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ON CONVEX FUNCTIONS OF HIGHER ORDER

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(*communicated by L.-E. Persson*)

Abstract. Based on J. L. W. V. Jensen's concept of convex functions as well on its generalization by E. M. Wright and related to T. Popoviciu's convexity notions, higher-order convexity properties of real functions are introduced and surveyed.

1. Introduction

Although convexity properties of functions were investigated already before the twentieth century by several authors (among others, M. O. Hölder [10], O. Stolz [24] and J. Hadamard [7], [8] achieved remarkable results connected to this field), the concept of convex functions was constructed and their first systematic study was carried out one hundred years ago by the Danish mathematician J. L. W. V. Jensen (cf. [11] and [12]). He recognized the importance of this notion already at that time: as he wrote '*Il me semble que la notion "fonction convexe" est à peu près aussi fondamentale que celles-ci "fonction positive", "fonction croissante". Si je ne me trompe pas en ceci la notion devra trouver sa place dans les expositions élémentaires de la théorie des fonctions réelles.*' (It seems to me that the notion "convex function" is just as fundamental as "positive function" or "increasing function". If I am not mistaken in this, the notion ought to find its place in elementary expositions of the theory of real functions.) Jensen was certainly not mistaken: the theory of convex functions has become a standard part of the subject-matter taught for students studying mathematics, alongside with thousands of scientific papers it is the topic of several books (to mention only some of the classicals: T. Popoviciu [21], R. T. Rockafellar [22], A. W. Roberts and D. E. Varberg [23], P. S. Bullen, D. S. Mitrinović and P. M. Vasić [3], D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, [17] C. E. Niculescu and L.-E. Persson, [18]), the basic concept was generalized in several directions (e.g., by T. Popoviciu [20], [21], E. F. Beckenbach [1], B. De Finetti [5], E. M. Wright [25]), and it has a fundamental role in several applications (for example in optimization theory, cf., e.g., S. Boyd and L. Vandenberghe [2] and the references therein).

In the present paper, based on Jensen's concept and on its generalization by Wright and motivated by Popoviciu's convexity notions, higher-order convexity properties of

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real functions are introduced and investigated. After describing the basic concepts our examination is connected to, in Section 3, we study the relationship of (higher-order) symmetric and cyclic convexity. In Section 4, we consider higher-order Wright-convex functions. In the next part, we describe how symmetric convexity and Wright-convexity are related, and we show an equivalence-theorem for different higher-order convexity concepts. Concluding the paper, we formulate some remarks and open problems connected to our results.

2. Basic concepts

A real valued function f defined on an interval $I \subseteq \mathbb{R}$ is called *convex on* I if it satisfies the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

for all $t \in]0,1[$ and $x, y \in I$. If this property holds for a fixed $t \in]0,1[$ (and for all $x, y \in I$) then f is said to be t*-convex on* I. In the case when $t = \frac{1}{2}$, a t-convex function is also called *Jensen-convex* (cf. [11], [12]). Obviously, any convex function is t-convex for all $t \in]0,1[$, however there are non-convex but t-convex functions for an arbitrary $t \in]0,1[$. (Concerning the construction of such functions with the aid of Hamel bases, we refer to [9], [14] and [13, Chapter V].)

According to E. M. Wright [25], a function $f : I \to \mathbb{R}$ is called *Wright-convex on* I if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y) \tag{1}$$

for all $t \in]0,1[$ and $x, y \in I$. Analogously to the definition of t-convexity, f is said to be t*-Wright-convex*, if $t \in]0,1[$ is fixed and (1) is valid for all $x, y \in I$. It is easy to see that Wright-convexity implies t-Wright-convexity for all $t \in]0,1[$. Furthermore, t-convexity yields t-Wright-convexity for an arbitrary $t \in]0,1[$, thus, Wright-convexity follows from convexity. (On further properties of convex and Wright-convex functions, cf. the books [13], [23] and the paper [15].)

3. On $(t_1, \dots, t_n)$-convex functions

In order to introduce the main terms of this section, we recall the notion of divided differences of real functions. The *divided difference* of the function $f : I \to \mathbb{R}$ with respect to the pairwise distinct points $x_0, \dots, x_n \in I$ is defined by

$$[x_0, \dots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}. \tag{2}$$

Obviously, divided differences are symmetric functions of $x_0, \dots, x_n$, furthermore, it is easy to prove that they have the recursive property

$$[x_0, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0} \tag{3}$$

for all positive integers n and pairwise distinct points $x_0, \dots, x_n \in I$.

In the following, let n be a fixed positive integer. According to T. Popoviciu (cf. [20] and [21]), a function $f : I \to \mathbb{R}$ is called *convex of order* $n-1$ *(or monotone of order* n*) on* I if

$$[x_0, \dots, x_n; f] \geqslant 0 \tag{4}$$

holds for all pairwise distinct elements $x_0, \dots, x_n \in I$. Due to the symmetry, we may assume $x_0 < \dots < x_n$ here.

Motivated by this definition, we introduce and investigate some more particular convexity concepts. Let $t_1, \dots, t_n$ be fixed positive real numbers. A function $f : I \to \mathbb{R}$ is said to be $(t_1, \dots, t_n)$*-convex on* I if inequality (4) holds for all $x_0, \dots, x_n \in I$ satisfying the properties $x_0 < \dots < x_n$ and

$$(x_1 - x_0) : \dots : (x_n - x_{n-1}) = t_1 : \dots : t_n$$

or equivalently,

$$[x, x + t_1 h, \dots, x + (t_1 + \dots + t_n)h; f] \geqslant 0$$

for all $h > 0$, $x \in I$ with $x + (t_1 + \dots + t_n)h \in I$. We call f *cyclically* $(t_1, \dots, t_n)$*-convex on* I if it is $(t_{i_1}, \dots, t_{i_n})$-convex for all cyclic permutations $(i_1, \dots, i_n)$ of $(1, \dots, n)$; finally, we call f *symmetrically* $(t_1, \dots, t_n)$*-convex on* I if it is $(t_{i_1}, \dots, t_{i_n})$-convex for all permutations $(i_1, \dots, i_n)$ of the integers $\{1, \dots, n\}$. In the case when $t_1 = \dots = t_n$, these definitions (are equivalent and) give the concept of Jensen-convexity of order $n-1$ (cf. [20] and [21]). It is easy to see that convexity of order 0 means monotonicity, convexity of order 1 is exactly convexity. Furthermore, cyclic (t_1, t_2)-convexity, symmetric (t_1, t_2)-convexity and t-convexity are equivalent for $t_1, t_2 > 0$ and $t = \frac{t_1}{t_1 + t_2}$. (Observe, that, in view of our notation, the meaning of (t)-convexity is different from that of t-convexity. Namely, (t)-convexity is a monotonicity property, while t-convexity is equivalent to symmetric and to cyclic $(t, 1-t)$-convexity.)

Obvious consequences of the definitions above are that convexity of order $n-1$ implies symmetric $(t_1, \dots, t_n)$-convexity, symmetric $(t_1, \dots, t_n)$-convexity implies cyclic $(t_1, \dots, t_n)$-convexity, and cyclic $(t_1, \dots, t_n)$-convexity implies $(t_1, \dots, t_n)$-convexity for all positive integers n and for all positive n-tuples $(t_1, \dots, t_n)$. In what follows, we investigate the implication between symmetric and cyclic $(t_1, \dots, t_n)$-convexity 'in the other direction'. By a result of N. Kuhn [14] and Z. Daróczy and Zs. Páles [4], t-convexity implies r-convexity for all real $t \in]0, 1[$ and rational $r \in]0, 1[$. As a generalization of this theorem, we prove that cyclic $(t_1, \dots, t_n)$-convexity implies symmetric $(r_1, \dots, r_n)$-convexity for arbitrary positive reals $t_1, \dots, t_n$ and positive rationals $r_1, \dots, r_n$. In our proof we use the following property of divided differences (cf. [13, Chapter XV, Lemma 2.6]).

LEMMA 3.1. *Let* n *be a positive integer* I *be a nonempty interval,* $x_0 < \dots < x_n \in I$*, and let* $j_0 < \dots < j_k$ *(fixed) elements of the set* $\{0, \dots, n\}$*. Then there exits non-negative real numbers* $c_0, \dots, c_{n-k}$ *such that*

$$[x_{j_0}, \dots, x_{j_k}; f] = \sum_{i=0}^{n-k} c_i [x_i, \dots, x_{i+k}; f]$$

is valid for all functions $f : I \to \mathbb{R}$.

THEOREM 3.2. *Let* n *be a positive integer,* $t_1, \dots, t_n$ *be positive real numbers and let* $f : I \to \mathbb{R}$ *be a function. If* f *is cyclically* $(t_1, \dots, t_n)$*-convex then it is symmetrically* $(r_1, \dots, r_n)$*-convex for all positive rational numbers* $r_1, \dots, r_n$.

Proof. Let $r_1, \dots, r_n > 0$ be fixed rationals and let $x \in I$ and $h > 0$ with $x + (r_1 + \cdots + r_n)h \in I$. There exist positive integers $p_1, \dots, p_n$ and q such that

$$r_1 = \frac{p_1}{q}, \quad \dots, \quad r_n = \frac{p_n}{q}. \tag{5}$$

Let us consider the elements

$$x_{kn+j} := x + \left(k + \frac{1}{T} \sum_{i=1}^{j} t_i \right) \frac{h}{q} \tag{6}$$

for $k = 0, \dots, (p_1 + \cdots + p_n)$, $j = 0, \dots, k$, where $T := \sum_{i=1}^{n} t_i$ and we use the convention $\sum_{i=1}^{0} t_i := 0$. It follows easily from this construction that

$$x_{i+1} - x_i = t_{i_1} \frac{h}{qT}, \quad x_{i+2} - x_i = (t_{i_1} + t_{i_2}) \frac{h}{qT}, \quad \dots, \quad x_{i+n} - x_i = (t_{i_1} + t_{i_2} + \cdots + t_{i_n}) \frac{h}{qT},$$

where $(i_1, \dots, i_n)$ is a cyclic permutation of $(1, \dots, n)$ for an arbitrary $i \in \{0, \dots, [(p_1 + \cdots + p_n) - 1]n\}$. Therefore, the cyclic $(t_1, \dots, t_n)$-convexity of f implies that

$$[x_i, x_{i+1}, \dots, x_{i+n}; f] \geqslant 0$$

for $i = 0, \dots, [(p_1 + \cdots + p_n) - 1]n$. By Lemma 3.1, there exist nonnegative integers $c_0, \dots, c_{[(p_1 + \cdots + p_n) - 1]n}$ such that

$$[x_0, x_{p_1 n}, \dots, x_{(p_1 + \cdots + p_n)n}; f] = \sum_{i=0}^{[(p_1 + \cdots + p_n) - 1]n} c_i \, [x_i, x_{i+1}, \dots, x_{i+n}; f],$$

which yields

$$[x_0, x_{p_1 n}, \dots, x_{(p_1 + \cdots + p_n)n}; f] \geqslant 0. \tag{7}$$

On the other hand, by (5) and (6), we have

$$x_0 = x, \quad x_{p_1 n} = x + r_1 h, \quad \dots, \quad x_{(p_1 + \cdots + p_n)n} = x + (r_1 + \cdots + r_n)h,$$

therefore, (7) gives

$$[x, x + r_1 h, \dots, x(r_1 + \cdots + r_n)h; f] \geqslant 0,$$

which implies the $(r_1, \dots, r_n)$-convexity of f on I. $\square$

COROLLARY 3.3. *Let n be a positive integer and let $r_1, \dots, r_n$ be positive rational numbers. A function $f : I \to \mathbb{R}$ is cyclically $(r_1, \dots, r_n)$-convex if and only if it is symmetrically $(r_1, \dots, r_n)$-convex.*

Proof. The definition of symmetric convexity and that of cyclic convexity gives that symmetric $(r_1, \dots, r_n)$-convexity always implies cyclic $(r_1, \dots, r_n)$-convex. The other direction of the statement follows from Theorem 3.2. □

COROLLARY 3.4. (Cf. [4], [14]). *Let $t \in]0,1[$ be a real and $r \in]0,1[$ be a rational number. If a function $f : I \to \mathbb{R}$ is t-convex then it is r-convex, too.*

4. On $(t_1, \dots, t_n)$-Wright-convex functions

The *(forward) difference* of the function $f : I \to \mathbb{R}$ at the point x with step-size h has the form

$$\Delta_h f(x) = f(x+h) - f(x)$$

whenever $x, x+h \in I$.

According to [6], the function $f : I \to \mathbb{R}$ is called *Wright-convex of order* $n-1$ *on* I if n is a positive integer and

$$\Delta_{h_1} \cdots \Delta_{h_n} f(x) \geqslant 0$$

for all $h_1, \dots, h_n > 0$, $x \in I$ with $x + h_1 + \cdots + h_n \in I$; it is said to be $(t_1, \dots, t_n)$-*Wright-convex on* I if $t_1, \dots, t_n$ are (fixed) positive numbers and

$$\Delta_{t_1 h} \cdots \Delta_{t_n h} f(x) \geqslant 0 \tag{8}$$

is valid for all $h > 0$, $x \in I$ with $x + (t_1 + \cdots + t_n)h \in I$. If $t_1 = \cdots = t_n$, this definition gives the notion of Jensen-convexity of order $n-1$ introduced by T. Popoviciu ([20], [21]). It is evident that Wright-convexity of order $n-1$ implies $(t_1, \dots, t_n)$-Wright-convexity for all $t_1, \dots, t_n > 0$. It can be easily shown that (t_1, t_2)-Wright-convexity is exactly t-Wright-convexity, where t_1 and t_2 are positive real numbers and $t = \frac{t_1}{t_1+t_2}$. (Note the difference between t-Wright-convexity and (t)-Wright-convexity.)

It is easy to see that the left hand side of (8) is a symmetric function of $(t_1, \dots, t_n)$, therefore, $(t_1, \dots, t_n)$-Wright-convexity implies $(t_{i_1}, \dots, t_{i_n})$-Wright-convexity for all permutations $(i_1, \dots, i_n)$ of the integers $\{1, \dots, n\}$. Another simple consequence of the definition above is that $(t_1, \dots, t_n)$-Wright-convexity yields $(ct_1, \dots, ct_n)$-Wright-convexity for arbitrary positive c and $t_1, \dots, t_n$. Gy. Maksa, K. Nikodem and Zs. Páles showed in [15] that t-Wright-convexity implies $\frac{kt}{kt+n(1-t)}$-Wright-convexity for every $t \in]0,1[$ and for all positive integers k, n. Motivated by these properties, we formulate and prove more general results on the relationship of $(t_1, \dots, t_n)$- and $(t'_1, \dots, t'_n)$-Wright-convex functions in the case of different n-tuples $(t_1, \dots, t_n)$ and $(t'_1, \dots, t'_n)$. Our statements contain the theorem cited above as special case.

THEOREM 4.1. *Let n be a positive integer, $T_1,\dots,T_n$ be (not necessarily finite) sets of positive real numbers, and denote ΣT_i the set of all finite linear combinations of the elements of T_i with positive rational coefficients $(i=1,\dots,n)$. If a function $f:I\to\mathbb{R}$ is $(t_1,\dots,t_n)$-Wright-convex for every $(t_1,\dots,t_n)\in T_1\times\cdots\times T_n$ then it is $(s_1,\dots,s_n)$-Wright-convex for all $(s_1,\dots,s_n)\in\Sigma T_1\times\cdots\times\Sigma T_n$, too.*

Proof. Suppose that the assumptions of the theorem are valid, and let $(s_1,\dots,s_n)$ be fixed. The $(s_1,\dots,s_n)$-Wright-convexity of f means that

$$\Delta_{s_1h}\cdots\Delta_{s_nh}f(x)\geqslant 0 \tag{9}$$

for all $h>0$ such that $x\in I$, $x+(s_1+\cdots+s_n)h\in I$. Since $(s_1,\dots,s_n)\in\Sigma T_1\times\cdots\times\Sigma T_n$, we have

$$s_i=\sum_{j=1}^{\ell_i} r_{ij}\tilde{t}_{ij}$$

where r_{ij} are positive rational numbers and $\tilde{t}_{ij}\in T_i$ for $j=1,\dots,\ell_i$, $i=1,\dots,n$. There exist positive integers q and p_{ij} such that $r_{ij}=\frac{p_{ij}}{q}$ for $j=1,\dots,\ell_i$, $i=1,\dots,n$. Thus, we may write

$$qs_i=\sum_{j=1}^{k_i} t_{ij}$$

where $t_{ij}\in T_i$ for $j=1,\dots,k_i$, $i=1,\dots,n$. Replacing h by qh in inequality (9), we obtain that it is equivalent to

$$\Delta_{(t_{11}+\cdots+t_{1k_1})h}\cdots\Delta_{(t_{n1}+\cdots+t_{nk_n})h}f(x)\geqslant 0 \tag{10}$$

for all $h>0$, $x\in I$, with $x+(t_{11}+\cdots+t_{1k_1}+\cdots+t_{n1}+\cdots+t_{nk_n})h\in I$.

In the following, we prove inequality (10). In order to do this, we consider the *translation operator* τ defined by $\tau_h f(x)=f(x+h)$ for $x\in I$, $h\in\mathbb{R}$ with $x+h\in I$. It is easy to see that

$$\begin{aligned}\Delta_{u_1+\cdots+u_m} &= \tau_{u_1+\cdots+u_m}-\tau_0\\ &= \tau_{u_1}-\tau_0+(\tau_{u_2}-\tau_0)\tau_{u_1}+\cdots+(\tau_{u_m}-\tau_0)\tau_{u_1+\cdots+u_{m-1}}\\ &= \Delta_{u_1}+\Delta_{u_2}\tau_{u_1}+\cdots+\Delta_{u_m}\tau_{u_1+\cdots+u_{m-1}}\end{aligned}$$

for all positive integers m and real numbers $u_1,\dots,u_m$. Using this property, we obtain that

$$\begin{aligned}&\Delta_{t_{11}+\cdots+t_{1k_1}}\cdots\Delta_{t_{n1}+\cdots+t_{nk_n}}\\ &\quad=(\Delta_{t_{11}}+\Delta_{t_{12}}\tau_{t_{11}}+\cdots+\Delta_{t_{1k_1}}\tau_{t_{11}+\cdots+t_{1k_1-1}})\cdots(\Delta_{t_{n1}}+\Delta_{t_{n2}}\tau_{t_{n1}}+\cdots+\Delta_{t_{nk_n}}\tau_{t_{n1}+\cdots+t_{nk_n-1}}).\end{aligned}$$

Applying the distributive law on the right hand side of this equation, we may write

$$\Delta_{t_{11}+\cdots+t_{1k_1}}\cdots\Delta_{t_{n1}+\cdots+t_{nk_n}}=\sum_{j_1=1}^{k_1}\cdots\sum_{j_n=1}^{k_n}\Delta_{t_{1j_1}}\cdots\Delta_{t_{nj_n}}\Theta_{j_1\dots j_n}, \tag{11}$$

where $\Theta_{j_1\dots j_n}$ denotes a product of some translation operators, more precisely,

$$\Theta_{j_1\dots j_n} = \tau_{\sum_{r=1}^{j_1-1} t_{1r}} \tau_{\sum_{r=1}^{j_2-1} t_{2r}} \cdots \tau_{\sum_{r=1}^{j_n-1} t_{nr}} \qquad (j_i = 1,\dots,k_i,\ i = 1,\dots,n).$$

Since $(t_{1j_1},\dots,t_{nj_n}) \in T_1 \times \cdots \times T_n$, f is $(t_{1j_1},\dots,t_{nj_n})$-Wright-convex for all $j_i = 1,\dots,k_i$, $i = 1,\dots,n$, thus, we have

$$\Delta_{t_{1j_1}h} \cdots \Delta_{t_{nj_n}h} f(x) \geqslant 0$$

for all $x \in I$, $h > 0$ such that $x + t_{1j_1}h + \cdots + t_{nj_n}h \in I$ where $j_i = 1,\dots,k_i$, $i = 1,\dots,n$. Therefore, using the notation

$$\tilde{\Theta}_{j_1\dots j_n} = \tau_{\sum_{r=1}^{j_1-1} t_{1r}h} \tau_{\sum_{r=1}^{j_2-1} t_{2r}h} \cdots \tau_{\sum_{r=1}^{j_n-1} t_{nr}h} \qquad (j_i = 1,\dots,k_i,\ i = 1,\dots,n),$$

we obtain that

$$\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \Delta_{t_{1j_1}h} \cdots \Delta_{t_{nj_n}h}\, \tilde{\Theta}_{j_1\dots j_n} f(x) \geqslant 0$$

for all $x \in I$, $h > 0$ with $x + \big((t_{11} + \cdots + t_{1k_1}) + \cdots + (t_{n1} + \cdots + t_{nk_n})\big)h \in I$ which, due to (11), implies (10), that is our statement. □

COROLLARY 4.2. *Let n be a positive integer, $t_1,\dots,t_n$ be positive real numbers, and let $f : I \to \mathbb{R}$ be a function. If f is $(t_1,\dots,t_n)$-Wright-convex then it is $(r_1t_1,\dots,r_nt_n)$-Wright-convex for all positive rationals $r_1,\dots,r_n$.*

Proof. The statement is contained in Theorem 4.1, with $T_1 = \{t_1\}, \dots, T_n = \{t_n\}$, as special case. □

COROLLARY 4.3. *Let n be a positive integer and let $r_1,\dots,r_n$ be positive rational numbers. A function $f : I \to \mathbb{R}$ is $(r_1,\dots,r_n)$-Wright-convex if and only if it is $(\underbrace{1,\dots,1}_{n-times})$-Wright-convex.*

Proof. The statement is an obvious consequence of Corollary 4.2. □

COROLLARY 4.4. (Cf. [15]). *If $t \in]0,1[$ is a real number and $f : I \to \mathbb{R}$ is a t-Wright-convex function then it is also $\frac{kt}{kt+n(1-t)}$-Wright-convex for all positive integers k, n.*

5. Connection between $(t_1,\dots,t_n)$-convexity and $(t_1,\dots,t_n)$-Wright-convexity

In this section, using the concepts defined in the previous parts of the paper, we prove that symmetric $(t_1,\dots,t_n)$-convexity implies $(t_1,\dots,t_n)$-Wright-convexity for arbitrary positive n-tuples $(t_1,\dots,t_n)$. A corollary of this result is the known theorem that convexity of order $n-1$ yields Wright-convexity of order $n-1$ for any positive integer n (cf. [13, Chapter XV, Theorem 7.1]). We also point out that the converse of the statement above is not valid in the case when $n \geqslant 2$. At the end of the section we formulate a theorem on the equivalence of different higher-order convexity concepts in special cases.

First we show a statement on divided and forward differences.

LEMMA 5.1. *Let n be a positive integer, $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then the equation*

$$\sum_{(i_1,\dots,i_n)} [x, x+t_{i_1}, \dots, x+t_{i_1}+\cdots+t_{i_n}; f] = \frac{\Delta_{t_1}\cdots\Delta_{t_n} f(x)}{t_1\cdots t_n} \tag{12}$$

is valid for all $x \in I$, $t_1, \dots, t_n > 0$ with $x+t_1+\cdots+t_n \in I$, where the summation is for all permutations $(i_1, \dots, i_n)$ of the integers $\{1, \dots, n\}$.

Proof. We prove the statement by induction on n. The statement is trivial for $n = 1$. Suppose, that it is valid for a positive integer n. Let $x \in I$ and $t_1, \dots, t_{n+1} > 0$ satisfying $x+t_1+\cdots+t_{n+1} \in I$ be given. Using the definition of divided differences and the inductive hypothesis, we obtain that

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{n+1})} [x, x+t_{i_1}, \dots, x+t_{i_1}+\cdots+t_{i_{n+1}}; f] \\
&= \sum_{(i_1,\dots,i_{n+1})} \frac{[x+t_{i_1}, \dots, x+t_{i_1}+\cdots+t_{i_{n+1}}; f] - [x, x+t_{i_1}, \dots, x+t_{i_1}+\cdots+t_{i_n}; f]}{t_1+\cdots+t_{n+1}} \\
&= \frac{1}{t_1+\cdots+t_{n+1}} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} f(x+t_i) - \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} f(x) \right)
\end{aligned}$$

where the first two summations are for all permutations $(i_1, \dots, i_{n+1})$ of the integers $\{1, \dots, n+1\}$, and we use the notation

$$\prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} = \frac{\Delta_1 \cdots \Delta_{i-1}\Delta_{i+1}\cdots\Delta_{n+1}}{t_1\cdots t_{i-1}t_{i+1}\cdots t_{n+1}}$$

for $i = 1, \dots, n+1$. Using $f(x+t_i) = \Delta_{t_i} f(x) + f(x)$, we can write

$$\begin{aligned}
\prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} f(x+t_i) - \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} f(x) &= \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} \Delta_{t_i} f(x) + \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} f(x) - \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\Delta_{t_j}}{t_j} f(x) \\
&= \frac{\Delta_{t_1}\cdots\Delta_{t_{n+1}} f(x)}{\prod_{\substack{j=1\\ j\neq i}}^{n+1} t_j},
\end{aligned}$$

therefore,

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{n+1})} [x, x+t_{i_1}, \dots, x+t_{i_1}+\cdots+t_{i_{n+1}}; f] \\
&\qquad = \frac{1}{t_1+\cdots+t_{n+1}} \sum_{i=1}^{n+1} t_i \frac{\Delta_{t_1}\cdots\Delta_{t_{n+1}} f(x)}{\prod_{j=1}^{n+1} t_j} \\
&\qquad = \frac{1}{t_1+\cdots+t_{n+1}} (t_1+\cdots+t_{n+1}) \frac{\Delta_{t_1}\cdots\Delta_{t_{n+1}} f(x)}{t_1\cdots t_{n+1}},
\end{aligned}$$

which yields our statement. □

REMARK 5.2. In the case when $t = t_1 = \cdots = t_n$, our lemma gives the well-known property (cf. eg. [13, Chapter XV, Lemma 2.5]) that if n is a positive integer, $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is a function then

$$n!\,[x, x+t, \ldots, x+nt; f] = \frac{\Delta_t^n f(x)}{t^n}$$

for all $x \in I$, $t > 0$ with $x + nt \in I$.

THEOREM 5.3. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function, n be a positive integer and $t_1, \ldots, t_n$ be positive real numbers. If f is symmetrically $(t_1, \ldots, t_n)$-convex on I then it is also $(t_1, \ldots, t_n)$-Wright-convex.*

Proof. Suppose that the assumptions in the theorem are valid. By the definition of symmetric $(t_1, \ldots, t_n)$-convexity, we have

$$[x, x + t_{i_1}h, \ldots, x + (t_{i_1} + \cdots + t_{i_n})h; f] \geqslant 0 \tag{13}$$

for arbitrary permutations $(i_1, \ldots, i_n)$ of the integers $\{1, \ldots, n\}$ and for all $h > 0$ and $x \in I$ satisfying $x + (t_1 + \cdots + t_n)h \in I$. Thus, the sum of the divided differences of the form in (13) is also nonnegative, therefore, by the positivity of h, $t_1, \ldots, t_n$ and using the previous lemma, we obtain that

$$\Delta_{t_1 h} \cdots \Delta_{t_n h} f(x) \geqslant 0$$

for all $h > 0$ and $x \in I$ with $x + (t_1 + \cdots + t_n)h \in I$, which completes the proof. □

COROLLARY 5.4. (Cf. [13, Chapter XV, Theorem 7.1]). *If n is a positive integer and the function $f : I \to \mathbb{R}$ is convex of order $n-1$, then f is also Wright-convex of order $n-1$.*

REMARK 5.5. It is easy to see that symmetric (t_1)-convexity is equivalent to (t_1)-Wright-convexity for every positive t_1. On the other hand, such an equivalence is not valid for integers $n \geqslant 2$: if I is an interval (with positive length) and $n \geqslant 2$ is an integer then there exist positive reals $(t_1, \ldots, t_n)$ for which there exists a function $f : I \to \mathbb{R}$ which is $(t_1, \ldots, t_n)$-Wright-convex on I but it is not symmetrically $(t_1, \ldots, t_n)$-convex there. Namely, let us consider a non-continuous function $f : I \to \mathbb{R}$ satisfying the inequality

$$\Delta_{h_1} \cdots \Delta_{h_n} f(x) \geqslant 0$$

for all $x \in I$, $h_1, \ldots, h_n > 0$ with $x + h_1 + \cdots + h_n \in I$. (Such a non-continuous function can be constructed using Hamel-bases, cf. eg. [9], [13].) By the non-continuity of f, there exist real numbers $x \in I$ and $\bar{h}_1, \ldots, \bar{h}_n > 0$ satisfying $x + \bar{h}_1 + \cdots + \bar{h}_n \in I$ for which

$$[x, x + \bar{h}_1, \ldots, x + \bar{h}_1 + \cdots + \bar{h}_n; f] < 0$$

(in the other case f would be continuous, cf. eg. [13, Chapter XV, §6]), thus, f is not symmetrically $(t_1,\dots,t_n)$-convex for $t_1=\bar{h}_1,\dots,t_n=\bar{h}_n$. However, there are positive n-tuples $(t_1,\dots,t_n)$ for every positive integer n, for which $(t_1,\dots,t_n)$-Wright-convexity implies symmetric $(t_1,\dots,t_n)$-convexity. For example, this statement is valid if $t_1=\cdots=t_n$ for an arbitrary n. These facts lead to the problem of characterizing those n-tuples for which symmetric $(t_1,\dots,t_n)$-convexity is equivalent to $(t_1,\dots,t_n)$-Wright-convexity. Concerning its (partial) solution in the case when $n=2$, we refer to [15] and [16].

Now, we present our equivalence-result on different higher-order convexity properties.

THEOREM 5.6. *Let $r_1,\dots,r_n$ be positive rationals and let $f:I\to\mathbb{R}$ be a function. Then the following conditions are equivalent:*

(i) f is symmetrically $(r_1,\dots,r_n)$-convex;
(ii) f is cyclically $(r_1,\dots,r_n)$-convex;
(iii) f is Jensen-convex of order $n-1$;
(iv) f is $(\underbrace{1,\dots,1}_{n-times})$-Wright-convex;
(v) f is $(r_1,\dots,r_n)$-Wright-convex.

Proof. According to Corollary 3.3, (i) is equivalent to (ii). Corollary 3.3 also implies that cyclic $(r_1,\dots,r_n)$-convexity is equivalent to symmetric $(\underbrace{1,\dots,1}_{n-times})$-convexity, thus (ii) is equivalent to (iii). The definition of higher order Wright-convexity gives the equivalence of (iii) and (iv). Finally, Corollary 4.3 yields the equivalence between (iv) and (v). □

In the classical setting, the statement of the theorem above reduces to various characterizations of (ordinary) Jensen-convexity.

COROLLARY 5.7. (Cf. [15]). *If $r\in]0,1[$ is a rational number then the following conditions are equivalent.*

(i) f is r-convex;
(ii) f is Jensen-convex;
(iii) f is $\frac{1}{2}$-Wright-convex;
(iv) f is r-Wright-convex.

6. Concluding Remarks

The notion of $(t_1,\dots,t_n)$-Wright-convexity has recently been introduced in [6]. In that paper, among others, it has been shown that $(t_1,\dots,t_n)$-Wright-convexity is a localizable property in the sense that if each point of the interval I has a neighborhood such that f restricted to that neighborhood is $(t_1,\dots,t_n)$-Wright-convex then f is $(t_1,\dots,t_n)$-Wright-convex on the entire interval I. Moreover, a particular n^{th}-order derivative has also been constructed in [6] whose nonnegativity characterizes $(t_1,\dots,t_n)$-Wright-convexity. Analogous results have been obtained for t-convexity

by Nikodem and Páles in [19]. In view of Theorem 5.6, $(r_1, \dots, r_n)$-convexity is also a localizable property if $r_1, \dots, r_n$ are positive rationals. However, it is not known if (cyclic/symmetric) $(t_1, \dots, t_n)$-convexity is localizable, or can be characterized via suitably constructed n^{th}-order generalized derivatives.

By Corollary 3.3, the notions of the cyclic and symmetric $(t_1, \dots, t_n)$-convexity coincide if $t_1, \dots, t_n$ are positive rationals. It is an open problem if this remains valid without assuming the rationality of $t_1, \dots, t_n$.

In [15], depending on the algebraic character of t, a t-Wright-convex but strictly Jensen-concave function was constructed. Thus, for such a choice of t, the t-Wright-convexity property does not imply Jensen-convexity. The exact description of the set of numbers t when t-Wright-convexity yields Jensen-convexity has not been found yet. It seems to be an even harder problem to find such a characterization in the higher-order setting.

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