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ON SANDWICH THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR

T. N. SHANMUGAM, S. SIVASUBRAMANIAN AND S. OWA

(*communicated by H. M. Srivastava*)

Abstract. The purpose of this present paper is to derive some subordination and superordination results for certain normalized analytic functions involving complex order in the open unit disk, acted upon by Carlson–Shaffer operator. Relations of the results, which are obtained in this paper, with various known results are also presented.

1. Introduction

Let $\mathcal{H}$ be the class of functions *analytic* in the *open* unit disk $\Delta := \{z : |z| < 1\}$. Let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$ and we let

$$\mathcal{A}_m := \left\{ f \in \mathcal{H}, f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \cdots \right\}.$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is *subordinate* to g if there exists a Schwarz function $\omega(z)$, analytic in Δ such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.1}$$

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then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is called to be superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently, Miller and Mocanu [7] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [7], Bulboacă [3] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [2]. Ali et al. [1] have used the results of Bulboacă [3] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [11] obtained sufficient conditions for a normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$ while, Obradović and Owa [8] obtained subordination results for the quantity $\left(\frac{f(z)}{z}\right)^{\mu}$. A detailed investigation of starlike functions of complex order and convex functions of complex order using Briot–Bouquet differential subordination technique has been studied very recently by Srivastava and Lashin [12].

Let the function $\varphi(a, c; z)$ be given by

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0, -1, -2, \ldots; z \in \Delta),$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n := \begin{cases} 1, & n = 0; \\ x(x+1)(x+2)\ldots(x+n-1), & n \in \mathbb{N} := \{1, 2, 3, \ldots\}. \end{cases}$$

Corresponding to the function $\varphi(a, c; z)$, Carlson and Shaffer [4] introduced a linear operator $L(a, c)$, which is defined by the following Hadamard product (or convolution):

$$L(a, c)f(z) := \varphi(a, c; z) * f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_n z^{n+1}.$$

We note that

$$L(a,a)f(z) = f(z), \quad L(2,1)f(z) = zf'(z), \quad L(\delta+1,1)f(z) = D^{\delta}f(z),$$

where $D^{\delta}f(z)$ is the *Ruscheweyh derivative* of $f(z)$ (see [10]).

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results involving the Carlson Shaffer Operator from which results of Srivastava and Lashin [12] and Obradović and Owa [8] are shown to be special cases .

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known results.

DEFINITION 2.1. ([7] Definition 2, p. 817) Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

THEOREM 2.2. ([6] Theorem 3.4h, p. 132) *Let the function q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in Δ, and*

(2) $\Re \frac{zh'(z)}{Q(z)} > 0$ *for $z \in \Delta$.*

If

$$\theta\{p(z)\} + zp'(z)\phi(p(z)) \prec \theta\{q(z)\} + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

LEMMA 2.3. ([5]) *Let g be a convex function in Δ and let*

$$h(z) = g(z) + m\alpha z g'(z),$$

where $\alpha > 0$ and m is a positive integer.

If $p(z) = g(0) + p_m z^m + \cdots$ is analytic in Δ and

$$p(z) + \alpha z p'(z) \prec h(z), \quad (z \in \Delta),$$

then

$$p(z) \prec g(z), \quad (z \in \Delta)$$

and this result is sharp.

LEMMA 2.4. ([6] Lemma 1, p. 71) *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}$ with $\Re(\gamma) \geqslant 0$. If $p \in \mathcal{H}$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z) \quad (z \in \Delta),$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \Delta)$$

where

$$q(z) = \frac{\gamma}{nz^{(\gamma/n)-1}} \int_0^z h(t)t^{(\gamma/n)-1}dt, \quad (z \in \Delta).$$

The function q is convex and is the best dominant.

THEOREM 2.5. ([3]) *Let the function q be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

1. $\Re\,[\vartheta'(q(z))/\varphi(q(z))] > 0$ *for* $z \in \Delta$,
2. $Q(z) = zq'(z)\varphi(q(z))$ *is starlike univalent in* Δ.

If $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta\{q(z)\} + zq'(z)\varphi(q(z)) \prec \vartheta\{p(z)\} + zp'(z)\varphi(p(z)), \tag{2.1}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. Subordination and superordination for analytic functions

We begin by proving involving differential subordination between analytic functions.

THEOREM 3.1. *Let the function $q(z)$ be analytic and univalent in Δ such that $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ. Let*

$$\Re\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \tag{3.1}$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0)$$

and

$$\begin{aligned}\Psi(a,c,\mu,\xi,\beta,\delta,f) := \alpha + \xi\left[\frac{L(a,c)f(z)}{z}\right]^{\mu} + \delta\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu} \\ + \beta\mu(1+a)\left[\frac{L(a+1,c)f(z)f(z)}{L(a,c)f(z)} - 1\right].\end{aligned} \tag{3.2}$$

If q satisfies the following subordination:

$$\Psi(a,c,\mu,\xi,\beta,\delta,f) \prec \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \tag{3.3}$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta,\mu \in \mathbb{C};\ \mu \neq 0;\ \beta \neq 0),$$

then

$$\left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \prec q(z) \quad (z \in \Delta;\ z \neq 0;\ \mu \in \mathbb{C};\ \mu \neq 0) \tag{3.4}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := \left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \quad (z \in \Delta;\ z \neq 0;\ f \in \mathcal{A}), \tag{3.5}$$

so that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right]. \tag{3.6}$$

By using the identity:

$$z(L(a,c)f(z))' = (1+a)L(a+1,c)f(z) - aL(a,c)f(z),$$

we obtain

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{(1+a)L(a+1,c)f(z)}{L(a,c)f(z)} - (1+a)\right].$$

By setting

$$\theta(\omega) := \alpha + \xi\omega + \delta\omega^2 \qquad \text{and} \qquad \phi(\omega) := \frac{\beta}{\omega},$$

it can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C}\setminus\{0\}).$$

Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)} \tag{3.7}$$

and

$$h(z) = \theta\{q(z)\} + Q(z) = \alpha + \xi p(z) + \delta(p(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{3.8}$$

we find that $Q(z)$ is starlike univalent in Δ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0,$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0).$$

The assertion (3.4) of Theorem 3.1 now follows by an application of Theorem 2.2. □

For the choices $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leqslant B < A \leqslant 1$ and $q(z) = \left\{\frac{1+z}{1-z}\right\}^{\gamma}$, $0 < \gamma \leqslant 1$, in Theorem 3.1, we get the following results (Corollaries 3.2 and 3.3 below).

COROLLARY 3.2. *Assume that* (3.1) *holds. If* $f \in \mathcal{A}$, *and*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f) \prec \alpha + \xi\frac{1+Az}{1+Bz} + \delta\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)}$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta,\mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

where $\Psi(a,c,\mu,\xi,\beta,\delta,f)$ *is as defined in* (3.2), *then*

$$\left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \prec \frac{1+Az}{1+Bz} \quad (z\in\Delta;\ z\neq 0;\ \mu\in\mathbb{C};\ \mu\neq 0)$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

COROLLARY 3.3. *Assume that* (3.1) *holds. If* $f\in\mathcal{A}$, *and*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f) \prec \alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\delta\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{(1-z^2)}$$

$$(z\in\Delta;\ \alpha,\ \delta,\ \xi,\ \beta,\mu\in\mathbb{C};\ \mu\neq 0;\beta\neq 0)$$

where $\Psi(a,c,\mu,\xi,\beta,\delta,f)$ *is as defined in* (3.2), *then*

$$\left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \prec \left\{\frac{1+z}{1-z}\right\}^{\gamma} \quad (z\in\Delta;\ \mu\in\mathbb{C};\ \mu\neq 0)$$

and $\left\{\frac{1+z}{1-z}\right\}^{\gamma}$ *is the best dominant.*

For a special case $q(z)=e^{\mu Az}$, with $|\mu A|<\pi$, Theorem 3.1 readily yields the following.

COROLLARY 3.4. *Assume that* (3.1) *holds. If* $f\in\mathcal{A}$, *and*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f) \prec \alpha+\xi e^{\mu Az}+\delta e^{2\mu Az}+\beta A\mu z$$

$$(z\in\Delta;\ \alpha,\ \delta,\ \xi,\ \beta,\mu\in\mathbb{C};\ \mu\neq 0;\beta\neq 0)$$

where $\Psi(a,c,\mu,\xi,\beta,\delta,f)$ *is as defined in* (3.2), *then*

$$\left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \prec e^{\mu Az} \quad (z\in\delta;\ z\neq 0;\ \mu\in\mathbb{C},\ \mu\neq 0)$$

and $e^{\mu Az}$ *is the best dominant.*

REMARK 3.5. Taking $a=c=1$, $\delta=\xi=0$, $\alpha=1$, $\beta=\frac{1}{\mu}$ in Corollary 3.4, we get the result obtained by Obradović and Owa [8].

For a special case when $q(z)=\frac{1}{(1-z)^{2b}}$ $(b\in\mathbb{C}\setminus\{0\})$, $a=c=1$, $\delta=\xi=0$, $\mu=\alpha=1$ and $\beta=\frac{1}{b}$, Theorem 3.1 reduces at once to the following known result obtained by Srivastava and Lashin [12].

COROLLARY 3.6. *Let* b *be a non zero complex number. If* $f\in\mathcal{A}$, *and*

$$1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ *is the best dominant.*

For $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$, $a = c = 1$, $\delta = \xi = 0$, $\alpha = 1, \beta = \frac{1}{\mu}$ in Theorem 3.1, we get the following known result obtained by Obradović and Owa [8].

COROLLARY 3.7. *Let* $-1 \leqslant B < A \leqslant 1$. *If* $f \in \mathcal{A}$, *and*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}} \quad (z \in \Delta;\ z \neq 0;\ \mu \in \mathbb{C};\ \mu \neq 0)$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ *is the best dominant.*

We remark here that $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$ is univalent if and only if either

$$\left|\frac{\mu(A-B)}{B} - 1\right| \leqslant 1 \quad \text{or} \quad \left|\frac{\mu(A-B)}{B} + 1\right| \leqslant 1.$$

Next,with the help of Lemma 2.4, we now prove the following .

THEOREM 3.8. *Let* $h \in \mathcal{H}, h(0) = 1$, $h'(0) \neq 0$ *which satisfy the inequality*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad (z \in \Delta).$$

If $f \in \mathcal{A}_m$ *satisfies the differential subordination*

$$\frac{L(a+2,c)f(z)}{z} \prec h(z), \quad (z \in \Delta; z \neq 0), \tag{3.9}$$

then

$$\frac{L(a+1,c)f(z)}{z} \prec g(z), \quad (z \in \Delta;\ z \neq 0), \tag{3.10}$$

where

$$g(z) = \frac{(a+1)}{mz^{((a+1)/m)-1}} \int_0^z h(t)t^{((1+a)/m)-1}dt, \quad (z \in \Delta). \tag{3.11}$$

The function g *is convex and is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{L(a+1,c)f(z)}{z} \quad (z \in \Delta,\ z \neq 0). \tag{3.12}$$

A straight forward computation gives,

$$\frac{zp'(z)}{p(z)} = \left[\frac{zL(a+1,c)f(z))'}{L(a+1,c)f(z)} - 1\right]. \tag{3.13}$$

By using the identity:

$$z(L(a,c)f(z))' = (1+a)L(a+1,c)f(z)f(z) - aL(a,c)f(z),$$

we have

$$\frac{zp'(z)}{p(z)} = \left[\frac{(1+a)L(a+2,c)f(z)}{L(a+1,c)f(z)} - (1+a)\right]$$

and hence,

$$p(z) + \frac{zp'(z)}{a+1} = \frac{L(a+2,c)f(z)}{z} \quad (z \in \Delta). \tag{3.14}$$

The assertion of Theorem 3.8 now follows from Lemma 2.4. □

Next, we prove Theorem 3.9.

THEOREM 3.9. *Let* $h \in \mathcal{H}, h(0) = 1,\ h'(0) \neq 0$ *which satisfy the inequality*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad (z \in \Delta).$$

If $f \in \mathcal{A}_m$ *satisfies the differential subordination*

$$\frac{L(a+1,c)f(z)}{z} \prec h(z), \quad (z \in \Delta;\ z \neq 0), \tag{3.15}$$

then

$$\frac{L(a,c)f(z)}{z} \prec g(z), \quad (z \in \Delta;\ z \neq 0), \tag{3.16}$$

where

$$g(z) = \frac{(1+a)}{mz^{((a+1)/m)-1}} \int_0^z h(t)t^{((1+a)/m)-1}dt, \quad (z \in \Delta). \tag{3.17}$$

The function g *is convex and is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{L(a,c)f(z)}{z} \quad (z \in \Delta;\ z \neq 0;\ z \neq 0), \tag{3.18}$$

so that, by a straight forward computation, we have

$$\frac{zp'(z)}{p(z)} = \left[\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right]. \tag{3.19}$$

By using the identity:

$$z(L(a,c)f(z))' = (1+a)L(a+1,c)f(z)f(z) - \lambda L(a,c)f(z),$$

we obtain

$$\frac{zp'(z)}{p(z)} = \left[\frac{(1+a)L(a+1,c)f(z)}{L(a,c)f(z)} - (1+a)\right]$$

and hence,

$$p(z) + \frac{zp'(z)}{a+1} = \frac{L(a,c)f(z)}{z} \quad (z \in \Delta). \tag{3.20}$$

The assertion of Theorem 3.9 now follows from Lemma 2.4. □

Next, by using Lemma 2.3, we prove the following Theorem 3.10.

THEOREM 3.10. *Let g be a convex function with $g(0) = 1$. Let h be a function, such that*

$$h(z) = g(z) + \frac{m}{\lambda + 1} z g'(z).$$

If $f \in \mathcal{A}_m$ satisfies the subordination

$$\frac{L(a+1, c)f(z)}{z} \prec h(z), \quad (z \in \Delta;\ z \neq 0), \tag{3.21}$$

then

$$\frac{L(a, c)f(z)}{z} \prec g(z), \quad (z \in \Delta;\ z \neq 0), \tag{3.22}$$

and this result is sharp.

Proof. The proof of the theorem is much akin to the proof of Theorem 3.9 and hence we omit the details involved. □

Next, by appealing to Theorem 2.5 of the preceding section, we prove Theorem 3.11 below.

THEOREM 3.11. *Let q be analytic and convex univalent in Δ such that $q(z) \neq 0$ and $\frac{zq'}{q}$ be starlike univalent in Δ. Further, let us assume that*

$$\Re\left[\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta} q(z)\right] > 0, \quad (z \in \Delta; \quad \delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0). \tag{3.23}$$

If $f \in \mathcal{A}$, $0 \neq \left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$, and $\Psi(a, c, \mu, \xi, \beta, \delta, f)$ is univalent in Δ, then

$$\alpha + \xi q(z) + \delta(q(z))^2 + \beta \frac{zq'(z)}{q(z)} \prec \Psi(a, c, \mu, \xi, \beta, \delta, f)$$
$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

implies

$$q(z) \prec \left(\frac{L(a, c)f(z)}{z}\right)^{\mu} \quad (z \in \Delta;\ z \neq 0;\ \mu \in \mathbb{C};\ \mu \neq 0) \tag{3.24}$$

and q is the best subordinant where $\Psi(a, c\mu, \xi, \beta, \delta, f)$ is as defined in (3.2).

Proof. By setting

$$\vartheta(w) := \alpha + \xi w + \delta w^2 \quad \text{and} \quad \varphi(w) := \beta \frac{w'}{w},$$

it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\varphi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\varphi(w) \neq 0, \quad (w \in \mathbb{C} \setminus \{0\}).$$

Since q is convex (univalent) function, it follows that

$$\Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] = \Re\left[\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta} q(z)\right] > 0,$$

$$(z \in \Delta;\ \delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0).$$

The assertion (3.23) of Theorem 3.11 follows by an application of Theorem 2.5.

We remark here that Theorem 3.11 can easily be restated, for different choices of the function $q(z)$. Combining Theorem 3.1 and Theorem 3.11, we get the following sandwich theorem.

THEOREM 3.12. *Let q_1 and q_2 be convex univalent in Δ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$. Suppose q_1 and q_2 satisfies* (3.23) *and* (3.1) *respectively. If $f \in \mathcal{A}$, $\left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\alpha + \xi\left[\frac{L(a,c)f(z)}{z}\right]^{\mu} + \delta\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu} + \mu(1+a)\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right],$$

$$(z \in \Delta;\ z \neq 0;\ \alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

is univalent in Δ, then

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \Psi(a,c,\mu,\xi,\beta,\delta,f)$$

$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

implies

$$q_1(z) \prec \left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \prec q_2(z) \quad (z \in \Delta;\ z \neq 0;\ \mu \in \mathbb{C};\ \mu \neq 0)$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

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T. N. Shanmugam
Department of Information Technology
Salalah College of Technology
P. O. Box No. 608
Salah
Sultanate of Oman
e-mail: drtns2001@yahoo.com

S. Sivasubramanian
Department of Mathematics
Easwari Engineering college
Ramapuram, Chennai 600 089
India
e-mail: sivasaisastha@rediffmail.com

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan
e-mail: owas@kindai.ac.jp