Mathematical
Inequalities
& Applications
Volume 10, Number 2 (2007), 351–358

A GENERALIZATION OF HYERS–ULAM–RASSIAS STABILITY OF THE G–FUNCTIONAL EQUATION

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(*communicated by Th. M. Rassias*)

Abstract. We investigate a generalization of the Hyers-Ulam-Rassias stability for the gamma type functional equation $f(x+p) = \Gamma(x)f(x) + \psi(x)$ and the stability in the sense of Ger for the functional equation of the form $f(x+p) = \Gamma(x)f(x)$. As a consequence, we obtain a stability result in the sense of Hyers-Ulam-Rassias for G-functional type equations.

1. Introduction

In 1940, the stability problem raised by S. M. Ulam [19] was solved by D. H. Hyers in [5]. The result of Hyers has been generalized to the unbounded case by Th. M. Rassias [16], and this has been extended by P. Găvruta [3] and R. Ger [4].

The gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad (x > 0)$$

is a solution of the gamma functional equation $g(x+1) = xg(x)$, whose stability is proved by S.-M. Jung ([8], [9], [10]) and G. H. Kim ([11], [12], [13]).

The G-function introduced by E. W. Barnes [2]

$$G(z) = (2\pi)^{\frac{z-1}{2}} e^{-\frac{z(z-1)}{2}} e^{-\gamma\frac{(z-1)^2}{2}} \prod_{k=1}^{\infty} \left[\left(1 + \frac{z-1}{k}\right)^k e^{1-z+\frac{(z-1)^2}{2k}}\right]$$

does satisfy the equation $G(x+1) = \Gamma(x)G(x)$ and $\Gamma(1) = G(1) = 1$, where γ is the Euler-Mascheroni's constant defined by $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) \cong 0.577215664\cdots$.

The properties and values of G-function depend on those of the gamma function. Since the double gamma function Γ_2 is defined by the reciprocal of the G-function(see [2]), $\Gamma_2(x) = 1/G(x)$, and its functional equation can be written in the form $\Gamma_2(x+1) = \Gamma_2(x)/\Gamma(x)$. Therefore the stability problem for the G-function is equivalent to the stability for the reciprocal of the double gamma function.

Mathematics subject classification (2000): 39B52, 39B72, 39B82.

Key words and phrases: functional equation, gamma and beta function, Hyers-Ulam stability, Hyers-Ulam-Rassias stability, p-series.

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Paper MIA-10-32

In this paper, we will investigate a generalization of the Hyers-Ulam stability in the sense of Găvruta and Ger for the functional equations

$$f(x+p) = \Gamma(x)f(x) + \psi(x), \tag{1.1}$$

$$f(x+p) = \Gamma(x)f(x), \tag{1.2}$$

$$f(x+1) = \Gamma(x)f(x), \tag{1.3}$$

where ψ is the given function, while f is the unknown function. The equation (1.3) will be called the G-functional equation because its solution is the G-function.

In section 2, we will study the stability in the sense of Găvruta for the functional equations (1.1), (1.2).

In section 3, we will consider the stability in the sense of Ger for the functional equations (1.2).

Throughout this paper, let R, R_+ and R_* denote the set of real numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively. Each positive real number δ is fixed. The functions $\psi : R_+ \to R_*$ and $\varepsilon : R_+ \to R_*$ are defined.

2. Generalization of Hyers-Ulam-Rassias stability

THEOREM 1. *Let ε be a given function such that*

$$\omega(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+kp)}{\prod_{j=0}^{k} |\Gamma(x+jp)|} < \infty \qquad \forall x \in R_+. \tag{2.1}$$

If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality

$$|f(x+p) - \Gamma(x)f(x) - \psi(x)| \leqslant \varepsilon(x) \qquad \forall x \in R_+, \tag{2.2}$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) with

$$|g(x) - f(x)| \leqslant \omega(x) \qquad \forall x \in R_+. \tag{2.3}$$

Proof. For any $x \in R_+$ and for every positive integer n, let $\omega_n : R_+ \longrightarrow R_*$ and $g_n : R_+ \longrightarrow R_+$ be the functions defined by

$$\omega_n(x) := \sum_{k=0}^{n-1} \frac{\varepsilon(x+kp)}{\prod_{j=0}^{k} |\Gamma(x+jp)|}$$

and

$$g_n(x) := \frac{f(x+np)}{\prod_{j=0}^{n-1} \Gamma(x+jp)} - \sum_{k=0}^{n-1} \frac{\psi(x+kp)}{\prod_{j=0}^{k} \Gamma(x+jp)}$$

for all $x \in R_+$, respectively.

By (2.2), it follows that

$$\left|\frac{f(x+p)}{\Gamma(x)} - f(x) - \frac{\psi(x)}{\Gamma(x)}\right| \leqslant \frac{\varepsilon(x)}{|\Gamma(x)|} \qquad \text{for all} \quad x \in R_+.$$

Substituting x by $x+np$ in this inequality, and then dividing both sides of the obtained inequality by $\prod_{j=0}^{n-1}|\Gamma(x+jp)|$, we get

$$|g_{n+1}(x)-g_n(x)| = \frac{\varepsilon(x+np)}{\prod_{j=0}^{n}|\Gamma(x+jp)|}. \tag{2.4}$$

By induction on n we prove that

$$|g_n(x)-f(x)| \leqslant \omega_n(x) \tag{2.5}$$

for all $x \in R_+$, and for all positive integers n. For the case $n=1$, the inequality (2.5) is an immediate consequence of (2.2).

Assume that the inequality (2.5) holds true for some n. Then we obtain the inequality for $n+1$ by (2.4) in the following way:

$$\begin{aligned}|g_{n+1}(x)-f(x)| &\leqslant |g_{n+1}(x)-g_n(x)| + |g_n(x)-f(x)| \\ &\leqslant \frac{\varepsilon(x+np)}{\prod_{j=0}^{n}|\Gamma(x+jp)|} + \omega_n(x) \\ &= \omega_{n+1}(x).\end{aligned}$$

We claim that $\{g_n(x)\}$ is a Cauchy sequence. Indeed, by (2.4) and (2.1), we have for $n>m$ that

$$\begin{aligned}|g_n(x)-g_m(x)| &\leqslant \sum_{k=m}^{n-1}|g_{k+1}(x)-g_k(x)| \\ &\leqslant \sum_{k=m}^{n-1}\frac{\varepsilon(x+kp)}{\prod_{j=0}^{k}|\Gamma(x+jp)|} \longrightarrow 0\end{aligned}$$

as $m \longrightarrow \infty$.

Hence, we can define a function $g : R_+ \longrightarrow R_+$ by

$$g(x) := \lim_{n\to\infty} g_n(x). \tag{2.6}$$

From the definition of g_n, we have $g_n(x+p) = \Gamma(x)g_{n+1}(x) + \psi(x)$, hence the function g satisfies (1.1).

We show from (2.5) that g satisfies the inequality (2.3) as follows:

$$\begin{aligned}|g(x)-f(x)| &= \lim_{n\to\infty}|g_n(x)-f(x)| \\ &\leqslant \lim_{n\to\infty}\omega_n(x) \\ &= \omega(x) \qquad \forall x \in R_+.\end{aligned}$$

If $h : R_+ \longrightarrow R_+$ is another such function, which satisfies (1.1) and (2.3), then we have

$$\begin{aligned}|g(x) - h(x)| &= |g(x+np) - h(x+np)| \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(x+jp)|} \\ &\leqslant 2\omega_n(x+np) \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(x+jp)|} \\ &= 2\Big(\sum_{k=0}^{\infty} \frac{\varepsilon(x+(n+k)p)}{\prod_{j=0}^{k} |\Gamma(x+(n+j)p)|}\Big) \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(x+jp)|} \\ &= 2\sum_{k=n}^{\infty} \frac{\varepsilon(x+kp)}{\prod_{j=0}^{k} |\Gamma(x+jp)|}\end{aligned}$$

for all $x \in R_+$ and all positive integers n, which tends to zero as $n \to \infty$, since $\omega(x)$ is bounded. This implies the uniqueness of g. $\square$

COROLLARY 1. *If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(x+p) - \Gamma(x)f(x) - \psi(x)| \leqslant \delta \qquad \forall x \in R_+, \tag{2.7}$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) with

$$|g(x) - f(x)| \leqslant \delta\mu(x) \qquad \forall x \in R_+, \tag{2.8}$$

where the function $\mu(x) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{\Gamma(x+jp)}$ for all $x \in R_+$.

In particular, if either $p \geqslant 1$ or $x+p > 2$ in the stability inequality (2.7), then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) with

$$|g(x) - f(x)| \leqslant \begin{cases} \dfrac{\delta\,(\Gamma(x+p)+1+e)}{\Gamma(x)\Gamma(x+p)} & \text{or} \\ \dfrac{\delta\Gamma(x+p)}{\Gamma(x)\,(\Gamma(x+p)-1)}. \end{cases} \tag{2.9}$$

Proof. Set $\varepsilon(x) = \delta$ in Theorem 1. The infinite series $\mu(x)$ satisfies the condition (2.1). Indeed, the sequence of partial sums $\{u_n(x)\}$ defined by

$$\mu_n(x) := \sum_{k=0}^{n} \prod_{j=0}^{k} \frac{1}{\Gamma(x+jp)}$$

is a Cauchy sequence with simple calculation.

In the case of $p \geqslant 1$, the defined function $\mu(x)$ implies

$$\begin{aligned}\sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{\Gamma(x+jp)} &\leqslant \frac{1}{\Gamma(x)\Gamma(x+p)} \left(\Gamma(x+p) + 1 + \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{\Gamma(x+j)}\right) \\ &< \frac{\Gamma(x+p)+1+e}{\Gamma(x)\Gamma(x+p)}.\end{aligned}$$

In the case of $x+p>2$, the defined function $\mu(x)$ implies

$$\sum_{k=0}^{\infty}\prod_{j=0}^{k}\frac{1}{\Gamma(x+jp)} \leqslant \frac{1}{\Gamma(x)}\left(1+\frac{1}{\Gamma(x+p)}+\frac{1}{\Gamma(x+p)^2}+\cdots\right)$$
$$= \frac{\Gamma(x+p)}{\Gamma(x)\left(\Gamma(x+p)-1\right)}. \quad \square$$

Theorem 2 and Corollary 2 follow immediately from Theorem 1 and Corollary 1 with $\psi(x)=0$.

THEOREM 2. *Let the function ε satisfies the condition* (2.1). *If a function $f: R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(x+p)-\Gamma(x)f(x)| \leqslant \varepsilon(x) \qquad \forall x \in R_+, \tag{2.10}$$

then there exists a unique solution $g: R_+ \longrightarrow R_+$ of the equation (1.2) *subject to the condition* (2.3) *for all $x \in R_+$.*

COROLLARY 2. *If a function $f: R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(x+p)-\Gamma(x)f(x)| \leqslant \delta \qquad \forall x \in R_+, \tag{2.11}$$

then there exists a unique solution $g: R_+ \longrightarrow R_+$ of the equation (1.2) *satisfying* (2.8) *for all $x \in R_+$.*

In particular, if either $p \geqslant 1$ or $x+p>2$ in the stability inequality (2.11), *then there exists a unique solution $g: R_+ \longrightarrow R_+$ of the equation* (1.2) *satisfying* (2.9).

REMARK 1. Theorem 2 and Corollary 2 in the case of $p=1$ provide the generalization of the Hyers-Ulam-Rassias stability in the sense of Găvruta and the Hyers-Ulam stability for the G-functional equation (1.3), respectively. For the latter the reader is referred in paper [14].

3. Stability in the sense of Ger for the equation (1.2)

THEOREM 3. *Let a function $f: R_+ \longrightarrow R_+$ satisfies the inequality*

$$\left|\frac{f(x+p)}{\Gamma(x)f(x)}-1\right| \leqslant \varepsilon(x) \qquad \forall x \in R_+, \tag{3.1}$$

where $\varepsilon: R_+ \longrightarrow (0,1)$ is a function such that

$$\sum_{j=0}^{\infty}\varepsilon(x+jp) < +\infty. \tag{3.2}$$

Then there exists a unique solution $g: R_+ \longrightarrow R_+$ of the equation (1.2) *with*

$$\alpha(x) \leqslant \frac{g(x)}{f(x)} \leqslant \beta(x), \tag{3.3}$$

where $\alpha(x) := \prod_{j=0}^{\infty}(1-\varepsilon(x+jp))$ and $\beta(x) := \prod_{j=0}^{\infty}(1+\varepsilon(x+jp))$ for all $x \in R_+$.

Proof. The condition (3.2) implies that $\prod_{j=0}^{\infty}(1 \pm \varepsilon(x+jp))$ converges. Hence, we can define the functions α, β for all $x \in R_+$ such that $0 < \alpha(x) := \prod_{j=0}^{\infty}(1 - \varepsilon(x+jp)) < \prod_{j=0}^{\infty}(1+\varepsilon(x+jp)) := \beta(x) < +\infty$, that is, these series are bounded.

For any $x \in R_+$ and for every positive integer n, we define

$$g_n(x) = \prod_{j=0}^{n-1} \frac{f(x+np)}{\Gamma(x+jp)}. \tag{3.3}$$

For all positive integers m, n with $n > m$, we have

$$\begin{aligned} \frac{g_n(x)}{g_m(x)} = \frac{f(x+(m+1)p)}{\Gamma(x+mp)f(x+mp)} \times \\ \times \frac{f(x+(m+2)p)}{\Gamma(x+(m+1)p)f(x+(m+1)p)} \times \\ \times \cdots \frac{f(x+np)}{\Gamma(x+(n-1)p)f(x+(n-1)p)}. \end{aligned} \tag{3.5}$$

It also follows from (3.1) that

$$0 < 1 - \varepsilon(x+jp) \leqslant \frac{f(x+(j+1)p)}{\Gamma(x+jp)\, f(x+jp)} \leqslant 1 + \varepsilon(x+jp) \tag{3.5}$$

for all $x \in R_+$ and $j = 0, 1, 2, \cdots$. From (3.5) and (3.6), we get

$$\prod_{j=m}^{n-1}(1 - \varepsilon(x+jp)) \leqslant \frac{g_n(x)}{g_m(x)} \leqslant \prod_{j=m}^{n-1}(1+\varepsilon(x+jp))$$

or

$$\begin{aligned} \sum_{j=m}^{n-1} \log(1 - \varepsilon(x+jp)) \leqslant \log g_n(x) - \log g_m(x) \\ \leqslant \sum_{j=m}^{n-1} \log(1+\varepsilon(x+jp)). \end{aligned}$$

Since $\sum_{j=0}^{\infty} \log(1-\varepsilon(x+jp)) = \log \alpha(x)$ and $\sum_{j=0}^{\infty} \log(1+\varepsilon(x+jp)) = \log \beta(x)$, it follows that $\lim_{m\to\infty} \sum_{j=m}^{\infty} \log(1-\varepsilon(x+jp)) = \lim_{m\to\infty} \sum_{j=m}^{\infty} \log(1+\varepsilon(x+jp)) = 0$ by boundedness of α, β. Hence, we note that $\{\log g_n(x)\}$ is a Cauchy sequence for all $x \in R_+$. It is reasonable to define a function $g : R_+ \to R_+$ by

$$g(x) = e^{L(x)} = \lim_{n\to\infty} g_n(x) \qquad \forall x \in R_+, \tag{3.6}$$

where $L(x) := \lim_{n\to\infty} \log g_n(x)$.

We get that

$$g(x+p) = \Gamma(x) g(x) \qquad \forall x \in R_+. \tag{3.7}$$

Since

$$\frac{g_n(x)}{f(x)} = \frac{f(x+p)}{\Gamma(x)f(x)} \cdot \frac{f(x+2p)}{\Gamma(x+p)f(x+p)} \cdots \frac{f(x+np)}{\Gamma(x+(n-1)p)\, f(x+(n-1)p)},$$

we get

$$\prod_{j=0}^{n-1}(1-\varepsilon(x+jp)) \leqslant \frac{g_n(x)}{f(x)} \leqslant \prod_{j=0}^{n-1}(1+\varepsilon(x+jp))$$

for all $x \in R_+$. This inequality implies (3.3) with the definition of α, β as $n \longrightarrow \infty$.

Assume $h : R_+ \longrightarrow R_+$ is a solution of equation (3.8) which satisfies the inequality (3.3). By (3.8), we have

$$\frac{g(x)}{h(x)} = \frac{g(x+np)}{h(x+np)} = \frac{g(x+np)}{f(x+np)} \cdot \frac{f(x+np)}{h(x+np)}$$

for any $x \in R_+$ and for any natural number n.

Hence, we have

$$\frac{\alpha(x+np)}{\beta(x+np)} \leqslant \frac{g(x)}{h(x)} \leqslant \frac{\beta(x+np)}{\alpha(x+np)}$$

for any natural number n. By the boundedness of the series ε,

$$\alpha(x+np) = \prod_{j=n}^{\infty}(1-\varepsilon(x+jp)) \longrightarrow 1$$

as $n \longrightarrow \infty$. Similarly $\beta(x+np) \longrightarrow 1$ as $n \longrightarrow \infty$.

Therefore, it is obvious that $h(x) \equiv g(x)$. $\square$

From the proof of Theorem 3, we can see that the assumption (3.2) is a weak condition for the convergence of α and β. The special case $p = 1$ has been considered in [14].

COROLLARY 3. *Let a function f satisfies inequality* (3.1), *in which* $\varepsilon : R_+ \longrightarrow (0,1)$ *is a function such that*

$$\alpha(x) := \prod_{j=0}^{\infty}(1-\varepsilon(x+jp)) \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}(1+\varepsilon(x+jp))$$

are bounded for all $x \in R_+$. Then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.2) *satisfying* (3.3) *for all $x \in R_+$.*

COROLLARY 4. *Let $\theta > 0$ be given. If a mapping $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$\left|\frac{f(x+1)}{\Gamma(x)f(x)} - 1\right| \leqslant \frac{\delta}{x^{1+\theta}} \qquad \forall x \in R_+,$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the gamma functional equation (1.3) *such that for any $x > \delta^{\frac{1}{1+\theta}}$ the following inequality is satisfied*

$$\alpha(x) \leqslant \frac{g(x)}{f(x)} \leqslant \beta(x),$$

where $\alpha(x) := \prod_{j=0}^{\infty}(1-\frac{\delta}{(x+j)^{1+\theta}})$ *and* $\beta(x) := \prod_{j=0}^{\infty}(1+\frac{\delta}{(x+j)^{1+\theta}})$.

Proof. Applying Theorem 3 with $p = 1, \varepsilon(x) = \frac{\delta}{x^{1+\theta}}$, if $x > \delta^{\frac{1}{1+\theta}}$, then $\sum_{j=0}^{\infty}\frac{\delta}{(x+j)^{1+\theta}}$ converges by the p-series method. Hence, we get the desired result. $\square$

4. Examples

We apply the result of Theorem 3 with $p = 1$.

EXAMPLE 1. $\varepsilon(1+i) = \frac{1}{(1+i)^q}$, for $q > 1$. Note that the series $\sum_{k=0}^{\infty} \frac{1}{k^q}$ in the case $q > 1$ converges.

EXAMPLE 2. $\varepsilon(1+i) = \frac{1}{(1+i)!}$. Note that $\sum_{i=0}^{\infty} \frac{1}{(1+i)!} = e - 1$.

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(Received September 1, 2003)

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