

ON THE CAUCHY—RASSIAS INEQUALITY AND LINEAR n -INNER PRODUCT PRESERVING MAPPINGS

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Abstract. We prove the Cauchy–Rassias stability of linear n -inner product preserving mappings in n -inner product Banach spaces. We apply the Cauchy–Rassias inequality that plays an influential role in the subject of functional equations. The inequality was introduced for the first time by Th. M. Rassias in his paper entitled: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72**, (1978), 297–300.

1. Introduction

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \rightarrow Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th. M. Rassias [23] introduced the following inequality, that we call *Cauchy–Rassias inequality*: Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad (*)$$

for all $x, y \in X$. Rassias [23] showed that there exists a unique \mathbb{R} -linear mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in X$. The inequality (*) has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. Beginning around the year 1980 the topic stability of linear functional equations has been studied by a number of mathematicians (see [7], [8] and [11]–[27]).

Jun and Lee [9] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \rightarrow [0, \infty)$ a function such that

$$\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty$$

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for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x))$$

for all $x \in X \setminus \{0\}$.

Let X and Y be complex Hilbert spaces. A mapping $f : X \rightarrow Y$ is called an *inner product preserving mapping* if f satisfies the *orthogonality equation*

$$\langle f(x), f(y) \rangle = \langle x, y \rangle$$

for all $x, y \in X$. The inner product preserving mapping problem has been investigated in several papers (see [2]–[4]).

In this paper, we prove the Cauchy–Rassias stability of inner product preserving mappings in Hilbert spaces, introduce the concept of n -inner product Banach space and establish the Cauchy–Rassias stability of linear n -inner product preserving mappings in n -inner product Banach spaces.

2. Cauchy–Rassias stability of inner product preserving mappings in Hilbert spaces

In this section, assume that X is a complex Hilbert space with norm $\|\cdot\|$, and that Y is a complex Hilbert space with norm $\|\cdot\|$.

THEOREM 2.1. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X \times X \rightarrow [0, \infty)$ such that*

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty, \quad (2.i)$$

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \varphi(x, y), \quad (2.ii)$$

$$|\langle f(x), f(y) \rangle - \langle x, y \rangle| \leq \varphi(x, y) \quad (2.iii)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y \in X$. Then there exists a unique inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x) \quad (2.iv)$$

for all $x \in X$.

Proof. By the same reasoning as in the proof of [21, Theorem 2.1], it follows from (2.i) and (2.ii) that there exists a unique \mathbb{C} -linear mapping $U : X \rightarrow Y$ satisfying (2.iv). The \mathbb{C} -linear mapping $U : X \rightarrow Y$ is given by

$$U(x) = \lim_{l \rightarrow \infty} \frac{1}{2^l} f(2^l x) \quad (2.1)$$

for all $x \in X$.

It follows from (2.iii) that

$$\begin{aligned} \left| \left\langle \frac{1}{2^l}f(2^l x), \frac{1}{2^l}f(2^l y) \right\rangle - \langle x, y \rangle \right| &= \frac{1}{2^{2l}} \left| \langle f(2^l x), f(2^l y) \rangle - \langle 2^l x, 2^l y \rangle \right| \\ &\leq \frac{1}{2^{2l}} \varphi(2^l x, 2^l y) \leq \frac{1}{2^l} \varphi(2^l x, 2^l y), \end{aligned}$$

which tends to zero as $l \rightarrow \infty$ for all $x, y \in X$ by (2.i). By (2.1),

$$\langle U(x), U(y) \rangle = \lim_{l \rightarrow \infty} \left\langle \frac{1}{2^l}f(2^l x), \frac{1}{2^l}f(2^l y) \right\rangle = \langle x, y \rangle$$

for all $x, y \in X$, as desired. \square

COROLLARY 2.2. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \left| \langle f(x), f(y) \rangle - \langle x, y \rangle \right| &\leq \theta(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X$. Then there exists a unique inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ to be Th. M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 2.1. \square

THEOREM 2.3. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X \setminus \{0\} \times X \setminus \{0\} \rightarrow [0, \infty)$ such that*

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty, \tag{2.v}$$

$$\|2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y)\| \leq \varphi(x, y), \tag{2.vi}$$

$$\left| \langle f(x), f(y) \rangle - \langle x, y \rangle \right| \leq \varphi(x, y) \tag{2.vii}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X \setminus \{0\}$. Then there exists a unique inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)) \tag{2.viii}$$

for all $x \in X \setminus \{0\}$.

Proof. By the same reasoning as in the proof of [11, Theorem 2.5], it follows from (2.v) and (2.vi) that there exists a unique \mathbb{C} -linear mapping $U : X \rightarrow Y$ satisfying (2.viii). The \mathbb{C} -linear mapping $U : X \rightarrow Y$ is given by

$$U(x) = \lim_{l \rightarrow \infty} \frac{1}{3^l} f(3^l x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.vii) that

$$\begin{aligned} \left| \left\langle \frac{1}{3^l}f(3^l x), \frac{1}{3^l}f(3^l y) \right\rangle - \langle x, y \rangle \right| &= \frac{1}{3^{2l}} \left| \langle f(3^l x), f(3^l y) \rangle - \langle 3^l x, 3^l y \rangle \right| \\ &\leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), \end{aligned}$$

which tends to zero as $l \rightarrow \infty$ for all $x, y \in X$ by (2.v). By (2.2),

$$\langle U(x), U(y) \rangle = \lim_{l \rightarrow \infty} \left\langle \frac{1}{3^l}f(3^l x), \frac{1}{3^l}f(3^l y) \right\rangle = \langle x, y \rangle$$

for all $x, y \in X$, as desired. \square

COROLLARY 2.4. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) \right\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \left| \langle f(x), f(y) \rangle - \langle x, y \rangle \right| &\leq \theta(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X \setminus \{0\}$. Then there exists a unique inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{3 + 3^p}{3 - 3^p} \theta \|x\|^p$$

for all $x \in X \setminus \{0\}$.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 2.3. \square

THEOREM 2.5. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X \setminus \{0\} \times X \setminus \{0\} \rightarrow [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} 3^{2j} \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) < \infty, \quad (2.ix)$$

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) \right\| \leq \varphi(x, y), \quad (2.x)$$

$$\left| \langle f(x), f(y) \rangle - \langle x, y \rangle \right| \leq \varphi(x, y) \quad (2.xi)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X \setminus \{0\}$. Then there exists a unique inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \tilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}\right) + \tilde{\varphi}\left(-\frac{x}{3}, x\right) \quad (2.xii)$$

for all $x \in X \setminus \{0\}$, where

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 3^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right)$$

for all $x, y \in X \setminus \{0\}$.

Proof. Note that

$$3^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) \leq 3^{2j} \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) \tag{2.3}$$

for all $x, y \in X$ and all positive integers j . By the Jun and Lee's theorem [9, Theorem 6], it follows from (2.ix), (2.3) and (2.x) that there exists a unique additive mapping $U : X \rightarrow Y$ satisfying (2.xii). The additive mapping $U : X \rightarrow Y$ is given by

$$U(x) = \lim_{l \rightarrow \infty} 3^l f\left(\frac{x}{3^l}\right) \tag{2.4}$$

for all $x \in X$. By the same method as in the proof of [11, Theorem 2.5], one can show that the additive mapping $U : X \rightarrow Y$ is \mathbb{C} -linear.

It follows from (2.xi) that

$$\begin{aligned} | \langle 3^l f\left(\frac{x}{3^l}\right), 3^l f\left(\frac{y}{3^l}\right) \rangle - \langle x, y \rangle | &= 3^{2l} | \langle f\left(\frac{x}{3^l}\right), f\left(\frac{y}{3^l}\right) \rangle - \langle \frac{x}{3^l}, \frac{y}{3^l} \rangle | \\ &\leq 3^{2l} \varphi\left(\frac{x}{3^l}, \frac{y}{3^l}\right), \end{aligned}$$

which tends to zero as $l \rightarrow \infty$ for all $x, y \in X$ by (2.ix). By (2.4),

$$\langle U(x), U(y) \rangle = \lim_{l \rightarrow \infty} \langle 3^l f\left(\frac{x}{3^l}\right), 3^l f\left(\frac{y}{3^l}\right) \rangle = \langle x, y \rangle$$

for all $x, y \in X$, as desired. \square

COROLLARY 2.6. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in (2, \infty)$ such that*

$$\begin{aligned} \|2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ | \langle f(x), f(y) \rangle - \langle x, y \rangle | &\leq \theta(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X \setminus \{0\}$. Then there exists a unique inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{3^p + 3}{3^p - 3} \theta \|x\|^p$$

for all $x \in X \setminus \{0\}$.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 2.5. \square

3. Cauchy–Rassias stability of linear n -inner product preserving mappings in n -inner product Banach spaces

We first recall the notion of n -inner product space.

DEFINITION 3.1 [5]. Let X be a complex linear space with $\dim X \geq n \geq 2$ and $\langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle : X^{n+1} \rightarrow \mathbb{C}$ be a function. Then $(X, \langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle)$ is called an n -inner product space if

- (nI₁) $\langle x, x \mid x_2, \dots, x_n \rangle \geq 0$,
- (nI₂) $\langle x, x \mid x_2, \dots, x_n \rangle = 0$ if and only if x, x_2, \dots, x_n are linearly dependent,
- (nI₃) $\langle x, y \mid x_2, \dots, x_n \rangle = \overline{\langle y, x \mid x_2, \dots, x_n \rangle}$,
- (nI₄) $\langle x, y \mid x_2, \dots, x_n \rangle = \langle x, y \mid x_{j_2}, \dots, x_{j_n} \rangle$ for every permutation (j_2, \dots, j_n) of $(2, \dots, n)$,
- (nI₅) $\langle x, x \mid x_2, x_3, \dots, x_n \rangle = \langle x_2, x_2 \mid x, x_3, \dots, x_n \rangle$,
- (nI₆) $\langle \alpha x, y \mid x_2, \dots, x_n \rangle = \alpha \langle x, y \mid x_2, \dots, x_n \rangle$,
- (nI₇) $\langle x + y, z \mid x_2, \dots, x_n \rangle \leq \langle x, z \mid x_2, \dots, x_n \rangle + \langle y, z \mid x_2, \dots, x_n \rangle$

for all $\alpha \in \mathbb{C}$ and all $x, y, z, x_1, \dots, x_n \in X$. The function $\langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle$ is called an n -inner product on X .

The concept of n -inner product was introduced in [1] for $n = 2$ and in [10] for $n \geq 2$.

For instance, on a given inner product space with inner product $\langle \cdot, \cdot \rangle$, one can put an n -inner product by defining $\langle x, y \mid x_2, \dots, x_n \rangle$ to be

$$\det \begin{pmatrix} \langle x, y \rangle & \langle x, x_2 \rangle & \dots & \langle x, x_n \rangle \\ \langle x_2, y \rangle & \langle x_2, x_2 \rangle & \dots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle x_n, y \rangle & \langle x_n, x_2 \rangle & \dots & \langle x_n, x_n \rangle \end{pmatrix}$$

DEFINITION 3.2. An n -inner product and normed (respectively, Banach) space X with n -inner product $\langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle_X$ and norm $\| \cdot \|$ is called an n -inner product normed (respectively, Banach) space.

For example, if $(X, \langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle)$ is an n -inner product space, $\{a_1, \dots, a_n\}$ is a fixed linearly independent set and $k > 0$ then

$$\langle x, y \rangle := k \sum_{\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}} \langle x, y \mid a_{i_2}, \dots, a_{i_n} \rangle$$

is an inner product on X (see [6]) and so X equipped with the norm induced by this inner product is an n -inner product normed space.

In the rest of this section, assume that X is an n -inner product Banach space with n -inner product $\langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle_X$ and norm $\| \cdot \|$, and that Y is an n -inner product Banach space with n -inner product $\langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle_Y$ and norm $\| \cdot \|$.

THEOREM 3.1. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^{n+1} \rightarrow [0, \infty)$ such that*

$$\tilde{\varphi}(x_0, \dots, x_n) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_0, \dots, 2^j x_n) < \infty, \tag{3.i}$$

$$\|f(\mu x_0 + \mu x_1) - \mu f(x_0) - \mu f(x_1)\| \leq \varphi(x_0, x_1, \underbrace{0, \dots, 0}_{n-1 \text{ times}}), \tag{3.ii}$$

$$|\langle f(x_0), f(x_1) \mid f(x_2), \dots, f(x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X| \leq \varphi(x_0, \dots, x_n) \tag{3.iii}$$

for all $\mu \in \mathbb{T}^1$ and all $x_0, \dots, x_n \in X$. Then there exists a unique \mathbb{C} -linear n -inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x, \underbrace{0, \dots, 0}_{n-1 \text{ times}}) \tag{3.iv}$$

for all $x \in X$.

Proof. By the same reasoning as in the proof of [20, Theorem 2.1], it follows from (3.i) and (3.ii) that there exists a unique \mathbb{C} -linear mapping $U : X \rightarrow Y$ satisfying (3.iv). The \mathbb{C} -linear mapping $U : X \rightarrow Y$ is given by

$$U(x) = \lim_{l \rightarrow \infty} \frac{1}{2^l} f(2^l x) \tag{3.1}$$

for all $x \in X$.

It follows from (3.iii) that

$$\begin{aligned} &|\langle \frac{1}{2^l} f(2^l x_0), \frac{1}{2^l} f(2^l x_1) \mid \frac{1}{2^l} f(2^l x_2), \dots, \frac{1}{2^l} f(2^l x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X| \\ &= \frac{1}{2^{2nl}} |\langle f(2^l x_0), f(2^l x_1) \mid f(2^l x_2), \dots, f(2^l x_n) \rangle_Y - \langle 2^l x_0, 2^l x_1 \mid 2^l x_2, \dots, 2^l x_n \rangle_X| \\ &\leq \frac{1}{2^{2nl}} \varphi(2^l x_0, \dots, 2^l x_n) \leq \frac{1}{2^l} \varphi(2^l x_0, \dots, 2^l x_n), \end{aligned}$$

which tends to zero as $l \rightarrow \infty$ for all $x_0, \dots, x_n \in X$ by (3.i). By (3.1),

$$\begin{aligned} &\langle U(x_0), U(x_1) \mid U(x_2), \dots, U(x_n) \rangle_Y \\ &= \lim_{l \rightarrow \infty} \langle \frac{1}{2^l} f(2^l x_0), \frac{1}{2^l} f(2^l x_1) \mid \frac{1}{2^l} f(2^l x_2), \dots, \frac{1}{2^l} f(2^l x_n) \rangle_Y \\ &= \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \end{aligned}$$

for all $x_0, \dots, x_n \in X$, as desired. \square

COROLLARY 3.2. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(\mu x_0 + \mu x_1) - \mu f(x_0) - \mu f(x_1)\| \leq \theta(\|x_0\|^p + \|x_1\|^p),$$

$$|\langle f(x_0), f(x_1) \mid f(x_2), \dots, f(x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X| \leq \theta \sum_{j=0}^n \|x_j\|^p$$

for all $\mu \in \mathbb{T}^1$ and all $x_0, \dots, x_n \in X$. Then there exists a unique \mathbb{C} -linear n -inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. Define $\varphi(x_0, \dots, x_n) = \theta \sum_{j=0}^n \|x_j\|^p$ to be Th. M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 3.1. \square

THEOREM 3.3. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : (X \setminus \{0\})^{n+1} \rightarrow [0, \infty)$ such that

$$\tilde{\varphi}(x_0, \dots, x_n) := \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x_0, \dots, 3^j x_n) < \infty, \quad (3.v)$$

$$\|2f\left(\frac{\mu x_0 + \mu x_1}{2}\right) - \mu f(x_0) - \mu f(x_1)\| \leq \varphi(x_0, x_1, \underbrace{0, \dots, 0}_{n-1 \text{ times}}), \quad (3.vi)$$

$$|\langle f(x_0), f(x_1) \mid f(x_2), \dots, f(x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X| \leq \varphi(x_0, \dots, x_n) \quad (3.vii)$$

for all $\mu \in \mathbb{T}^1$ and all $x_0, \dots, x_n \in X \setminus \{0\}$. Then there exists a unique \mathbb{C} -linear n -inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{1}{3} \left(\tilde{\varphi}(x, -x, \underbrace{0, \dots, 0}_{n-1 \text{ times}}) + \tilde{\varphi}(-x, 3x, \underbrace{0, \dots, 0}_{n-1 \text{ times}}) \right) \quad (3.viii)$$

for all $x \in X \setminus \{0\}$.

Proof. By the same reasoning as in the proof of [20, Theorem 2.5], it follows from (3.v) and (3.vi) that there exists a unique \mathbb{C} -linear mapping $U : X \rightarrow Y$ satisfying (3.viii). The \mathbb{C} -linear mapping $U : X \rightarrow Y$ is given by

$$U(x) = \lim_{l \rightarrow \infty} \frac{1}{3^l} f(3^l x) \quad (3.2)$$

for all $x \in X$.

It follows from (3.vii) that

$$\begin{aligned} & \left| \left\langle \frac{1}{3^l}f(3^l x_0), \frac{1}{3^l}f(3^l x_1) \mid \frac{1}{3^l}f(3^l x_2), \dots, \frac{1}{3^l}f(3^l x_n) \right\rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \right| \\ &= \frac{1}{3^{2nl}} \left| \langle f(3^l x_0), f(3^l x_1) \mid f(3^l x_2), \dots, f(3^l x_n) \rangle_Y - \langle 3^l x_0, 3^l x_1 \mid 3^l x_2, \dots, 3^l x_n \rangle_X \right| \\ &\leq \frac{1}{3^{2nl}} \varphi(3^l x_0, \dots, 3^l x_n) \leq \frac{1}{3^l} \varphi(3^l x_0, \dots, 3^l x_n), \end{aligned}$$

which tends to zero as $l \rightarrow \infty$ for all $x_0, \dots, x_n \in X$ by (3.v). By (3.2),

$$\begin{aligned} & \langle U(x_0), U(x_1) \mid U(x_2), \dots, U(x_n) \rangle_Y \\ &= \lim_{l \rightarrow \infty} \left\langle \frac{1}{3^l}f(3^l x_0), \frac{1}{3^l}f(3^l x_1) \mid \frac{1}{3^l}f(3^l x_2), \dots, \frac{1}{3^l}f(3^l x_n) \right\rangle_Y \\ &= \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \end{aligned}$$

for all $x_0, \dots, x_n \in X$, as desired. \square

COROLLARY 3.4. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|2f\left(\frac{\mu x_0 + \mu x_1}{2}\right) - \mu f(x_0) - \mu f(x_1)\| \leq \theta(\|x_0\|^p + \|x_1\|^p),$$

$$\left| \langle f(x_0), f(x_1) \mid f(x_2), \dots, f(x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \right| \leq \theta \sum_{j=0}^n \|x_j\|^p$$

for all $\mu \in \mathbb{T}^1$ and all $x_0, \dots, x_n \in X \setminus \{0\}$. Then there exists a unique \mathbb{C} -linear n -inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{3 + 3^p}{3 - 3^p} \theta \|x\|^p$$

for all $x \in X$.

Proof. Define $\varphi(x_0, \dots, x_n) = \theta \sum_{j=0}^n \|x_j\|^p$, and apply Theorem 3.3. \square

THEOREM 3.5. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : (X \setminus \{0\})^{n+1} \rightarrow [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} 3^{2nj} \varphi\left(\frac{x_0}{3^j}, \dots, \frac{x_n}{3^j}\right) < \infty, \tag{3.ix}$$

$$\|2f\left(\frac{\mu x_0 + \mu x_1}{2}\right) - \mu f(x_0) - \mu f(x_1)\| \leq \varphi(x_0, x_1, \underbrace{0, \dots, 0}_{n-1 \text{ times}}), \tag{3.x}$$

$$\left| \langle f(x_0), f(x_1) \mid f(x_2), \dots, f(x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \right| \leq \varphi(x_0, \dots, x_n) \tag{3.xi}$$

for all $\mu \in \mathbb{T}^1$ and all $x_0, \dots, x_n \in X \setminus \{0\}$. Then there exists a unique \mathbb{C} -linear n -inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \tilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}, \underbrace{0, \dots, 0}_{n-1 \text{ times}}\right) + \tilde{\varphi}\left(-\frac{x}{3}, x, \underbrace{0, \dots, 0}_{n-1 \text{ times}}\right) \tag{3.xii}$$

for all $x \in X \setminus \{0\}$, where

$$\tilde{\varphi}(x_0, \dots, x_n) := \sum_{j=0}^{\infty} 3^j \varphi\left(\frac{x_0}{3^j}, \dots, \frac{x_n}{3^j}\right)$$

for all $x_0, \dots, x_n \in X$.

Proof. Note that

$$3^j \varphi\left(\frac{x_0}{3^j}, \dots, \frac{x_n}{3^j}\right) \leq 3^{2nj} \varphi\left(\frac{x_0}{3^j}, \dots, \frac{x_n}{3^j}\right) \tag{3.3}$$

for all $x_0, \dots, x_n \in X$ and all positive integers j . By the same reasoning as in the proof of Theorem 2.5, it follows from (3.ix), (3.3) and (3.x) that there exists a unique \mathbb{C} -linear mapping $U : X \rightarrow Y$ satisfying (3.xii). The \mathbb{C} -linear mapping $U : X \rightarrow Y$ is given by

$$U(x) = \lim_{l \rightarrow \infty} 3^l f\left(\frac{x}{3^l}\right) \tag{3.4}$$

for all $x \in X$.

It follows from (3.xi) that

$$\begin{aligned} & \left| \left\langle 3^l f\left(\frac{x_0}{3^l}\right), 3^l f\left(\frac{x_1}{3^l}\right) \mid 3^l f\left(\frac{x_2}{3^l}\right), \dots, 3^l f\left(\frac{x_n}{3^l}\right) \right\rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \right| \\ &= 3^{2nl} \left| \langle f\left(\frac{x_0}{3^l}\right), f\left(\frac{x_1}{3^l}\right) \mid f\left(\frac{x_2}{3^l}\right), \dots, f\left(\frac{x_n}{3^l}\right) \rangle_Y - \langle \frac{x_0}{3^l}, \frac{x_1}{3^l} \mid \frac{x_2}{3^l}, \dots, \frac{x_n}{3^l} \rangle_X \right| \\ &\leq 3^{2nl} \varphi\left(\frac{x_0}{3^l}, \dots, \frac{x_n}{3^l}\right), \end{aligned}$$

which tends to zero as $l \rightarrow \infty$ for all $x_0, \dots, x_n \in X$ by (3.ix). By (3.4),

$$\begin{aligned} & \langle U(x_0), U(x_1) \mid U(x_2), \dots, U(x_n) \rangle_Y \\ &= \lim_{l \rightarrow \infty} \left\langle 3^l f\left(\frac{x_0}{3^l}\right), 3^l f\left(\frac{x_1}{3^l}\right) \mid 3^l f\left(\frac{x_2}{3^l}\right), \dots, 3^l f\left(\frac{x_n}{3^l}\right) \right\rangle_Y \\ &= \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \end{aligned}$$

for all $x_0, \dots, x_n \in X$, as desired. \square

COROLLARY 3.6. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in (2n, \infty)$ such that

$$\|2f\left(\frac{\mu x_0 + \mu x_1}{2}\right) - \mu f(x_0) - \mu f(x_1)\| \leq \theta(\|x_0\|^p + \|x_1\|^p),$$

$$\left| \langle f(x_0), f(x_1) \mid f(x_2), \dots, f(x_n) \rangle_Y - \langle x_0, x_1 \mid x_2, \dots, x_n \rangle_X \right| \leq \theta \sum_{j=0}^n \|x_j\|^p$$

for all $\mu \in \mathbb{T}^1$ and all $x_0, \dots, x_n \in X \setminus \{0\}$. Then there exists a unique \mathbb{C} -linear n -inner product preserving mapping $U : X \rightarrow Y$ such that

$$\|f(x) - U(x)\| \leq \frac{3^p + 3}{3^p - 3} \theta \|x\|^p$$

for all $x \in X \setminus \{0\}$.

Proof. Define $\varphi(x_0, \dots, x_n) = \theta \sum_{j=0}^n \|x_j\|^p$, and apply Theorem 3.5. \square

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REFERENCES

- [1] C. DIMINNIE, S. GÄHLER AND A. WHITE, *2-inner product spaces*, Demonstratio Math. **6** (1973), 525–536.
- [2] J. CHMIELIŃSKI, *On a singular case in the Hyers–Ulam–Rassias stability of the Wigner equation*, J. Math. Anal. Appl. **289** (2004), 571–583.
- [3] J. CHMIELIŃSKI, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl. (to appear).
- [4] J. CHMIELIŃSKI, S. JUNG, *The stability of the Wigner equation on a restricted domain*, J. Math. Anal. Appl. **254** (2001), 309–320.
- [5] Y. J. CHO, P. C. S. LIN, S. S. KIM AND A. MISIAK, *Theory of 2-Inner Product Spaces*, Nova Science Publ., New York, 2001.
- [6] H. GUNAWAN, *An inner product space that makes a set of vectors orthonormal*, J. Austral. Math. Soc. Gaz. **28** (2001), 194–197.
- [7] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *Stability of Functional Equations in Several Variables*, Birkhauser, Boston, Basel, Berlin, 1998.
- [8] D. H. HYERS, TH. M. RASSIAS, *Approximate homomorphisms*, Aequationes Mathematicae **44** (1992), 125–153.
- [9] K. JUN, Y. LEE, *A generalization of the Hyers–Ulam–Rassias stability of Jensen’s equation*, J. Math. Anal. Appl. **238** (1999), 305–315.
- [10] A. MISIAK, *n -inner product spaces*, Math. Nachr. **140** (1989), 299–319.
- [11] C. PARK, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.
- [12] C. PARK, *Linear functional equations in Banach modules over a C^* -algebra*, Acta Appl. Math. **77** (2003), 125–161.
- [13] C. PARK, *Modified Trif’s functional equations in Banach modules over a C^* -algebra and approximate algebra homomorphisms*, J. Math. Anal. Appl. **278** (2003), 93–108.
- [14] C. PARK, *Lie $*$ -homomorphisms between Lie C^* -algebras and Lie $*$ -derivations on Lie C^* -algebras*, J. Math. Anal. Appl. **293** (2004), 419–434.
- [15] C. PARK, *Universal Jensen’s equations in Banach modules over a C^* -algebra and its unitary group*, Acta Math. Sinica **20** (2004), 1047–1056.
- [16] C. PARK, *Homomorphisms between Lie JC^* -algebras and Cauchy–Rassias stability of Lie JC^* -algebra derivations*, J. Lie Theory **15** (2005), 393–414.
- [17] C. PARK, *Isomorphisms between unital C^* -algebras*, Math. Ineq. Appl. (to appear).
- [18] C. PARK, *Positive linear operators in C^* -algebras*, Math. Ineq. Appl. (to appear).
- [19] C. PARK, *Homomorphisms between C^* -algebras and linear derivations on C^* -algebras*, Math. Ineq. Appl. (to appear).
- [20] C. PARK, *Homomorphisms between Poisson JC^* -algebras*, Bull. Braz. Math. Soc. (to appear).
- [21] C. PARK, J. HOU, *Homomorphisms between C^* -algebras associated with the Trif functional equation and linear derivations on C^* -algebras*, J. Korean Math. Soc. **41** (2004), 461–477.
- [22] A. PRASTARO, TH. M. RASSIAS, *Ulam stability in geometry of PDE’s*, Nonlinear Functional Analysis and Applications **8 yr 2003**, 259–278.

- [23] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [24] TH. M. RASSIAS, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
- [25] TH. M. RASSIAS, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
- [26] TH. M. RASSIAS, *On the stability of functional equations and a problem of Ulam*, Acta Applicandae Mathematicae **62** (2000), 23–130.
- [27] TH. M. RASSIAS, P. ŠEMRL, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.

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