

SOME NEW NONLINEAR RETARDED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

QING-HUA MA AND JOSIP PEČARIĆ

(communicated by D. Bainov)

Abstract. In the present paper, priori bound on certain new retarded integral inequalities in two independent variables are established. Applications are given to illustrate the usefulness of these inequalities.

1. Introduction

It is well known that integral inequalities play a fundamental role in the theory of differential and integral equations. Over decades, many investigators have discovered a lot of useful and interesting integral inequalities in order to achieve various goals, see [1-15] and the references cited therein. Recently, Lipovan [4, 5] proved some retarded Gronwall-like and Gronwall-Ou-Iang type inequalities and presented some of their applications to quantitative study of delay differential equations. In [14], Sun generalized Lipovan's results and also given some interesting applications to delay differential equations. The aim of the present paper is to establish some more general two independent variables retarded version of above inequalities which on the one hand generalize some existing results and the other hand can be used as tools to study the behavior of solutions of certain general retarded differential and integral equations. Some applications of our results are also given.

2. Main results

Throughout, we define $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, $I_1 = [x_0, X)$ and $I_2 = [y_0, Y)$ two given intervals of R ; $\Delta = I_1 \times I_2$ and denote by $C^i(M, S)$ the class of all i -times continuously differentiable functions defined on set M with range in the set S ($i = 1, 2, \dots$) and $C^0(M, S) = C(M, S)$. The first-order partial derivative of a function $Z(x, y)$ for $x, y \in R^2$ with respect to x and y are denoted as usual by $D_1Z(x, y)$ and $D_2Z(x, y)$, respectively.

Mathematics subject classification (2000): 26D15, 26D20.

Key words and phrases: retarded integral inequality, two independent variables, priori bound, partial differential equation.

The first author's research was supported by NSF of Guangdong Province (Project No.011471) and the Research Grants Council of the Guangdong University of Foreign Studies (Project No. gw2005-1-04) of China.

THEOREM 2.1. Let $a_i, b_j \in C(\Delta, \mathbb{R}_+)$, $\alpha_{1i}, \alpha_{2j} \in C^1(I_1, I_1)$, $\beta_{1i}, \beta_{2j} \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_{1i}(x), \alpha_{2j}(x) \leq x$ on I_1 , $\beta_{1i}(y), \beta_{2j}(y) \leq y$ on I_2 , $i = 1, \dots, m$, $j = 1, \dots, n$, $w \in C(\mathbb{R}_0, \mathbb{R}_+)$ be a nondecreasing function with $w(u) > 0$ for $u > 0$ and $p > q \geq 0$ and $k \geq 0$ be constants.

If $u \in C(\alpha, \mathbb{R}_+)$ and

$$\begin{aligned} u^p(x, y) \leq & k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^q(s, t) dt ds \\ & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^q(s, t) w(u(s, t)) dt ds \end{aligned} \quad (1)$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \left\{ G^{-1} \left[G \left(k^{\frac{p-q}{p}} + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) \right) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right] \right\}^{\frac{1}{p-q}} \quad (2)$$

for $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1$, where

$$A_{1i}(x, y) = \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) dt ds, \quad i = 1, \dots, m, \quad (3)$$

$$B_{2j}(x, y) = \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) dt ds, \quad j = 1, \dots, n, \quad (4)$$

$$G(r) = \int_{r_0}^r \frac{1}{w(s^{\frac{1}{p-q}})} ds, \quad r \geq r_0 > 0, \quad (5)$$

G^{-1} denotes the inverse function of G , and real numbers $x_1 \in I_1, y_1 \in I_2$ are chosen so that the quantity in the square brackets of (2) is in the range of G .

Proof. Let $k > 0$ and define a positive nondecreasing function

$$\begin{aligned} r_1(x, y) = & k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^q(s, t) dt ds \\ & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^q(s, t) w(u(s, t)) dt ds, \end{aligned} \quad (6)$$

for $(x, y) \in \Delta$, then from (1) we have

$$u(x, y) \leq r_1^{\frac{1}{p}}(x, y) \quad (7)$$

and

$$\begin{aligned} D_1 r_1(x, y) = & \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) u^q(\alpha_{1i}(x), t) dt \\ & + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) u^q(\alpha_{2j}(x), t) w(u(\alpha_{2j}(x), t)) dt \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) r_1^{\frac{q}{p}}(\alpha_{1i}(x), t) dt \\
 &\quad + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) r_1^{\frac{q}{p}}(\alpha_{2j}(x), t) w(r_1^{\frac{1}{p}}(\alpha_{2j}(x), t)) dt \\
 &\leq r_1^{\frac{q}{p}}(\alpha_{1i}(x), \beta_{1i}(y)) \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \\
 &\quad + r_1^{\frac{q}{p}}(\alpha_{2j}(x), \beta_{2j}(y)) \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) w(r_1^{\frac{1}{p}}(\alpha_{2j}(x), t)) dt \\
 &\leq r_1^{\frac{q}{p}}(x, y) \left[\sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \right. \\
 &\quad \left. + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) w(r_1^{\frac{1}{p}}(\alpha_{2j}(x), t)) dt \right],
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 \frac{D_1 r_1(x, y)}{r_1^{\frac{q}{p}}(x, y)} &\leq \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \\
 &\quad + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) w(r_1^{\frac{1}{p}}(\alpha_{2j}(x), t)) dt.
 \end{aligned} \tag{8}$$

Keeping y fixed in (8), setting $x = \sigma$ and then integrating from x_0 to x and changing the variable we obtain

$$\begin{aligned}
 \frac{p}{p-q} r_1^{\frac{p-q}{p}}(x, y) &\leq \frac{p}{p-q} k^{\frac{p-q}{p}} + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) dt ds \\
 &\quad + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w(r_1^{\frac{1}{p}}(s, t)) dt ds.
 \end{aligned} \tag{9}$$

Setting $v_1(x, y) = r_1^{\frac{p-q}{p}}(x, y)$, (9) can be rewritten as

$$\begin{aligned}
 v_1(x, y) &\leq k^{\frac{p-q}{p}} + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) \\
 &\quad + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w(v_1^{\frac{1}{p-q}}(s, t)) dt ds \\
 &:= \bar{A}(x, y) + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w(v_1^{\frac{1}{p-q}}(s, t)) dt ds.
 \end{aligned} \tag{10}$$

Fixing any numbers $\bar{x}_1 (x_0 < \bar{x}_1 \leq x_1)$ and $\bar{y}_1 (x_0 < \bar{y}_1 \leq y_1)$, from (10) we have

$$v_1(x, y) \leq \bar{A}(\bar{x}_1, \bar{y}_1) + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w(v_1^{\frac{1}{p-q}}(s, t)) dt ds, \quad (11)$$

for $x_0 \leq x \leq \bar{x}_1, y_0 \leq y \leq \bar{y}_1$. Define

$$\bar{r}_1(x, y) = \bar{A}(\bar{x}_1, \bar{y}_1) + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w(v_1^{\frac{1}{p-q}}(s, t)) dt ds, \quad (12)$$

for $x_0 \leq x \leq \bar{x}_1, y_0 \leq y \leq \bar{y}_1$, then from (11), (12) we observe that

$$v_1(x, y) \leq \bar{r}_1(x, y) \quad (13)$$

and

$$\begin{aligned} D_1 \bar{r}_1(x, y) &= \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) w(v_1^{\frac{1}{p-q}}(\alpha_{2j}(x), t)) dt \\ &\leq \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) w(v_1^{\frac{1}{p-q}}(\alpha_{2j}(x), \beta_{2j}(y))) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \\ &\leq \frac{p-q}{p} w(v_1^{\frac{1}{p-q}}(x, y)) \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \\ &\leq \frac{p-q}{p} w(\bar{r}_1^{\frac{1}{p-q}}(x, y)) \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \end{aligned}$$

i.e.,

$$\frac{D_1 \bar{r}_1(x, y)}{w(\bar{r}_1^{\frac{1}{p-q}}(x, y))} \leq \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt. \quad (14)$$

From the definition of G and (14) we have

$$D_1 G(\bar{r}_1(x, y)) = \frac{D_1 \bar{r}_1(x, y)}{w(\bar{r}_1^{\frac{1}{p-q}}(x, y))} \leq \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt. \quad (15)$$

Keeping y fixed in (15), setting $x = \sigma$ and then integrating from x_0 to $x, x \in I_1$, making the change of variable $s = \alpha_{2j}(\sigma)$ and using the definition of $\bar{r}_1(x, y)$ we have

$$G(\bar{r}_1(x, y)) \leq G(\bar{A}(\bar{x}_1, \bar{y}_1)) + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) dt ds$$

for $x_0 \leq x \leq \bar{x}_1, y_0 \leq y \leq \bar{y}_1$. Taking $x = \bar{x}_1, y = \bar{y}_1$ in the last inequality and (13), we obtain

$$G(\bar{r}_1(\bar{x}_1, \bar{y}_1)) \leq G(\bar{A}(\bar{x}_1, \bar{y}_1)) + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(\bar{x}_1)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(\bar{y}_1)} b_j(s, t) dt ds$$

and

$$v_1(\bar{x}_1, \bar{y}_1) \leq \bar{r}_1(\bar{x}_1, \bar{y}_1).$$

Since $x_0 < \bar{x}_1 \leq x_1, y_0 < \bar{y}_1 \leq y_1$ are arbitrary, from the last relations we have

$$G(\bar{r}_1(x, y)) \leq G(\bar{A}(x, y)) + \frac{p - q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) dt ds$$

or

$$\bar{r}_1(x, y) \leq G^{-1} \left[G(\bar{A}(x, y)) + \frac{p - q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) dt ds \right] \tag{16}$$

and

$$v_1(x, y) \leq \bar{r}_1(x, y) \tag{17}$$

for $x_0 < x \leq x_1, y_0 < y \leq y_1$. Hence from (16), (17) and (7) we have

$$\begin{aligned} u(x, y) &\leq r_1^{\frac{1}{p}}(x, y) = v_1^{\frac{1}{p-q}}(x, y) \leq \bar{r}_1^{\frac{1}{p-q}}(x, y) \\ &\leq \left\{ G^{-1} \left[G(\bar{A}(x, y)) + \frac{p - q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) dt ds \right] \right\}^{\frac{1}{p-q}}, \end{aligned} \tag{18}$$

for $x_0 < x \leq x_1, y_0 < y \leq y_1$. By (1), (18) holds also when $x = x_0$ or $y = y_0$. Hence (2) holds for all $k > 0$. By continuity, (2) also holds for any $k \geq 0$. \square

REMARK 1. When $m = n = 1, \alpha_{11}(x) = \alpha_{21}(x) = \alpha(x), \beta_{11}(y) = \beta_{21}(y) = \beta(y)$, Theorem 2.1 is a two independent-variable version of Theorem 2.1 of Sun [14], which can be used in the qualitative analysis of hyperbolic partial differential equations with retarded arguments.

REMARK 2. If

$$\int_{r_0}^{\infty} \frac{1}{w(s^{\frac{1}{p-q}})} ds = \infty,$$

then $G(\infty) = \infty$ and (2) is valid on R_+^2 .

If we let $q = p - 1$ in Theorem 2.1, then we have the following corollaries.

COROLLARY 2.1. Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j} (i = 1, \dots, m; j = 1, \dots, n)$ and w , and the constants p and k be defined as in Theorem 2.1, and

$$\begin{aligned} u^p(x, y) &\leq k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^{p-1}(s, t) dt ds \\ &\quad + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^{p-1}(s, t) w(u(s, t)) dt ds \end{aligned} \tag{19}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \bar{G}^{-1} \left[\bar{G} \left(k^{\frac{1}{p}} + \frac{1}{p} \sum_{i=1}^m A_{1i}(x, y) \right) + \frac{1}{p} \sum_{j=1}^n B_{2j}(x, y) \right] \tag{20}$$

for $x_0 \leq x \leq x_2, y_0 \leq y \leq y_2$, where $A_{1i}(x, y)$ and $B_{2j}(x, y)$ are defined as in (3) and (4) respectively,

$$\bar{G}(r) = \int_{r_0}^r \frac{1}{w(s)} ds, r \geq r_0 > 0, \tag{21}$$

\overline{G}^{-1} denotes the inverse function of \overline{G} , and real numbers $x_2 \in I_1, y_2 \in I_2$ are chosen so that the quantity in the square brackets of (20) is in the range of \overline{G} .

COROLLARY 2.2. *Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$) and constants p and k be defined as in Theorem 2.1, and*

$$\begin{aligned}
 u^p(x, y) \leq & k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^{p-1}(s, t) dt ds \\
 & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^p(s, t) dt ds
 \end{aligned} \tag{22}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \left(k^{\frac{1}{p}} + \frac{1}{p} \sum_{i=1}^m A_{1i}(x, y) \right) \exp \left(\frac{1}{p} \sum_{j=1}^n B_{2j}(x, y) \right) \tag{23}$$

for $(x, y) \in \Delta$, where $A_{1i}(x, y)$ and $B_{2j}(x, y)$ are defined as in (3) and (4) respectively.

REMARK 3. (i) It is interesting to note here that if $k = c^p, a_i(s, t) = p\overline{a}_i(s, t)$ and $b_j(s, t) = p\overline{b}_j(s, t)$ in (19) and (22), then the bound appeared in (20) and (23) on the unknown function $u(x, y)$ has not relation with the parameter p , respectively.

(ii) When $p = 1, a_i(x, y) = 0, i = 1, \dots, m, n = 2, \alpha_{21}(x) = x, \beta_{21}(y) = y$ in Corollary 2.2, we have a Pachpatte’s result [11, Theorem 3 (c_1)].

COROLLARY 2.3. *Let the functions $u, a_i, b_j, \alpha_{1i}, \beta_{1i}$ ($i = 1, \dots, m$) and w , and the constants p, q and k be defined as in Theorem 2.1, and*

$$u^p(x, y) \leq k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^q(s, t) dt ds, \tag{24}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \left(k^{\frac{p-q}{p}} + \frac{p-q}{p} \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) dt ds \right)^{\frac{1}{p-q}} \tag{25}$$

for any $(x, y) \in \Delta$.

REMARK 4. For $q = 1, m = 2, \alpha_{11}(x) = x, \beta_{11}(y) = y$ in Corollary 2.3, we have a Pachpatte’s result [11, Theorem 3 (c_3)].

THEOREM 2.2. *Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$) and w , and the constants p, q and k be defined as in Theorem 2.1, and*

$$\begin{aligned}
 u^p(x, y) \leq & k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^q(s, t) w(u(s, t)) dt ds \\
 & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^q(s, t) w(u(s, t)) dt ds
 \end{aligned} \tag{26}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \left\{ G^{-1} \left[G \left(k \frac{p-q}{p} \right) + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right] \right\}^{\frac{1}{p-q}} \tag{27}$$

for $x_0 \leq x \leq x_3, y_0 \leq y \leq y_3$, where $A_{1i}(x, y)$ and $B_{2j}(x, y)$ are defined as in (3) and (4) respectively, G and G^{-1} are defined as in Theorem 2.1, and real numbers $x_3 \in I_1, y_3 \in I_2$ are chosen so that the quantity in the square brackets of (27) is in the range of G .

Proof. Let $k > 0$, define $r_2(x, y)$ to denote the right-hand side of (26) and $v_2(x, y) = r_2^{\frac{p-q}{p}}(x, y)$, then by same steps in the proofs of Theorem 2.1, we have

$$u(x, y) \leq r_2^{\frac{1}{p}}(x, y) \tag{28}$$

and

$$\begin{aligned} v_2(x, y) \leq & k \frac{p-q}{p} + \frac{p-q}{p} \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) w(v_2^{\frac{1}{p-q}}(s, t)) dt ds \\ & + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w(v_2^{\frac{1}{p-q}}(s, t)) dt ds. \end{aligned} \tag{29}$$

Setting $\bar{r}_2(x, y)$ as the right-hand side of (29), then we have $\bar{r}_2(x_0, y) = k \frac{p-q}{p}$,

$$v_2(x, y) \leq \bar{r}_2(x, y), \tag{30}$$

and

$$\begin{aligned} D_1 \bar{r}_2(x, y) &= \frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) w(v_2^{\frac{1}{p-q}}(\alpha_{1i}(x), t)) dt \\ &+ \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) w(v_2^{\frac{1}{p-q}}(\alpha_{2j}(x), t)) dt \\ &\leq \frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) w(v_2^{\frac{1}{p-q}}(\alpha_{1i}(x), \beta_{1i}(y))) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \\ &+ \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) w(v_2^{\frac{1}{p-q}}(\alpha_{2j}(x), \beta_{2j}(y))) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \\ &\leq \frac{p-q}{p} w(v_2^{\frac{1}{p-q}}(x, y)) \left(\sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \right. \\ &\quad \left. + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{p-q}{p} w(\bar{r}_2^{\frac{1}{p-q}}(x, y)) \left(\sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \right. \\ &\quad \left. + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \right), \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{D_1 \bar{r}_2(x, y)}{w(\bar{r}_2^{\frac{1}{p-q}}(x, y))} &\leq \frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \\ &\quad + \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt. \end{aligned} \tag{31}$$

From the definition of G and (31) we have

$$\begin{aligned} D_1 G(\bar{r}_2(x, y)) &= \frac{D_1 \bar{r}_2(x, y)}{w(\bar{r}_2^{\frac{1}{p-q}}(x, y))} \\ &\leq \frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \\ &\quad + \frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt. \end{aligned} \tag{32}$$

Keeping y fixed in (32), setting $x = \sigma$ and then integrating from x_0 to $x, x \in I_1$, making the change of variables and using the definition of $\bar{r}_2(x, y)$ we get

$$G(\bar{r}_2(x, y)) \leq G(k^{\frac{p-q}{p}}) + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y),$$

i.e.,

$$\bar{r}_2(x, y) \leq G^{-1} \left[G(k^{\frac{p-q}{p}}) + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right], \tag{33}$$

for $x_0 \leq x \leq x_3, y_0 \leq y \leq y_3$. Hence from (28), (29), (30) and (33) we get the desired inequality (27). By continuity, (27) also holds for any $k \geq 0$. \square

REMARK 5. In [11], Pachpatte have studied following nonlinear delay integral inequality

$$\begin{aligned} u(x, y) &\leq k + \int_{x_0}^x \int_{y_0}^y a(s, t) g_1(u(s, t)) dt ds \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s, t) g_2(u(s, t)) dt ds, \end{aligned} \tag{*}$$

but because he added a condition $g_1 \leq g_2$ or $g_2 \leq g_1$, (*) only contain one nonlinear term $g_2(u)$ or $g_1(u)$, respectively. By this fact, for $p = 1, q = 0, m = n = 1, \alpha_{11}(x) = x, \beta_{11}(y) = y, \alpha_{21}(x) = \alpha(x), \beta_{21}(y) = \beta(y)$ in Theorem 2.2, we have a

Pachpatte’s result [11, Theorem 4, (d_1)]; For $p > 1, q = 0, m = n = 1, \alpha_{11}(x) = x, \beta_{11}(y) = y, \alpha_{21}(x) = \alpha(x), \beta_{21}(y) = \beta(y)$ in Theorem 2.2, we get the other one result of Pachpatte [11, Theorem 4 (d_3)].

Let $q = p - 1$ in Theorem 2.2, then we have the following corollaries.

COROLLARY 2.4. *Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$) and w , and the constants p and k be defined as in Theorem 2.2, and*

$$u^p(x, y) \leq k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^{p-1}(s, t) w(u(s, t)) dt ds + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^{p-1}(s, t) w(u(s, t)) dt ds \tag{34}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \overline{G}^{-1} \left[\overline{G}(k^{\frac{1}{p}}) + \frac{1}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right] \tag{35}$$

for $x_0 \leq x \leq x_4, y_0 \leq y \leq y_4$, where $A_{1i}(x, y)$ and $B_{2j}(x, y)$ are defined as in (3) and (4) respectively, \overline{G} and \overline{G}^{-1} are defined as in Corollary 2.1, and real numbers $x_4 \in I_1, y_4 \in I_2$ are chosen so that the quantity in the square brackets of (35) is in the range of \overline{G} .

COROLLARY 2.5. *Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$) and constants p and k be defined as in Theorem 2.1, and*

$$u^p(x, y) \leq k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^p(s, t) dt ds + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^p(s, t) dt ds \tag{36}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq k^{\frac{1}{p}} \exp \left(\frac{1}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{1}{p} \sum_{j=1}^n B_{2j}(x, y) \right) \tag{37}$$

for $(x, y) \in \Delta$, where $A_{1i}(x, y)$ and $B_{2j}(x, y)$ are defined as in (3) and (4) respectively.

REMARK 6. It is interesting to note here that if $k = c^p, a_i(s, t) = p\overline{a}_i(s, t)$ and $b_j(s, t) = p\overline{b}_j(s, t)$ in (34) and (36), then the bound appeared in (35) and (37) on the unknown function $u(x, y)$ has not relation with the parameter p , respectively.

THEOREM 2.3. *Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$) and the constants p, q and k be defined as in Theorem 2.1, $w_i \in C(R_+, R_+)$ be a nondecreasing function with $w_i(u) > 0$ for $u > 0, i = 1, 2$, and*

$$u^p(x, y) \leq k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^q(s, t) w_1(u(s, t)) dt ds + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^q(s, t) w_2(u(s, t)) dt ds \tag{38}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \left\{ \Omega^{-1} \left[\Omega \left(k \frac{p-q}{p} \right) + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right] \right\}^{\frac{1}{p-q}} \quad (39)$$

for $x_0 \leq x \leq x_5, y_0 \leq y \leq y_5$, where $A_{1i}(x, y)$ and $B_{2j}(x, y)$ are defined as in (3) and (4) respectively,

$$\Omega(r) = \int_{r_0}^r \frac{1}{w_1(s^{\frac{1}{p-q}}) + w_2(s^{\frac{1}{p-q}})} ds, r \geq r_0 > 0,$$

Ω^{-1} is the inverse function of Ω and real numbers $x_5 \in I_1, y_5 \in I_2$ are chosen so that the quantity in the square brackets of (39) is in the range of Ω .

Proof. Let $k > 0$, define $r_3(x, y)$ to denote the right-hand side of (38) and $v_3(x, y) = r_3^{\frac{p-q}{p}}(x, y)$, then by same steps in the proofs of Theorem 2.1, we have

$$u(x, y) \leq r_3^{\frac{1}{p}}(x, y) \quad (40)$$

and

$$\begin{aligned} v_3(x, y) &\leq k \frac{p-q}{p} + \frac{p-q}{p} \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) w_1(v_3^{\frac{1}{p-q}}(s, t)) dt ds \\ &\quad + \frac{p-q}{p} \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) w_2(v_3^{\frac{1}{p-q}}(s, t)) dt ds. \end{aligned} \quad (41)$$

Setting $\bar{r}_3(x, y)$ as the right-hand side of (41), then we have $\bar{r}_3(x_0, y) = k \frac{p-q}{p}$,

$$v_3(x, y) \leq \bar{r}_3(x, y), \quad (42)$$

and

$$\begin{aligned} &\frac{D_1 \bar{r}_3(x, y)}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\ &= \frac{\frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) w_1(v_3^{\frac{1}{p-q}}(\alpha_{1i}(x), t)) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\ &\quad + \frac{\frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) w_2(v_3^{\frac{1}{p-q}}(\alpha_{2j}(x), t)) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\ &\leq \frac{\frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) w_1(v_3^{\frac{1}{p-q}}(\alpha_{1i}(x), \beta_{1i}(y))) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \end{aligned}$$

$$\begin{aligned}
 & + \frac{\frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) w(v_3^{\frac{1}{p-q}}(\alpha_{2j}(x), \beta_{2j}(y))) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\
 & \leq \frac{\frac{p-q}{p} \sum_{i=1}^m \alpha'_{1i}(x) w_1(\bar{r}_3^{\frac{1}{p-q}}(\alpha_{1i}(x), \beta_{1i}(y))) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\
 & + \frac{\frac{p-q}{p} \sum_{j=1}^n \alpha'_{2j}(x) w(\bar{r}_3^{\frac{1}{p-q}}(\alpha_{2j}(x), \beta_{2j}(y))) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\
 & \leq \frac{\frac{p-q}{p} w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) \sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\
 & + \frac{\frac{p-q}{p} w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y)) \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt}{w_1(\bar{r}_3^{\frac{1}{p-q}}(x, y)) + w_2(\bar{r}_3^{\frac{1}{p-q}}(x, y))} \\
 & \leq \frac{p-q}{p} \left(\sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \right. \\
 & \quad \left. + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \right).
 \end{aligned}$$

By the definition of Ω , we observe that from the last inequality

$$\begin{aligned}
 D_1 \Omega(\bar{r}_3(x, y)) & \leq \frac{p-q}{p} \left(\sum_{i=1}^m \alpha'_{1i}(x) \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(\alpha_{1i}(x), t) dt \right. \\
 & \quad \left. + \sum_{j=1}^n \alpha'_{2j}(x) \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(\alpha_{2j}(x), t) dt \right).
 \end{aligned}$$

Keeping y fixed, setting $x = \sigma$ and then integrating with respecting to σ form x_0 to x and changing the variables we obtain

$$\Omega(\bar{r}_3(x, y)) \leq \Omega(\bar{r}_3(x_0, y)) + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y).$$

By the definition of $\bar{r}_3(x, y)$ and the last inequality we have

$$\bar{r}_3(x, y) \leq \Omega^{-1} \left[\Omega(k^{\frac{p-q}{p}}) + \frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right] \tag{43}$$

for $x_0 \leq x \leq x_5, y_0 \leq y \leq y_5$.

By (40-43), we get the desired inequality (39). By continuity, (39) also holds for any $k \geq 0$. \square

COROLLARY 2.6. *Let the functions $u, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$) and the constants p, q and k be defined as in Theorem 2.1, and*

$$\begin{aligned}
 u^p(x, y) \leq & k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s, t) u^p(s, t) dt ds \\
 & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s, t) u^{\frac{p+q}{2}}(s, t) dt ds
 \end{aligned} \tag{44}$$

for any $(x, y) \in \Delta$, then

$$u(x, y) \leq \left[(1 + k^{\frac{p-q}{2p}}) \exp \left(\frac{p-q}{p} \sum_{i=1}^m A_{1i}(x, y) + \frac{p-q}{p} \sum_{j=1}^n B_{2j}(x, y) \right) - 1 \right]^2 \tag{45}$$

for any $(x, y) \in \Delta$.

Proof. Let $w_1(u) = u^{p-q}, w_2(u) = u^{\frac{p-q}{2}}$, then we have

$$\Omega(r) = \int_{r_0}^r \frac{ds}{s + s^{\frac{1}{2}}} = 2 \ln \frac{1 + r^{\frac{1}{2}}}{1 + r_0^{\frac{1}{2}}}$$

and

$$\Omega^{-1}(r) = \left[\left(1 + r_0^{\frac{1}{2}} \right) \exp \left(\frac{r}{2} \right) - 1 \right]^2.$$

Now by Theorem 2.3, we have the desired inequality (45).

3. Applications

Consider the initial boundary value problem of hyperbolic partial delay equation

$$\begin{aligned}
 D_2 D_1 u^p(x, y) = & F \left[x, y, u(x-h_{11}(x), y-g_{11}(y)), \dots, u(x-h_{1m}(x), y-g_{1m}(y)), \right. \\
 & \left. u(x-h_{21}(x), y-g_{21}(y)), \dots, u(x-h_{2n}(x), y-g_{2n}(y)) \right],
 \end{aligned} \tag{46}$$

with the given initial boundary conditions

$$u(x, y_0) = c_1(x), u(x_0, y) = c_2(y), c_1(x_0) = c_2(y_0) = 0, \tag{47}$$

where $p \geq 1$ is a constant, $F \in C(\Delta \times R^{m+n}, R)$, $c_1 \in C^1(I_1, R)$, $c_2 \in C^1(I_2, R)$; $h_{1i} \in C(I_1, R_+)$, $g_{2j} \in C(I_2, R_+)$ are nonincreasing, and such that $x - h_{1i}(x) \geq 0$, $x - h_{1i}(x) \in C^1(I_1, I_1)$, $y - g_{2j}(y) \geq 0$, $y - g_{2j}(y) \in C^1(I_2, I_2)$, $h'_{1i}(x) < 1$, $g'_{2j}(y) < 1$, $h_{1i}(x_0) = 0$, $g_{2j}(y_0) = 0$ for $i = 1, \dots, m, j = 1, \dots, n; x \in I_1, y \in I_2$ and

$$M_{1i} = \max_{x \in I_1} \frac{1}{1 - h'_{1i}(x)}, N_{1i} = \max_{y \in I_2} \frac{1}{1 - g'_{1i}(y)}. \tag{48}$$

$$M_{2j} = \max_{x \in I_1} \frac{1}{1 - h'_{2j}(x)}, N_{2j} = \max_{y \in I_2} \frac{1}{1 - g'_{2j}(y)}. \tag{49}$$

Our first result gives the bound on the solution of the problem (46)-(47).

THEOREM 3.1. *Suppose that*

$$|F(x, y, u_{11}, \dots, u_{1m}, u_{21}, \dots, u_{2n})| \leq \sum_{i=1}^m a_i(x, y) |u_{1i}|^{p-1} + \sum_{j=1}^n b_j(x, y) |u_{2j}|^p, \tag{50}$$

$$|c_1(x) + c_2(y)| \leq k, \tag{51}$$

where $a_i(x, y), b_j(x, y), k$ are as in Theorem 2.1. If $u(x, y)$ is any solution of (46) - (47), then

$$|u(x, y)| \leq \left(k^{\frac{1}{p}} + \frac{1}{p} \sum_{i=1}^m \bar{A}_{1i}(x, y) \right) \exp \left(\frac{1}{p} \sum_{j=1}^n \bar{B}_{2j}(x, y) \right) \tag{52}$$

where

$$\bar{A}_{1i}(x, y) = \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} \bar{a}_i(s, t) dt ds, \tag{53}$$

$$\bar{B}_{2j}(x, y) = \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} \bar{b}_j(s, t) dt ds, \tag{54}$$

where $\bar{a}_i(x, y) = M_{1i} N_{1i} a_i(\alpha_{1i}^{-1}(x), \beta_{1i}^{-1}(y)), \bar{b}_j(x, y) = M_{2j} N_{2j} b_j(\alpha_{2j}^{-1}(x), \beta_{2j}^{-1}(y)), \alpha_{1i}(x) = x - h_{1i}, \beta_{1i}(y) = y - g_{1i}(y), \alpha_{2j}(x) = x - h_{2j}(x), \beta_{2j}(y) = y - g_{2j}(y)$.

Proof. It is ease to observe that every solution $u(x, y)$ of (46)-(47) satisfies the equivalent integral equation

$$\begin{aligned} u^p(x, y) = & (x) + c_2(y) + \int_{x_0}^x \int_{y_0}^y F \left[s, t, u(s - h_{11}(s), t - g_{11}(t)), \dots, \right. \\ & u(s - h_{1m}(s), t - g_{1m}(t)), u(s - h_{21}(s), t - g_{21}(t)), \dots, \\ & \left. u(s - h_{2n}(s), t - g_{2n}(t)) \right] dt ds. \end{aligned} \tag{55}$$

Applying (50), (51) to (55) and changing the variables we obtain

$$\begin{aligned} |u(x, y)|^p \leq & k + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} \bar{a}_i(s, t) |u(s, t)|^{p-1} dt ds \\ & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} \bar{b}_j(s, t) |u(s, t)|^p dt ds \end{aligned} \tag{56}$$

An application of Corollary 2.2 to (56) yields (52). \square

The next theorem is about the uniqueness of solutions of (46)-(47).

THEOREM 3.2. Let $M_{1i}, N_{1i}, M_{2j}, N_{2j}, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$), \bar{a}_i, \bar{b}_j be as in Theorem 3.1. Suppose that the function F in (46) satisfies the condition

$$\begin{aligned} & |F(x, y, u_{11}, \dots, u_{1m}, u_{21}, \dots, u_{2n}) - F(x, y, v_{11}, \dots, v_{1m}, v_{21}, \dots, v_{2n})| \\ & \leq \sum_{i=1}^m a_i(x, y) |u_{1i}^p - v_{1i}^p| + \sum_{j=1}^n b_j(x, y) |u_{2j}^p - v_{2j}^p|, \end{aligned} \quad (57)$$

Then the problem (46) - (47) has at most one solution on Δ

Proof. Let $u(x, y)$ and $v(x, y)$ be two solutions of (46)-(47) on Δ , then we have

$$\begin{aligned} u^p(x, y) - v^p(x, y) &= c_1(x) + c_2(y) + \int_{x_0}^x \int_{y_0}^y \left\{ F[s, t, u(s - h_{11}(s), t - g_{11}(t)), \dots, \right. \\ & u(s - h_{1m}(s), t - g_{1m}(t)), u(s - h_{21}(s), t - g_{21}(t)), \dots, u(s - h_{2n}(s), t - g_{2n}(t))] \\ & - F[s, t, v(s - h_{11}(s), t - g_{11}(t)), \dots, v(s - h_{1m}(s), t - g_{1m}(t)), \\ & v(s - h_{21}(s), t - g_{21}(t)), \dots, v(s - h_{2n}(s), t - g_{2n}(t))] \left. \right\} dt ds. \end{aligned} \quad (58)$$

From (58) and (57), making the changing of variables we get

$$\begin{aligned} |u^p(x, y) - v^p(x, y)| &\leq \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} \bar{a}_i(s, t) |u^p(s, t) - v^p(s, t)| dt ds \\ &+ \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} \bar{b}_j(s, t) |u^p(s, t) - v^p(s, t)| dt ds. \end{aligned} \quad (59)$$

An application of Corollary 2.5 to the function $|u^p(x, y) - v^p(x, y)|^{\frac{1}{p}}$ shows that

$$|u^p(x, y) - v^p(x, y)|^{\frac{1}{p}} \leq 0$$

for all $(x, y) \in \Delta$. Hence $u(x, y) = v(x, y)$. \square

Finally, we investigate the continuous dependence of the solutions of problems of (46)-(47) on the function F and the boundary data c_1 and c_2 .

THEOREM 3.3. Let $u(x, y)$ and $v(x, y)$ be the solutions of (46) with the given initial boundary data

$$u(x, y_0) = c_1(x), u(x_0, y) = c_2(y), c_1(x_0) = c_2(y_0) = 0, \quad (60)$$

$$u(x, y_0) = d_1(x), u(x_0, y) = d_2(y), d_1(x_0) = d_2(y_0) = 0, \quad (61)$$

respectively, where $c_1, d_1 \in C^1(I_1, R)$, $c_2, d_2 \in C^1(I_2, R)$. Suppose that the function F satisfies the condition (57) in Theorem 3.2. Let

$$|(c_1(x) - d_1(x)) + (c_2(y) - d_2(y))| < \varepsilon^p \quad (62)$$

where ε is an arbitrary positive number. Let $M_{1i}, N_{1i}, M_{2j}, N_{2j}, a_i, b_j, \alpha_{1i}, \alpha_{2j}, \beta_{1i}, \beta_{2j}$ ($i = 1, \dots, m; j = 1, \dots, n$), \bar{a}_i, \bar{b}_j be as in Theorem 3.1. Then

$$|u^p(x, y) - v^p(x, y)| \leq \varepsilon \exp \left(\frac{1}{p} \sum_{i=1}^m \bar{A}_{1i}(x, y) + \frac{1}{p} \sum_{j=1}^n \bar{B}_{2j}(x, y) \right) \tag{63}$$

for $(x, y) \in \Delta$, where $\bar{A}_{1i}(x, y)$ and $\bar{B}_{2j}(x, y)$ are defined as in (53) and (54) respectively. Hence u^p depends continuously on c_1 and c_2 . In particular, if u does not change sign, it depends continuously on the initial boundary data.

Proof. Let $u(x, y)$ and $v(x, y)$ be solutions of (46)-(60) and (46)-(61), respectively. Then we have

$$\begin{aligned} u^p(x, y) - v^p(x, y) = & c_1(x) - d_1(x) + c_2(y) - d_2(y) + \int_{x_0}^x \int_{y_0}^y \left\{ F \left[s, t, u(s - h_{11}(s), t - g_{11}(t)), \right. \right. \\ & \dots, u(s - h_{1m}(s), t - g_{1m}(t)), u(s - h_{21}(s), t - g_{21}(t)), \dots, u(s - h_{2n}(s), t - g_{2n}(t)) \left. \right] \\ & - F \left[s, t, v(s - h_{11}(s), t - g_{11}(t)), \dots, v(s - h_{1m}(s), t - g_{1m}(t)), \right. \\ & \left. \left. v(s - h_{21}(s), t - g_{21}(t)), \dots, v(s - h_{2n}(s), t - g_{2n}(t)) \right] \right\} dt ds \end{aligned} \tag{64}$$

for $(x, y) \in \Delta$. From (64), (62) and (57), making change of variables we have

$$\begin{aligned} |u^p(x, y) - v^p(x, y)| \leq & \varepsilon^p + \sum_{i=1}^m \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} \bar{a}_i(s, t) |u^p(s, t) - v^p(s, t)| dt ds \\ & + \sum_{j=1}^n \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} \bar{b}_j(s, t) |u^p(s, t) - v^p(s, t)| dt ds. \end{aligned} \tag{65}$$

An application of Corollary 2.5 to the function $|u^p(x, y) - v^p(x, y)|^{\frac{1}{p}}$ shows that

$$|u^p(x, y) - v^p(x, y)| \leq \varepsilon \exp \left(\frac{1}{p} \sum_{i=1}^m \bar{A}_{1i}(x, y) + \frac{1}{p} \sum_{j=1}^n \bar{B}_{2j}(x, y) \right) \tag{66}$$

for $(x, y) \in \Delta$, where $\bar{A}_{1i}(x, y)$ and $\bar{B}_{2j}(x, y)$ are defined as in (53) and (54) respectively.

Now if the functions $\bar{A}_{1i}(x, y)$ and $\bar{B}_{2j}(x, y)$ are bounded for $(x, y) \in \Delta$, so

$$|u^p(x, y) - v^p(x, y)| \leq \varepsilon \cdot K$$

for some $K > 0$ for all $(x, y) \in \Delta$. Hence u^p depends continuously on c_1 and c_2 . \square

In conclusion, we note that the inequalities and applications can be extended easily to functions involving many independent variables.

Acknowledgment. The authors are grateful to the referee for correcting misprints in the initial manuscript and helpful suggestions.

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(Received July 7, 2005)

Qing-Hua Ma
Faculty of Information Science and Technology
Guangdong University of Foreign Studies
Guangzhou 510420
P. R. China
e-mail: gdqhma@21cn.com

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia
e-mail: pecaric@element.hr