

## ON THE STABILITY OF AN $n$ -DIMENSIONAL QUADRATIC AND ADDITIVE FUNCTIONAL EQUATION

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(communicated by T. M. Rassias)

*Abstract.* In this paper, we investigate the generalized Hyers-Ulam stability problem of a quadratic and additive type functional equation

$$f\left(\sum_{i=1}^n x_i\right) + (n-2) \sum_{i=1}^n f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j), \quad (n > 2)$$

for the even or odd case in the  $n$  variables.

### 1. Introduction

In 1940, S. M. Ulam [16] raised a question concerning the stability of group homomorphisms:

*Let  $G_1$  be a group and let  $G_2$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\epsilon > 0$ , does there exist a  $\delta > 0$  such that if a function  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(xy), h(x)h(y)) < \delta$  for all  $x, y \in G_1$ , then there exists a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(x)) < \epsilon$  for all  $x \in G_1$ ?*

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it.

It is easy to see that the quadratic function  $f(x) = cx^2$  on real field is a solution of the following equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

So, it is natural that the equation (1.1) is called a *quadratic functional equation*. In particular, every solution of the quadratic equation (1.1) is said to be a *quadratic function*. It is well known that a function  $f$  between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function  $B$  such that  $f(x) = B(x, x)$  for all  $x$  (see [1]). The biadditive function  $B$  is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.2}$$

*Mathematics subject classification* (2000): 39A11, 39B72.

*Key words and phrases:* Hyers-Ulam stability, quadratic function.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-070-C00009).

During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors and we may refer lots of papers concerning the stability results for various functional equations [4, 6, 7, 8, 10, 14, 15]. A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [3, 5, 11]. Further, Jun and Lee [9] proved the generalized Hyers-Ulam stability of the pexiderized quadratic equation (1.1).

Now, we introduce the following generalized quadratic and additive type functional equation,

$$f\left(\sum_{i=1}^n x_i\right) + (n-2) \sum_{i=1}^n f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j), \quad (n > 2) \quad (1.3)$$

in the class of functions between real vector spaces.

For  $n = 3$ , P. Kannappan proved that a function  $f$  satisfies the functional equation (1.3) if and only if there exist a symmetric biadditive function  $B$  and an additive function  $A$  such that  $f(x) = B(x, x) + A(x)$  for all  $x$  (see [13]). The Hyers-Ulam stability problem for the equation of  $n = 3$  was proved by S.-M. Jung [12]. The Hyers-Ulam-Rassias stability problem for the equation of  $n = 4$  was also investigated by Chang, Lee and Kim [2].

In this paper, we obtain the general solution of the equation (1.3) with  $n$  variables in the class of functions between real vector spaces and we establish the generalized Hyers-Ulam-Rassias stability problem for the equation (1.3) for the case even or odd of  $n$  variables in the sense of Hyers, Ulam, Rassias and Gávruta since it has slightly different upper bounds depending on the even or odd case of  $n$ .

## 2. General solution of (1.3)

We here present the general solution of (1.3).

**THEOREM 2.1.** *Let  $X$  and  $Y$  be real vector spaces. A function  $f : X \rightarrow Y$  satisfies the functional equation (1.3) if and only if there exist functions  $B : X \times X \rightarrow Y$ ,  $A : X \rightarrow Y$  such that  $f(x) = B(x, x) + A(x)$  for all  $x \in X$ , where  $B$  is symmetric biadditive, and  $A$  is additive.*

*Proof.* First we assume that  $f$  is a solution of the equation (1.3). If we put  $x_i = 0$  for all  $i$  in (1.3), we see that  $f(0) = 0$ . Putting  $x_4 = x_5 = \cdots = x_n = 0$  in (1.3), we get

$$f(x_1 + x_2 + x_3) + f(x_1) + f(x_2) + f(x_3) = f(x_1 + x_2) + f(x_2 + x_3) + f(x_1 + x_3) \quad (2.1)$$

for all  $x_1, x_2, x_3 \in X$ . Therefore according to [13], the assertion is true.

Conversely, if there exist a symmetric biadditive function  $B : X \times X \rightarrow Y$  and an additive function  $A : X \rightarrow Y$  such that  $f(x) = B(x, x) + A(x)$  for all  $x \in X$ , then it is obvious that  $f$  satisfies the equation (1.3).  $\square$

### 3. Stability of (1.3)

Throughout this section  $X$  and  $Y$  will be a real linear space and a real Banach space, respectively, unless we give any specific reference. Given  $f : X \rightarrow Y$ , we set

$$Df(x_1, x_2, \dots, x_n) := f\left(\sum_{i=1}^n x_i\right) + (n-2) \sum_{i=1}^n f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j), \quad (n > 2)$$

for all  $x_i \in X$  ( $i = 1, 2, \dots, n$ ).

Let  $\varphi : \underbrace{X \times X \times \dots \times X}_{n\text{-times}} \rightarrow [0, \infty)$  be a mapping satisfying one of the conditions

(A), (B) and one of the conditions (C), (D):

$$\Phi_1(x_1, \dots, x_n) := \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} \varphi(2^k x_1, \dots, 2^k x_n) < \infty \tag{A}$$

$$\Phi_2(x_1, \dots, x_n) := \sum_{k=0}^{\infty} 4^k \varphi\left(\frac{x_1}{2^{k+1}}, \dots, \frac{x_n}{2^{k+1}}\right) < \infty \tag{B}$$

$$\Psi_1(x_1, \dots, x_n) := \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x_1, \dots, 2^k x_n) < \infty \tag{C}$$

$$\Psi_2(x_1, \dots, x_n) := \sum_{k=0}^{\infty} 2^k \varphi\left(\frac{x_1}{2^{k+1}}, \dots, \frac{x_n}{2^{k+1}}\right) < \infty \tag{D}$$

for all  $x_1, \dots, x_n \in X$ . For convenience we use the following notation:

$$\varphi_{\lfloor \frac{n}{2} \rfloor}(x) := \underbrace{\varphi(x, \dots, x)}_{\lfloor \frac{n}{2} \rfloor + 1}, -x, \dots, -x) + \varphi(-x, \dots, -x, \underbrace{x, \dots, x}_{\lfloor \frac{n}{2} \rfloor + 1})$$

for all  $x_i \in X$  ( $i = 1, 2, \dots, n$ ), where  $\lfloor k \rfloor$  denotes the greatest integer less than or equal to  $k \in \mathbb{R}$ .

Now we examine the situations that the functional equation (1.3) is stable in the sense of Hyers, Ulam, Rassias and Găvruta, and construct a true function for a given approximate function differing from a control function  $\varphi$ . One of the conditions (A), (B) will be needed to derive a quadratic function and one of the conditions (C), (D) will be requested to derive an additive function in the following theorem. We start from the odd case of  $n$  variables.

**THEOREM 3.1.** *Let  $\varphi$  be as above. Assume that a function  $f : X \rightarrow Y$  satisfies*

$$\|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n) \tag{3.1}$$

for all  $x_1, \dots, x_n \in X$  and for some odd  $n := 2m + 1$ . Then there exist a unique additive function  $A : X \rightarrow Y$  and a unique quadratic function  $Q : X \rightarrow Y$  satisfying

(1.3) such that

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{1}{2 \lfloor \frac{n}{2} \rfloor^2} \Phi_{i_{\lfloor \frac{n}{2} \rfloor}}(x) + \frac{1}{2 \lfloor \frac{n}{2} \rfloor} \Psi_{j_{\lfloor \frac{n}{2} \rfloor}}(x), \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{1}{2 \lfloor \frac{n}{2} \rfloor^2} \Phi_{i_{\lfloor \frac{n}{2} \rfloor}}(x), \end{aligned}$$

and

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leq \frac{1}{2 \lfloor \frac{n}{2} \rfloor} \Psi_{j_{\lfloor \frac{n}{2} \rfloor}}(x)$$

for all  $x \in X$  and for  $i = 1$  or  $2$ ,  $j = 1$  or  $2$ .

The functions  $Q, A$  are given by

$$\begin{cases} Q(x) = \lim_{k \rightarrow \infty} \frac{f(2^k x) + f(-2^k x)}{2 \cdot 4^k} & \text{if (A) holds} \\ Q(x) = \lim_{k \rightarrow \infty} \frac{4^k}{2} [f(\frac{x}{2^k}) + f(-\frac{x}{2^k})], f(0) = 0 & \text{if (B) holds} \\ A(x) = \lim_{k \rightarrow \infty} \frac{f(2^k x) - f(-2^k x)}{2^{k+1}} & \text{if (C) holds} \\ A(x) = \lim_{k \rightarrow \infty} 2^{k-1} [f(\frac{x}{2^k}) - f(-\frac{x}{2^k})], f(0) = 0 & \text{if (D) holds} \end{cases}$$

for all  $x \in X$

*Proof.* Let  $f_1 : X \rightarrow Y$  be a function defined by  $f_1(x) := \frac{f(x) + f(-x)}{2}$  for all  $x \in X$ . Then  $f_1(0) = f(0)$ ,  $f_1(x) = f_1(-x)$ ,  $\lfloor \frac{n}{2} \rfloor = m$ , and

$$\|Df_1(x_1, \dots, x_n)\| \leq \frac{1}{2} [\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n)] \tag{3.2}$$

holds for all  $x_i \in X$ . Putting  $x_i = x$  for  $i = 1, \dots, m + 1$  and  $x_i = -x$  for  $i = m + 2, \dots, n$  in (3.2) yields

$$\begin{aligned} &\left\| f_1(x) + (2m + 1)(2m - 1)f_1(x) \right. \\ &\quad \left. - \binom{m+1}{2} f_1(2x) \binom{m}{2} f_1(2x) - m(m+1)f_1(0) \right\| \leq \frac{1}{2} \varphi_{\lfloor \frac{n}{2} \rfloor}(x), \end{aligned} \tag{3.3}$$

which is written by

$$\|4m^2 f_1(x) - m^2 f_1(2x) - m(m+1)f_1(0)\| \leq \frac{1}{2} \varphi_{\lfloor \frac{n}{2} \rfloor}(x) \tag{3.4}$$

for all  $x \in X$ .

*Case 1.* Assume that the control function  $\varphi$  satisfies the condition (A).

Dividing both sides of (3.4) by  $4m^2$ , we have

$$\begin{aligned} \left\| f_1(x) - \frac{f_1(2x)}{4} - \frac{m+1}{4m} f_1(0) \right\| &= \left\| \left[ f_1(x) - \frac{m+1}{3m} f_1(0) \right] - \frac{1}{4} \left[ f_1(2x) - \frac{m+1}{3m} f_1(0) \right] \right\| \\ &\leq \frac{1}{8m^2} \varphi_{\lfloor \frac{n}{2} \rfloor}(x) \end{aligned} \tag{3.5}$$

for all  $x \in X$ . Replacing  $x$  by  $2^{k-1}x$  and dividing by  $4^{k-1}$  in (3.5) we obtain

$$\left\| \frac{[f_1(2^{k-1}x) - \frac{m+1}{3m}f_1(0)]}{4^{k-1}} - \frac{[f_1(2^kx) - \frac{m+1}{3m}f_1(0)]}{4^k} \right\| \leq \frac{1}{2m^2 4^k} \varphi_{[\frac{m}{2}]}(2^{k-1}x) \quad (3.6)$$

for all  $x \in X$  and for all  $k \in \mathbb{N}$ .

The triangle inequality and induction argument implies easily by (3.6) that

$$\left\| \left[ f_1(x) - \frac{m+1}{3m}f_1(0) \right] - \frac{[f_1(2^kx) - \frac{m+1}{3m}f_1(0)]}{4^k} \right\| \leq \frac{1}{2m^2} \sum_{i=0}^{k-1} \frac{\varphi_{[\frac{m}{2}]}(2^i x)}{4^{i+1}} \quad (3.7)$$

for all  $x \in X$  and for all  $k \in \mathbb{N}$ .

In order to prove convergence of the sequence  $\left\{ F_k(x) := \frac{f_1(2^kx) - \frac{m+1}{3m}f_1(0)}{4^k} \right\}_{k \in \mathbb{N}}$ , we show that the sequence is a Cauchy sequence in  $Y$ . By (3.7) we obtain that

$$\begin{aligned} \|F_k(x) - F_{k+l}(x)\| &= \frac{1}{4^k} \left\| [f_1(2^kx) - \frac{m+1}{3m}f_1(0)] - \frac{[f_1(2^{l+2^k}x) - \frac{m+1}{3m}f_1(0)]}{4^l} \right\| \\ &\leq \frac{1}{2m^2} \sum_{i=0}^{l-1} \frac{\varphi_{[\frac{m}{2}]}(2^{k+i}x)}{4^{k+i+1}} \end{aligned} \quad (3.8)$$

for all  $x \in X$  and for all  $k, l \in \mathbb{N}$ . Since the right hand side of (3.8) tends to zero as  $k \rightarrow \infty$ ,  $\{F_k(x)\}$  is a Cauchy sequence for all  $x \in X$  and thus converges by the completeness of  $Y$ . Therefore we can define a function  $Q : X \rightarrow Y$  by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f_1(2^kx)}{4^k} = \lim_{k \rightarrow \infty} \frac{f(2^kx) + f(-2^kx)}{2 \cdot 4^k}, \quad x \in X.$$

Note that  $Q(0) = 0$ ,  $Q(-x) = Q(x)$  for all  $x \in X$ .

Replacing  $x_i$  in (3.2) by  $2^kx_i$  for all  $i = 1, \dots, n$  and dividing both sides by  $4^k$ , and after then taking the limit in the resulting inequality, we have

$$\begin{aligned} &\left\| Q\left(\sum_{i=1}^n x_i\right) + (n-2) \sum_{i=1}^n Q(x_i) - \sum_{1 \leq i < j \leq n} Q(x_i + x_j) \right\| \\ &\leq \lim_{k \rightarrow \infty} \frac{\varphi(2^kx_1, \dots, 2^kx_n) + \varphi(-2^kx_1, \dots, -2^kx_n)}{2 \cdot 4^k} = 0. \end{aligned} \quad (3.9)$$

Since  $Q$  is even and  $Q(2^l x) = 4^l Q(x)$  for all  $l \in \mathbb{N}$ , the function  $Q$  satisfies the equation (1.3) and is quadratic by Theorem 2.1.

Taking the limit in (3.7) as  $k \rightarrow \infty$ , we obtain

$$\begin{aligned} \left\| f_1(x) - \frac{m+1}{3m}f_1(0) - Q(x) \right\| &\leq \frac{1}{2m^2} \sum_{i=0}^{\infty} \frac{\varphi_{[\frac{m}{2}]}(2^i x)}{4^{i+1}} \\ &= \frac{1}{2[\frac{m}{2}]^2} \Phi_{1, [\frac{m}{2}]}(x) \end{aligned} \quad (3.10)$$

for all  $x \in X$ .

To prove the uniqueness, let  $Q'$  be another quadratic function satisfying (3.10). Then  $Q'(0) = 0$ ,  $Q'(2^l x) = 4^l Q'(x)$ , and  $Q'(-x) = Q'(x)$  for all  $x \in X$ . Thus we have

$$\begin{aligned} \|Q(x) - Q'(x)\| &\leq \frac{1}{4^l} \left\| Q(2^l x) - f_1(2^l x) + \frac{m+1}{3m} f_1(0) \right\| \\ &\quad + \frac{1}{4^l} \left\| f_1(2^l x) - \frac{m+1}{3m} f_1(0) - Q'(2^l x) \right\| \\ &\leq \frac{1}{4^{l \lfloor \frac{l}{2} \rfloor}} \Phi_{1, \lfloor \frac{l}{2} \rfloor}(2^l x) \end{aligned}$$

for all  $l \in \mathbb{N}$ . Taking the limit as  $l \rightarrow \infty$ , we conclude that  $Q(x) = Q'(x)$  for all  $x \in X$ .

*Case 2.* If  $\varphi$  satisfies the condition (B) (and hence implies (D)), the proof is analogous to that of *Case 1.* and we sketch the proof as follows.

Indeed, putting  $x_i = 0$  for all  $i$  in (3.1) we have  $f(0) = 0$  since  $\Phi_2(0, \dots, 0) < \infty$  and  $\varphi(0, \dots, 0) = 0$ .

Replacing  $x$  by  $\frac{x}{2}$  in (3.4) we get

$$\left\| f_1(x) - 4f_1\left(\frac{x}{2}\right) \right\| \leq \frac{1}{2m^2} \varphi_{\lfloor \frac{l}{2} \rfloor}\left(\frac{x}{2}\right) \tag{3.11}$$

for all  $x \in X$ .

An induction argument implies from (3.11) that

$$\begin{aligned} \left\| f_1(x) - 4^k f_1\left(\frac{x}{2^k}\right) \right\| &\leq \frac{1}{2m^2} \sum_{i=0}^{k-1} 4^i \varphi_{\lfloor \frac{l}{2} \rfloor}\left(\frac{x}{2^{i+1}}\right) \\ &\leq \frac{1}{2m^2} \sum_{i=0}^{\infty} 4^i \varphi_{\lfloor \frac{l}{2} \rfloor}\left(\frac{x}{2^{i+1}}\right) \\ &= \frac{1}{2 \lfloor \frac{l}{2} \rfloor^2} \Phi_{2, \lfloor \frac{l}{2} \rfloor}(x) \end{aligned} \tag{3.12}$$

for all  $x \in X$  and for all  $k \in \mathbb{N}$ .

Using the similar argument to that of *Case 1.*, we can show that the sequence  $\{4^k f_1(\frac{x}{2^k})\}_{k \in \mathbb{N}}$  is a Cauchy sequence in  $Y$  for all  $x \in X$  and thus converges. Therefore we can define a function  $Q : X \rightarrow Y$  by

$$Q(x) = \lim_{k \rightarrow \infty} 4^k f_1\left(\frac{x}{2^k}\right) = \lim_{k \rightarrow \infty} \frac{4^k}{2} \left[ f\left(\frac{x}{2^k}\right) + f\left(-\frac{x}{2^k}\right) \right], \quad x \in X.$$

Taking the limit in (3.12) as  $k \rightarrow \infty$ , we obtain

$$\|f_1(x) - Q(x)\| \leq \frac{1}{2 \lfloor \frac{l}{2} \rfloor^2} \Phi_{2, \lfloor \frac{l}{2} \rfloor}(x) \tag{3.13}$$

for all  $x \in X$ .

Note that  $Q(0) = 0$ ,  $Q(-x) = Q(x)$  for all  $x \in X$  and thus  $Q$  is the unique quadratic function satisfying the equation (1.3) and the inequality (3.13).

Now let  $f_2 : X \rightarrow Y$  be a function defined by  $f_2(x) := \frac{1}{2}[f(x) - f(-x)]$  for all  $x \in X$ . Then  $f_2(0) = 0$ ,  $f_2(-x) = -f_2(x)$ , and the relation (3.1) can be written by

$$\|Df_2(x_1, \dots, x_n)\| \leq \frac{1}{2}[\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n)] \tag{3.14}$$

for all  $x_i \in X$ . Applying the process (3.3) to (3.14) yields

$$\|2f_2(x) - f_2(2x)\| \leq \frac{1}{2m} \varphi_{[\frac{n}{2}]}(x) \tag{3.15}$$

for all  $x \in X$ .

*Case 3.* Assume that  $\varphi$  satisfies the condition (C) (and hence implies (A)).

Dividing the last inequality by 2 we have

$$\left\|f_2(x) - \frac{f_2(2x)}{2}\right\| \leq \frac{1}{4m} \varphi_{[\frac{n}{2}]}(x) \tag{3.16}$$

for all  $x \in X$ .

It follows by an induction argument that

$$\left\|f_2(x) - \frac{f_2(2^k x)}{2^k}\right\| \leq \frac{1}{2m} \sum_{i=0}^{k-1} \frac{\varphi_{[\frac{n}{2}]}(2^i x)}{2^{i+1}} \tag{3.17}$$

for all  $x \in X$  and for all  $k \in \mathbb{N}$ .

Using the similar argument to that of *Case 1.*, we can show that the sequence  $\{\frac{f_2(2^k x)}{2^k}\}_{k \in \mathbb{N}}$  is a Cauchy sequence in  $Y$  for all  $x \in X$  and thus converges. Therefore we can define a function  $A : X \rightarrow Y$  by

$$A(x) = \lim_{k \rightarrow \infty} \frac{f_2(2^k x)}{2^k} = \lim_{k \rightarrow \infty} \frac{f(2^k x) - f(-2^k x)}{2^{k+1}}, \quad x \in X.$$

Taking the limit in (3.17) as  $k \rightarrow \infty$ , we obtain

$$\|f_2(x) - A(x)\| \leq \frac{1}{2[\frac{n}{2}]} \Psi_{1[\frac{n}{2}]}(x) \tag{3.18}$$

for all  $x \in X$ .

Observe that  $A(0) = 0$ ,  $A(-x) = -A(x)$  for all  $x \in X$  and thus  $A$  is the unique additive function satisfying the equation (1.3) and the inequality (3.18), which is similarly proved like as *Case 1.*

*Case 4.* Assume that  $\varphi$  satisfies the condition (D), then the proof is analogous to that of *Case 2.* Indeed, putting  $x_i = 0$  for all  $i$  in (3.1) we have  $f(0) = 0$  since  $\Psi_2(0, \dots, 0) < \infty$  and  $\varphi(0, \dots, 0) = 0$ .

Replacing  $x$  by  $\frac{x}{2}$  in (3.15) we get

$$\left\|f_2(x) - 2f_2\left(\frac{x}{2}\right)\right\| \leq \frac{1}{2m} \varphi_{[\frac{n}{2}]} \left(\frac{x}{2}\right) \tag{3.19}$$

for all  $x \in X$ .

Using the induction argument, we get that

$$\left\| f_2(x) - 2^k f_2\left(\frac{x}{2^k}\right) \right\| \leq \frac{1}{2^m} \sum_{i=0}^{k-1} 2^i \varphi_{\lfloor \frac{n}{2} \rfloor}\left(\frac{x}{2^{i+1}}\right) \quad (3.20)$$

for all  $x \in X$  and for all  $k \in \mathbb{N}$ .

Using the similar argument to that of *Case 2*, we can show that the sequence  $\{2^k f_2(\frac{x}{2^k})\}_{k \in \mathbb{N}}$  is a Cauchy sequence in  $Y$  for all  $x \in X$  and thus converges. Therefore we can define a function  $A : X \rightarrow Y$  by

$$A(x) = \lim_{k \rightarrow \infty} 2^k f_2\left(\frac{x}{2^k}\right) = \lim_{k \rightarrow \infty} 2^{k-1} [f\left(\frac{x}{2^k}\right) - f\left(-\frac{x}{2^k}\right)], \quad x \in X.$$

Taking the limit in (3.20) as  $k \rightarrow \infty$ , we obtain

$$\|f_2(x) - A(x)\| \leq \frac{1}{2^{\lfloor \frac{n}{2} \rfloor}} \Psi_{2^{\lfloor \frac{n}{2} \rfloor}}(x) \quad (3.21)$$

for all  $x \in X$ .

Similarly, noting that  $A(0) = 0$ ,  $A(-x) = -A(x)$  for all  $x \in X$ , we conclude that  $A$  is the unique additive mapping subject to (1.3) and (3.21). The proof is complete.  $\square$

We remark that if the control function  $\varphi$  satisfies conditions (A) and (C), the proof of *Case 1* and *Case 3* is still true under the condition

$$\|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n)$$

for all  $x_1, \dots, x_n \in X \setminus \{0\}$  and thus conclusions in Theorem 3.1 work for all  $x \in X \setminus \{0\}$ . We state the alternative theorem for the even case of  $n$  variables.

**THEOREM 3.2.** *Let  $\varphi$  be as in Theorem 3.1. Assume that a function  $f : X \rightarrow Y$  satisfies*

$$\|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n) \quad (3.22)$$

for all  $x_1, \dots, x_n \in X$  and for some even  $n := 2m$ . Then there exist a unique additive function  $A : X \rightarrow Y$  and a unique quadratic function  $Q : X \rightarrow Y$  satisfying (1.3) such that

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3^{\lfloor \frac{n}{2} \rfloor}} f(0) \right\| &\leq \frac{\Phi_{i_{\lfloor \frac{n}{2} \rfloor}}(x)}{2^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor - 1)} + \frac{\Psi_{j_{\lfloor \frac{n}{2} \rfloor}}(x)}{4^{\lfloor \frac{n}{2} \rfloor - 1}}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3^{\lfloor \frac{n}{2} \rfloor}} f(0) \right\| &\leq \frac{1}{2^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor - 1)} \Phi_{i_{\lfloor \frac{n}{2} \rfloor}}(x), \end{aligned}$$

and

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leq \frac{1}{4^{\lfloor \frac{n}{2} \rfloor - 1}} \Psi_{j_{\lfloor \frac{n}{2} \rfloor}}(x)$$

for all  $x \in X$  and for  $i = 1$  or  $2$ ,  $j = 1$  or  $2$ .

The functions  $Q, A$  are given by

$$\left\{ \begin{array}{ll} Q(x) = \lim_{k \rightarrow \infty} \frac{f(2^k x) + f(-2^k x)}{2 \cdot 4^k} & \text{if (A) holds} \\ Q(x) = \lim_{k \rightarrow \infty} \frac{4^k}{2} [f(\frac{x}{2^k}) + f(-\frac{x}{2^k})], f(0) = 0 & \text{if (B) holds} \\ A(x) = \lim_{k \rightarrow \infty} \frac{f(2^k x) - f(-2^k x)}{2^{k+1}} & \text{if (C) holds} \\ A(x) = \lim_{k \rightarrow \infty} 2^{k-1} [f(\frac{x}{2^k}) - f(-\frac{x}{2^k})], f(0) = 0 & \text{if (D) holds} \end{array} \right.$$

for all  $x \in X$

*Proof.* We use the same notations as Theorem 3.1. The relation (3.22) can be written by the form of  $Df_1, Df_2$  as follows:

$$\|Df_1(x_1, \dots, x_n)\| \leq \frac{1}{2} [\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n)] \tag{3.23}$$

$$\|Df_2(x_1, \dots, x_n)\| \leq \frac{1}{2} [\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n)] \tag{3.24}$$

for all  $x_i \in X$ .

Putting  $x_i = x$  for  $i = 1, \dots, m + 1$  and  $x_i = -x$  for  $i = m + 2, \dots, n$  in (3.23) yields

$$\left\| 4f_1(x) - f_1(2x) - \frac{m + 1}{m} f_1(0) \right\| \leq \frac{1}{2m(m - 1)} \varphi_{[\frac{n}{2}]}(x) \tag{3.25}$$

for all  $x \in X$ , where  $[\frac{n}{2}] = m$ . Applying the same process to (3.24), one obtains that

$$\|2f_2(x) - f_2(2x)\| \leq \frac{1}{4(m - 1)} \varphi_{[\frac{n}{4}]}(x) \tag{3.26}$$

for all  $x \in X$ .

From (3.25) and (3.26), we come to the conclusions, which the arguments used in Theorem 3.1 carry over almost verbatim.  $\square$

From the main Theorem 3.1 and Theorem 3.2, we obtain the following corollary concerning the stability of the equation (1.3). Let  $X$  and  $Y$  be a real normed linear space and a real Banach space, respectively, in the following corollaries.

**COROLLARY 3.3.** *Let  $p \neq 1, p \neq 2$  and  $\epsilon \geq 0$  be real numbers. Assume that a function  $f : X \rightarrow Y$  satisfies the inequality*

$$\|Df(x_1, \dots, x_n)\| \leq \epsilon \left( \sum_{i=1}^n \|x_i\|^p \right), \quad (n > 2) \tag{3.27}$$

for all  $x_i \in X$  ( $x_i \in X \setminus \{0\}$  if  $p < 0$ ). Then for each three cases  $p < 1, 1 < p < 2$  and  $2 < p$ , we can find a unique additive function  $A : X \rightarrow Y$  and a unique quadratic

function  $Q : X \rightarrow Y$  which satisfy (1.3) and the inequalities in case  $n := 2m + 1$

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{(2 \lfloor \frac{n}{2} \rfloor + 1)}{\lfloor \frac{n}{2} \rfloor^2} \frac{\epsilon \|x\|^p}{|4 - 2^p|} + \frac{(2 \lfloor \frac{n}{2} \rfloor + 1)}{\lfloor \frac{n}{2} \rfloor} \frac{\epsilon \|x\|^p}{|2 - 2^p|}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{(2 \lfloor \frac{n}{2} \rfloor + 1)}{\lfloor \frac{n}{2} \rfloor^2} \frac{\epsilon \|x\|^p}{|4 - 2^p|}, \\ \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| &\leq \frac{(2 \lfloor \frac{n}{2} \rfloor + 1)}{\lfloor \frac{n}{2} \rfloor} \frac{\epsilon \|x\|^p}{|2 - 2^p|}, \end{aligned}$$

or in case  $n := 2m$

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{2}{(\lfloor \frac{n}{2} \rfloor - 1) |4 - 2^p|} \frac{\epsilon \|x\|^p}{|4 - 2^p|} + \frac{\lfloor \frac{n}{2} \rfloor}{(\lfloor \frac{n}{2} \rfloor - 1)} \frac{\epsilon \|x\|^p}{|2 - 2^p|}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{2}{(\lfloor \frac{n}{2} \rfloor - 1) |4 - 2^p|} \frac{\epsilon \|x\|^p}{|4 - 2^p|}, \\ \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| &\leq \frac{\lfloor \frac{n}{2} \rfloor}{(\lfloor \frac{n}{2} \rfloor - 1)} \frac{\epsilon \|x\|^p}{|2 - 2^p|}, \end{aligned}$$

for all  $x \in X$  ( $x \in X \setminus \{0\}$  if  $p < 0$ ), where  $f(0) = 0$  if  $p > 1$ .

*Proof.* Let  $\varphi(x_1, \dots, x_n) := \epsilon (\sum_{i=1}^n \|x_i\|^p)$  for all  $x_i \in X$  ( $x \in X \setminus \{0\}$  if  $p < 0$ ) ( $i = 1, \dots, n$ ).

If  $p < 1$ , the control function  $\varphi$  satisfies (A) and (C). If  $1 < p < 2$ , the function  $\varphi$  satisfies (A) and (D). If  $2 < p$ , the function  $\varphi$  satisfies (B) and (D). Thus applying Theorem 3.1 and Theorem 3.2 to the three cases  $p < 1$ ,  $1 < p < 2$  and  $2 < p$ , we obtain easily the following Hyers-Ulam stability results.  $\square$

**COROLLARY 3.4.** Assume that for some  $\theta > 0$ , a function  $f : X \rightarrow Y$  satisfies the inequality

$$\|Df(x_1, \dots, x_n)\| \leq \theta \tag{3.28}$$

for all  $x_i \in X$ . Then there exist a unique additive function  $A : X \rightarrow Y$  and a unique quadratic function  $Q : X \rightarrow Y$  which satisfy (1.3) and the inequalities in case  $n := 2m + 1$

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{\theta}{3 \lfloor \frac{n}{2} \rfloor^2} + \frac{\theta}{\lfloor \frac{n}{2} \rfloor}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3 \lfloor \frac{n}{2} \rfloor} f(0) \right\| &\leq \frac{\theta}{3 \lfloor \frac{n}{2} \rfloor^2}, \\ \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| &\leq \frac{\theta}{\lfloor \frac{n}{2} \rfloor}, \end{aligned}$$

or in case  $n := 2m$

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3^{\lfloor \frac{n}{2} \rfloor}} f(0) \right\| &\leq \frac{\theta}{3^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor - 1)} + \frac{\theta}{2(\lfloor \frac{n}{2} \rfloor - 1)}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\lfloor \frac{n}{2} \rfloor + 1}{3^{\lfloor \frac{n}{2} \rfloor}} f(0) \right\| &\leq \frac{\theta}{3^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor - 1)}, \\ \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| &\leq \frac{\theta}{2(\lfloor \frac{n}{2} \rfloor - 1)}, \end{aligned}$$

for all  $x \in X$ .

*Proof.* Putting  $\varphi(x, y) := \theta$ , we get immediately the results.  $\square$

Let  $H : \overbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}^{n\text{-times}} \rightarrow \mathbb{R}_+$  and  $\varphi_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be mappings such that

$$\begin{aligned} \varphi_0(\lambda) &> 0, \quad \text{for all } \lambda > 0, \\ \varphi_0(2) &< 2, \end{aligned}$$

$$\varphi_0(2\lambda) \leq \varphi_0(2)\varphi_0(\lambda), \quad \text{for all } \lambda > 0,$$

$$H(\lambda t_1, \dots, \lambda t_n) \leq \varphi_0(\lambda)H(t_1, \dots, t_n), \quad \text{for all } t_1, \dots, t_n \in \mathbb{R}_+, \quad \lambda > 0.$$

We consider in the next corollary

$$\varphi(x_1, \dots, x_n) := H(\|x_1\|, \dots, \|x_n\|).$$

Then

$$\begin{aligned} \varphi(2^i x_1, \dots, 2^i x_n) &= H(2^i \|x_1\|, \dots, 2^i \|x_n\|) \\ &\leq \varphi_0(2^i)H(\|x_1\|, \dots, \|x_n\|) \\ &\leq (\varphi_0(2))^i H(\|x_1\|, \dots, \|x_n\|), \end{aligned}$$

and according to  $\varphi_0(2) < 2$  we have

$$\begin{aligned} \Phi_1(x_1, \dots, x_n) &\leq \sum_{i=0}^{\infty} \frac{(\varphi_0(2))^i H(\|x_1\|, \dots, \|x_n\|)}{4^{i+1}} \\ &= \frac{H(\|x_1\|, \dots, \|x_n\|)}{4 - \varphi_0(2)}, \end{aligned}$$

and

$$\begin{aligned} \Psi_1(x_1, \dots, x_n) &\leq \sum_{i=0}^{\infty} \frac{(\varphi_0(2))^i H(\|x_1\|, \dots, \|x_n\|)}{2^{i+1}} \\ &= \frac{H(\|x_1\|, \dots, \|x_n\|)}{2 - \varphi_0(2)}. \end{aligned}$$

Hence, we see that the following corollary holds by Theorem 3.1 and Theorem 3.2.

COROLLARY 3.5. Assume that a function  $f : X \rightarrow Y$  satisfies

$$\|Df(x_1, \dots, x_n)\| \leq H(\|x_1\|, \dots, \|x_n\|)$$

for all  $x_1, \dots, x_n \in X$ . Then there exist a unique additive function  $A : X \rightarrow Y$  and a unique quadratic function  $Q : X \rightarrow Y$  which satisfy (1.3) and the inequalities in case  $n := 2m + 1$

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\left[\frac{n}{2}\right] + 1}{3\left[\frac{n}{2}\right]} f(0) \right\| &\leq \frac{H(\|x_1\|, \dots, \|x_n\|)}{\left[\frac{n}{2}\right]^2(4 - \varphi_0(2))} + \frac{H(\|x_1\|, \dots, \|x_n\|)}{2\left[\frac{n}{2}\right](2 - \varphi_0(2))}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\left[\frac{n}{2}\right] + 1}{3\left[\frac{n}{2}\right]} f(0) \right\| &\leq \frac{H(\|x_1\|, \dots, \|x_n\|)}{\left[\frac{n}{2}\right]^2(4 - \varphi_0(2))}, \\ \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| &\leq \frac{H(\|x_1\|, \dots, \|x_n\|)}{2\left[\frac{n}{2}\right](2 - \varphi_0(2))} \end{aligned}$$

or in case  $n := 2m$

$$\begin{aligned} \left\| f(x) - Q(x) - A(x) - \frac{\left[\frac{n}{2}\right] + 1}{3\left[\frac{n}{2}\right]} f(0) \right\| &\leq \frac{H(\|x_1\|, \dots, \|x_n\|)}{\left[\frac{n}{2}\right](\left[\frac{n}{2}\right] - 1)(4 - \varphi_0(2))} \\ &\quad + \frac{H(\|x_1\|, \dots, \|x_n\|)}{2(\left[\frac{n}{2}\right] - 1)(2 - \varphi_0(2))}, \\ \left\| \frac{f(x) + f(-x)}{2} - Q(x) - \frac{\left[\frac{n}{2}\right] + 1}{3\left[\frac{n}{2}\right]} f(0) \right\| &\leq \frac{H(\|x_1\|, \dots, \|x_n\|)}{\left[\frac{n}{2}\right](\left[\frac{n}{2}\right] - 1)(4 - \varphi_0(2))}, \\ \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| &\leq \frac{H(\|x_1\|, \dots, \|x_n\|)}{2(\left[\frac{n}{2}\right] - 1)(2 - \varphi_0(2))} \end{aligned}$$

for all  $x \in X$ .

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(Received May 19, 2003)

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