0$ then for $f \in C[0, 1]$ one has*

$$\lim_{n \rightarrow \infty} P_n^{\alpha, \beta} f = f$$

uniformly on the compact $[0, 1]$, $\|f\| = \sup_{x \in [0, 1]} |f(x)|$.

Proof. We have (see [4])

$$p_{n-1}^{\alpha, \beta}(1) \leq (G^{t-2} p_{n-2}^{\alpha, \beta})(1) \leq p_n^{\alpha, \beta}(1), \quad n = 2, 3, \dots$$

and we can write

$$\frac{p_{n-1}^{\alpha, \beta}(1)}{q_n^{\alpha}(1 + n\beta)} \leq \frac{\sum_{i=2}^n \binom{n-2}{i-2} q_{n-i}^{\alpha}(n\beta) (G^{t-2} p_{i-2}^{\alpha, \beta})(1)}{q_n^{\alpha}(1 + n\beta)} = d_n \leq 1.$$

Taking into account that

$$\frac{1}{1+n\beta} \cdot \frac{1}{1+(n-1)\alpha+n\beta} \left(\frac{1+\alpha+(n-1)\beta}{1+\alpha+n\beta} \right)^{n-2} \leq \frac{P_{n-1}^{\alpha,\beta}(1)}{q_n^\alpha(1+n\beta)} \leq 1$$

we have

$$\lim_{n \rightarrow \infty} \frac{P_{n-1}^{\alpha,\beta}(1)}{q_n^\alpha(1+n\beta)} = 1,$$

and hence

$$\lim_{n \rightarrow \infty} d_n = 1, \quad \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n = 1.$$

Now according to the Bohman–Korovkin theorem it is sufficient to show that

$$\lim_{n \rightarrow \infty} \|e_k - P_n^{\alpha,\beta} e_k\| = 0, \quad k = 0, 1, 2.$$

We have

$$\begin{aligned} \|e_0 - P_n^{\alpha,\beta} e_0\| &= 0 \\ \|e_1 - P_n^{\alpha,\beta} e_1\| &= \|x - a_n x\| = 1 - a_n \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \|e_1 - P_n^{\alpha,\beta} e_1\| = 0$$

Finally

$$\|e_2 - P_n^{\alpha,\beta} e_2\| \leq (1 - b_n) + \frac{c_n}{n} + \frac{1}{4} \left(b_n - \frac{n-1}{n} d_n \right)$$

where

$$1 - \frac{1}{n} \leq 1 \leq \frac{b_n}{d_n}$$

Therefore

$$\lim_{n \rightarrow \infty} \|e_2 - P_n^{\alpha,\beta} e_2\| = 0$$

THEOREM 2.3. *If $P_n^{\alpha,\beta}$ is defined by (1) then for each $x \in [0, 1]$ the following inequality*

$$\left| (P_n^{\alpha,\beta}(x) - f(x)) \right| \leq \frac{4}{3} w_1 \left(f; \sqrt{3A_n(x)} \right) \tag{3}$$

holds, where

$$A_n(x) = \left(1 - 2a_n + \frac{n-1}{n} d_n \right) x^2 + \left(\frac{a_n}{n} + b_n + \frac{c_n}{n} - \frac{n-1}{n} d_n - \frac{1}{n} \right) x + \frac{1}{3n^2}$$

and

$$w_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, 1], |x - y| \leq \delta, \delta > 0\}$$

is the usual first modulus of continuity of a function $f : [0, 1] \rightarrow \mathbb{R}$.

Proof. We have

$$\begin{aligned} (P_n^{\alpha,\beta}f)(x) - f(x) &= \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x)f\left(\frac{k}{n}\right) - \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x)f(x) \\ &= n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(\frac{k}{n}\right) dt - n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dt \\ &= n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[f\left(\frac{k}{n}\right) - f(x) \right] dt \end{aligned}$$

and

$$|(P_n^{\alpha,\beta})(x) - f(x)| \leq n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left| f\left(\frac{k}{n}\right) - f(x) \right| dt.$$

Using the method of [1] we obtain for $|t - x| < \delta$

$$\left| f\left(\frac{k}{n}\right) - f(x) \right| \leq w_1(f; |t - x|) \leq (1 + \delta^{-2}(t - x)^2)w_1(f; \delta)$$

where $w_1(f; \delta)$ is the first modulus of continuity. For $|t - x| \geq \delta$ we have

$$w_1\left(f; \frac{|t - x|}{\delta} \delta\right) \leq \left(1 + \frac{(t - x)^2}{\delta^2}\right) w_1(f; \delta)$$

Hence

$$|(P_n^{\alpha,\beta})(x) - f(x)| \leq n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} [1 + \delta^{-2}(t - x)^2] w_1(f; \delta) dt$$

Now we observe that

$$n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} w_1(f; \delta) dt = \left(\sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \right) w_1(f; \delta) = w_1(f; \delta),$$

and

$$\begin{aligned} n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \delta^{-2}(t - x)^2 w_1(f; \delta) dt \\ = \left[P_n^{\alpha,\beta} e_2 + \left(\frac{1}{n} - 2x\right) P_n^{\alpha,\beta} e_1 + \left(x^2 - \frac{1}{n}x + \frac{1}{3n^2}\right) P_n^{\alpha,\beta} e_0 \right] \delta^{-2} w_1(f; \delta) \\ = \delta^{-2} A_n(x) w_1(f; \delta), \end{aligned}$$

where

$$A_n(x) = \left(1 - 2a_n + \frac{n-1}{n}d_n\right) x^2 + \left(\frac{1}{n}a_n + b_n + \frac{1}{n}c_n - \frac{n-1}{n}d_n - \frac{1}{n}\right) x + \frac{1}{3n^2},$$

$A_n(x) \geq 0$, for each $x \in [0, 1]$.

Hence

$$|(P_n^{\alpha,\beta}f)(x) - f(x)| \leq [1 + \delta^{-2}A_n(x)]w_1(f; \delta).$$

Choosing $\delta = \sqrt{3A_n(x)}$ we obtain

$$|(P_n^{\alpha,\beta}f)(x) - f(x)| \leq \frac{4}{3}w_1(f; \sqrt{3A_n(x)}).$$

REMARK 2.1. For $\beta = 0$, $\alpha \rightarrow 0$ we have $a_n \rightarrow 1$, $b_n \rightarrow 1$, $d_n \rightarrow 1$, $c_n = 0$ and

$$A_n(x) \rightarrow -\frac{1}{n}x^2 + \frac{1}{n}x + \frac{1}{3n^2}.$$

Hence of (3) we obtain

$$|(B_n f)(x) - f(x)| \leq \frac{4}{3}w_1\left(f; \sqrt{\frac{3}{n}x(1-x) + \frac{1}{n^2}}\right). \quad (4)$$

Next we observe that

$$\frac{3}{n}x(1-x) + \frac{1}{n^2} \leq \frac{3}{4n} + \frac{1}{n^2} \leq \frac{1}{n} \quad \text{for } n \geq 4,$$

and of the inequality (4) we obtain the inequality of Popoviciu (see [11]) corresponding to the Bernstein polynomial

$$|(B_n f)(x) - f(x)| \leq \frac{3}{2}w_1\left(f; \frac{1}{\sqrt{n}}\right).$$

THEOREM 2.4. Let $P_n^{\alpha,\beta}$ defined by (1). Then for $f \in C^1[0, 1]$ one has

$$|(P_n^{\alpha,\beta}f)(x) - f(x)| \leq (1 - a_n)x\|f'\| + [1 + (1 - a_n)x]w_1\left(f'; \sqrt{A_n(x)B_n(x)}\right) \quad (5)$$

where $A_n(x)$ is defined by Theorem 2.3, $B_n(x) \geq 0$ is defined by

$$B_n(x) = \left(1 - 2a_n + \frac{n-1}{n}d_n\right)x^2 + \left(b_n + \frac{1}{n}c_n - \frac{n-1}{n}d_n\right)x,$$

and

$$\|f'\| = \max_{x \in [0,1]} |f'(x)|$$

Proof. We have

$$|(P_n^{\alpha,\beta}f)(x) - f(x)| \leq \sum_{k=0}^n P_{n,k}^{\alpha,\beta}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right|$$

and we can write

$$f\left(\frac{k}{n}\right) - f(x) = \left(\frac{k}{n} - x\right)f'(x) + \left(\frac{k}{n} - x\right)(f'(\xi) - f'(x))$$

where $\xi = \xi(x, k, n)$ is a point of the interval determined by x and $\frac{k}{n}$.

Next

$$|(P_n^{\alpha,\beta} f)(x) - f(x)| \leq \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| |f'(x)| + \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| |f'(\xi) - f'(x)|$$

and

$$\sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| |f'(x)| \leq \|f'\| \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| \leq \|f'\| (1 - a_n)x$$

Now

$$|f'(\xi) - f'(x)| \leq \sup |f'(t) - f'(x)| = w_1(f'; |t - x|) = w_1\left(f'; \delta \frac{|t - x|}{\delta}\right) \leq (1 + \delta^{-1}|t - x|)w_1(f'; \delta)$$

Hence

$$\begin{aligned} \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| |f'(\xi) - f'(x)| &= n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f'(\xi) - f'(x)| dt \\ &\leq n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| \int_{\frac{k}{n}}^{\frac{k+1}{n}} (1 + \delta^{-1}|t - x|)w_1(f'; \delta) dt \\ &= w_1(f'; \delta) \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| + n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| \delta^{-1} w_1(f'; \delta) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t - x| dt \\ &\leq (1 - a_n)xw_1(f'; \delta) + n\delta^{-1}w_1(f'; \delta) \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t - x| dt. \end{aligned}$$

According to Cauchy's inequality we have

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} |t - x| dt \leq \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt \right)^{1/2} \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t - x)^2 dt \right)^{1/2} = \frac{1}{\sqrt{n}} \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t - x)^2 dt \right)^{1/2}$$

and

$$\sqrt{n}\delta^{-1}w_1(f'; \delta) \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left| \frac{k}{n} - x \right| \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t - x)^2 dt \right)^{1/2}$$

$$\begin{aligned} &\leq \sqrt{n}\delta^{-1}w_1(f'; \delta) \left(\sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left(\frac{k}{n} - x\right)^2 \right)^{1/2} \left(\sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2 dt \right)^{1/2} \\ &= \delta^{-1}w_1(f'; \delta) \left(\sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left(\frac{k}{n} - x\right)^2 \right)^{1/2} \left(n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2 dt \right)^{1/2}. \end{aligned}$$

Now using (2)

$$\sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \left(\frac{k}{n} - x\right)^2 = \left(1 - 2a_n + \frac{n-1}{n}d_n\right)x^2 + \left(b_n + \frac{1}{n}c_n - \frac{n-1}{n}d_n\right)x$$

and

$$n \sum_{k=0}^n l_{n,k}^{\alpha,\beta}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2 dt = A_n(x)$$

where $A_n(x)$ is defined by Theorem 2.3.

We note

$$B_n(x) = \left(1 - 2a_n + \frac{n-1}{n}d_n\right)x^2 + \left(b_n + \frac{1}{n}c_n - \frac{n-1}{n}d_n\right)x, \quad B_n(x) \geq 0$$

and we have

$$|(P_n^{\alpha,\beta}f)(x) - f(x)| \leq (1 - a_n)x\|f'\| + \left[(1 - a_n)x + \delta^{-1}\sqrt{A_n(x)B_n(x)}\right]w_1(f'; \delta).$$

For $\delta = \sqrt{A_n(x)B_n(x)}$ we obtain

$$|(P_n^{\alpha,\beta}f)(x) - f(x)| \leq (1 - a_n)x\|f'\| + [1 + (1 - a_n)x]w_1\left(f'; \sqrt{A_n(x)B_n(x)}\right)$$

and the proof of our theorem is complete.

REMARK 2.2. For $\beta = 0, \alpha \rightarrow 0$ we have

$$\begin{aligned} A_n(x) &\rightarrow \frac{1}{n}x(1-x) + \frac{1}{3n^2}, \\ B_n(x) &\rightarrow \frac{1}{n}x(1-x), \end{aligned}$$

and of (5) we obtain

$$|(B_n f)(x) - f(x)| \leq w_1\left(f'; \sqrt{\frac{1}{n^2}x^2(1-x)^2 + \frac{1}{3n^3}x(1-x)}\right).$$

Whence

$$|(B_n f)(x) - f(x)| \leq w_1\left(f'; \sqrt{\frac{1}{16n^2} + \frac{1}{12n^3}}\right). \tag{6}$$

We observe that

$$\frac{1}{16n^2} + \frac{1}{12n^3} \leq \frac{1}{n}$$

and of (6) we have the inequality

$$| (B_n f)(x) - f(x) | \leq \frac{3}{2} w_1 \left(f'; \sqrt{\frac{1}{n}} \right)$$

due to Lorentz (see [11]).

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