Mathematical
Inequalities
& Applications
Volume 11, Number 4 (2008), 647–653

L'HOSPITAL–TYPE RULES FOR MONOTONICITY: DISCRETE CASE

IOSIF PINELIS

(*communicated by N. Elezović*)

Abstract. Let $f := (f_n)$ and $g := (g_n)$ be two sequences, and let $r := \frac{f}{g}$ and $\rho := \frac{\delta f}{\delta g}$, where $(\delta f)_n := \delta f_n := f_n - f_{n-1}$. In particular, it is shown that, if r is monotonic, then the monotonicity pattern of ρ may switch at most once (from increase to decrease or vice versa). "Continuous" versions of such results were obtained earlier by the author.

1. Introduction

Let $-\infty \leqslant a < b \leqslant \infty$, let f and g be continuously differentiable functions defined on the interval (a,b), and let $r = f/g$ and $\rho = f'/g'$. In [15], general "rules" for monotonicity patterns, resembling the usual l'Hospital rules for limits, were given. For example, according to Proposition 1.9 in [15], one has the following: if ρ is increasing and $gg' > 0$ on (a,b), then $r \searrow\nearrow$, which means that there is some c in $[a,b]$ such that r is decreasing on (a,c) and increasing on (c,b). In particular, if c is either a or b, the result is that r is either increasing or decreasing on the entire interval (a,b). If one also knows whether r is increasing or decreasing in a right neighborhood of a and in a left neighborhood of b, then one can discriminate with certainty between these three patterns. Using such rules, one can generally determine ([15, 18]) the monotonicity pattern of r given that of ρ, however complicated the latter might be.

Clearly, these l'Hospital-type rules for monotonicity patterns are helpful wherever the l'Hospital rules for limits are. Moreover, the monotonicity rules apply even outside such contexts, because they do not require that both f and g (or either of them) tend to 0 or ∞ at any point. (In the special case when both f and g vanish at an endpoint of the interval (a,b), l'Hospital-type rules for monotonicity can be found, in different forms and with different proofs, in [1]–[4], [8]–[10], [12], and [14]–[17].) Thus, it should not be surprising that a wide variety of applications of the l'Hospital-type rules for monotonicity patterns were given: in areas of analytic inequalities [5, 14, 15, 18]; approximation theory [16]; differential geometry [8, 9, 10, 19], information theory [14, 15]; (quasi)conformal mappings [1, 2, 3, 4]; probability and statistics [12, 15, 16, 17, 21, 22, 23, 24], including the very recent papers [21, 22, 23, 24], where these mononicity rules have become a

Mathematics subject classification (2000): 26A48, 11B99, 39A70.
Key words and phrases: l'Hospital-type rules for monotonicity, sequences.

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Paper MIA-11-54

standard tool. In the present note, discrete analogues of l'Hospital-type rules both for monotonicity and limits are given.

2. Rules

Let $f := (f_n : n \in \overline{a,b})$ and $g := (g_n : n \in \overline{a,b})$ be two real sequences or, equivalently, functions defined on an interval $\overline{a,b} := \{n \in \mathbb{Z} : a \leqslant n \leqslant b\}$ of integers, where a and b are in $\mathbb{Z} \cup \{-\infty, \infty\}$ (so that $\overline{a,b} = \emptyset$ if $a > b$, and no infinite endpoint belongs to $\overline{a,b}$). Let

$$r := \frac{f}{g} \quad \text{and} \quad \rho := \frac{\Delta f}{\Delta g},$$

where $(\Delta f)_n := \Delta f_n := f_n - f_{n-1}$ for $n \in \overline{a+1,b}$, so that the function Δf is defined on $\overline{a+1,b}$ (with $\pm\infty + 1 := \pm\infty$). It is assumed throughout that g and Δg do not take on the zero value and do not change their respective signs.

THEOREM 1. *Suppose that ρ is either nondecreasing or nonincreasing. Then the dependence of the monotonicity pattern of r on that of ρ (and also on the sign of $g\,\Delta g$) is given by the following table:*

ρ	$g\,\Delta g$	r
$\nearrow$	> 0	$\searrow\nearrow$
$\searrow$	> 0	$\nearrow\searrow$
$\nearrow$	< 0	$\nearrow\searrow$
$\searrow$	< 0	$\searrow\nearrow$

Table 1.

Here, for instance, the statement $r \searrow\nearrow$ can be taken to mean that there is some k in $\overline{a,b} \cup \{a,b\}$ such that r is nonincreasing ($\searrow$) on $\overline{a,k}$ and nondecreasing ($\nearrow$) on $\overline{k,b}$. In particular, if $k = a$ then $r \searrow\nearrow$ will imply $r \nearrow$ on the entire interval $\overline{a,b}$; similarly, if $k = b$ then $r \searrow\nearrow$ will imply $r \searrow$ on $\overline{a,b}$.

REMARK 1. To discriminate between these three possibilities ($k = a$, $k = b$, and $a < k < b$) in the case when (say) a and b are finite, it suffices to know whether $r_{a+1} \geqslant r_a$ and $r_{b-1} \geqslant r_b$; if (say) $b = \infty$, then one may instead want to know the monotocity pattern of r in a neighborhood of ∞. $\diamondsuit$

Proof of Theorem 1. Without loss of generality, a and b are finite. In view of the "horizontal" and "vertical" reflections $\mathbb{Z} \ni n \leftrightarrow (-n)$ and $f \leftrightarrow (-f)$, it suffices to consider only the first line of Table 1, with $\rho \nearrow$ and $g\,\Delta g > 0$. Next, it suffices to show that there is some $k \in \overline{a,b}$ such that $\Delta r \leqslant 0$ on $\overline{a+1,k}$ and $\Delta r > 0$ on $\overline{k+1,b}$.

Let now $k := \sup\{n \in \overline{a+1,b} \colon \Delta r_n \leqslant 0\}$ (here, $\sup\emptyset := a$). Then $\Delta r > 0$ on $\overline{k+1,b}$. If $\Delta r \leqslant 0$ on $\overline{a+1,k}$, the proof is completed. To obtain a contradiction, assume the contrary, that $\Delta r_n > 0$ for some $n \in \overline{a+1,k}$ (so that $k \neq a$). Hence, there exists $m := \max\{n \in \overline{a+1,k} \colon \Delta r_n > 0\}$. Then in fact $m \in \overline{a+1,k-1}$; this follows because $k \neq a$ and hence, by the definition of k, one has $\Delta r_k \leqslant 0$, while $\Delta r_m > 0$. Now it also follows from the definition of m that $\Delta r_{m+1} \leqslant 0$.

The key observation is that for all $n \in \overline{a+1,b}$

$$g_n g_{n-1}\,\Delta r_n = (\rho_n - r_n)\,g_n\,\Delta g_n = (\rho_n - r_{n-1})\,g_{n-1}\,\Delta g_n. \tag{1}$$

Using these identities (with $n = m$ and $n = m+1$) together with the obtained above inequalities $\Delta r_m > 0$ and $\Delta r_{m+1} \leqslant 0$, one concludes that $\rho_m > r_m \geqslant \rho_{m+1}$, which contradicts the condition that ρ is nondecreasing. $\square$

REMARK 2. (i) For the case given by the first line of Table 1, when $\rho \nearrow$ and $g\,\Delta g > 0$, the above proof shows that there is some $k \in \overline{a,b}$ such that $\Delta r \leqslant 0$ on $\overline{a+1,k}$ and $\Delta r > 0$ on $\overline{k+1,b}$. Using the horizontal reflection $\mathbb{Z} \ni n \leftrightarrow (-n)$, one can then see that there also exists some $\ell \in \overline{a,b}$ such that $\Delta r < 0$ on $\overline{a+1,\ell}$ and $\Delta r \geqslant 0$ on $\overline{\ell+1,b}$. Hence, the conclusion $r \searrow\!\nearrow$ for this first-line case in Table 1 can actually be understood in a slightly stronger sense: that there are some k and ℓ in $\overline{a,b} \cup \{a,b\}$ such that r is (strictly) decreasing on $\overline{a,\ell}$, constant on $\overline{\ell,k}$, and increasing on $\overline{k,b}$. Moreover, identities (1) show that r equals a constant C on an interval $\overline{\ell,k}$ of integers only if $\rho = C$ on $\overline{\ell+1,k}$ (indeed, if $r = C$ on $\overline{\ell,k}$, then $\Delta r = 0$ on $\overline{\ell+1,k}$, and so, $\rho_n = r_n = C$ for all $n \in \overline{\ell+1,k}$, because of the requirement that neither g nor Δg vanish at any point).

(ii) It follows that, if ρ is (strictly) increasing or decreasing, then the statement $r \searrow\!\nearrow$ can be taken to mean that there is some k in $\overline{a,b} \cup \{a,b\}$ such that r is decreasing on $\overline{a,k}$ and increasing on $\overline{k+1,b}$. For further details on this point, see the example at the end of this note.

(iii) Similar comments are valid for the other three cases given by the Table 1. $\diamondsuit$

A "continuous" counterpart of Remark 2 is developed in [25].

Proposition 1.9 in [15], mentioned earlier, was in fact a corollary of a general result, Proposition 1.2 in [15], which also contains the special case when both f and g vanish at an endpoint of the interval (a,b) (as pointed out in Remark 1.5 in [15]). Here, we shall treat the discrete analogue of that special case separately, as follows.

PROPOSITION 1. *Suppose that either (i) $b = \infty$, $f_\infty := \lim_{n\to\infty} f_n = 0$, and $g_\infty = 0$ or (ii) $a = -\infty$, $f_{-\infty} := \lim_{n\to-\infty} f_n = 0$, and $g_{-\infty} = 0$. Suppose also that ρ is nondecreasing or nonincreasing or increasing or decreasing; then r is so, respectively.*

Proof. Consider the case when $b = \infty$, $f_\infty = 0$, and $g_\infty = 0$. Then the result follows immediately from the identity

$$g_n g_{n-1}\,\Delta r_n = \sum_{j=n}^{\infty} \Delta g_n\,\Delta g_j\,(\rho_j - \rho_n)$$

for all $n \in \overline{a+1,\infty}$. The other case can now be proved by the horizontal reflection; compare with the proof of Theorem 1. $\square$

"Discrete" analogues of l'Hospital's rules for limits are well known – see e.g. [7]. We present a version of such rules here, along with a proof, for the sake of completeness and for the readers' convenience:

PROPOSITION 2. *Suppose that $b = \infty$, and either (i) $|g_\infty| = \infty$ or (ii) $f_\infty = g_\infty = 0$. Then $r_\infty = \rho_\infty$ provided that the latter limit, ρ_∞, exists.*

Proof. Let us first consider part (i), assuming that $|g_\infty| = \infty$. For $m < n$, one has $r_n = \dfrac{f_m}{g_n} + r_{m,n}\Big(1 - \dfrac{g_m}{g_n}\Big)$, where $r_{m,n} := \dfrac{f_n - f_m}{g_n - g_m} = \sum\limits_{j=m+1}^{n} \rho_j\,|\Delta g_j| \Big/ \sum\limits_{j=m+1}^{n} |\Delta g_j|$, since Δg is assumed not to change sign. It follows that $r_{m,n}$ is between the minimum and maximum values of ρ over the interval $\overline{m+1,n}$. To complete the proof of part (i), it remains to let $n \to \infty$ and then $m \to \infty$.

Part (ii) is even simpler to prove, since here $r_m = r_{m,\infty}$ for all m. $\square$

Note that condition $|f_\infty| = \infty$ (which is imposed in [7]) is not in fact needed in part (i) of Proposition 2. Moreover, the above proof can be obviously adapted (using, say, the Mean-Value Theorem for the ratio corresponding to $r_{m,n}$) to l'Hospital's original "differentiable" case, even though the condition $|f_\infty| = \infty$ is traditionally included into the formulation of the corresponding l'Hospital rule for limits.

3. Illustrations

Let $p := (p_j : j \in \mathbb{Z})$ be a positive sequence. Then p is called log-convex on $\overline{a,b}$ if $q := \ln p$ is convex on $\overline{a,b}$, in the sense that Δq is nondecreasing on $\overline{a+1,b}$, which is equivalent to the condition that the ratio p_n/p_{n+1} be nonincreasing in $n \in \overline{a,b-1}$. The log-concavity of a sequence and the strict versions of log-convexity and log-concavity are defined similarly.

COROLLARY 1. *Suppose that p is log-convex or log-concave on $\overline{a,\infty}$, and $f_n := \sum_{j=n}^{\infty} p_j < \infty$ for all $n \in \mathbb{Z}$. Then f is, respectively, log-convex or log-concave on $\overline{a,\infty}$. More generally, the same conclusion holds for any natural k if $f = R^k p$, where $R^k p$ is given by the formula*

$$(R^k p)_n := \sum_{j=n}^{\infty} \binom{j-n+k-1}{j-n} p_j \quad \textit{for all } n \in \mathbb{Z}.$$

Proof. Let $g_n := f_{n+1}$ for all $n \in \mathbb{Z}$. Then the first part of Corollary 1 follows immediately from Proposition 1. In turn, this yields the second part of the corollary, because it is easy to see that $(R^k\colon k \in \overline{1,\infty})$ is a semigroup of operators with $(R^1 p)_n = \sum_{j=n}^{\infty} p_j$ for all n; cf. Remark 5 in [13]. $\square$

Corollary 1 is essentially well-known. For the "log-concave" part, see, for example, [13] (where k was allowed to be any positive real number) and, for the continuous counterpart in the case $k = 1$, [6] and [11]. The "log-convex" part can also be obtained from the well-known fact that any linear combination with positive coefficients of log-convex functions is log-convex, having also in mind that the log-convexity is preserved under the shift $n \mapsto n+1$.

COROLLARY 2. *Suppose that p is log-concave on $\overline{0,\infty}$, and $f_n := \sum_{j=0}^{n} p_j$ for all $n \in \overline{0,\infty}$. Then f is strictly log-concave on $\overline{0,\infty}$. More generally, the same conclusion holds for any natural k if $f = L^k p$, where $L^k p$ is given by the formula*

$$(L^k p)_n := \sum_{j=0}^{n} \binom{n-j+k-1}{n-j} p_j \quad \textit{for all } n \in \overline{0,\infty}.$$

Proof. Let again $g_n := f_{n+1}$ for all $n \in \overline{0,\infty}$. Then the first part of Corollary 2 follows immediately from part (i) of Remark 2 (since $g_0 g_1 \Delta r_1 > p_1^2 - p_0 p_2 \geqslant 0$, and so, $r_1 > r_0$). In turn, this yields the second part of the corollary, since $(L^k\colon k \in \overline{1,\infty})$ is a semigroup with $(L^1 p)_n = \sum_{j=0}^{n} p_j$ for all n. (One can note that $(L^k p)_n = (T^{-1} R^k T p)_n$ for all $n \in \overline{0,\infty}$, all natural k, and all p such that $p = 0$ on $\overline{-\infty,-1}$, where $(Tp)_n := p_{-n}$ for all n.) $\square$

However, the "log-convex" analogue of Corollary 2 does not hold. Indeed, if a sequence p is both log-convex and log-concave (that is, geometric) then, by Corollary 2, f is strictly log-concave and hence not log-convex. Of course, this observation does not contradict Theorem 1 or Remark 2, according to which the log-convexity of p implies that f may switch at most once from log-concavity to log-convexity.

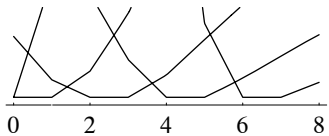
Let us conclude this paper with another illustration of presented results and methods; note the use of Remarks 1 and 2 and identities (1).

EXAMPLE. Let sequences $f := f^{(\alpha)}$ and g on $\overline{0,\infty}$ be given by the formulas $f_n^{(\alpha)} := \alpha + \sum_{j=0}^{n} p_j$ and $g_n := \sum_{j=0}^{n} q_j$ for any $\alpha \geqslant 0$ and all $n \in \overline{0,\infty}$, where p and q are positive sequences such that $\rho = p/q$ is increasing on $\overline{1,\infty}$ and $r_0^{(0)} < r_1^{(0)}$, where in turn $r^{(\alpha)} := f^{(\alpha)}/g$. (For instance, one can take $p_j = j!$ and $q_j = (j/e)^j$ for all $j \in \overline{0,\infty}$, assuming that $0^0 = 1$.)

Then, by Theorem 1 (or, rather, by part (ii) of Remark 2; cf. Remark 1), the sequence $r^{(0)}$ is increasing on the entire interval $\overline{0,\infty}$. Moreover, for every $\alpha > 0$ there is some k_α in $\overline{0,\infty} \cup \{\infty\}$ such that $r^{(\alpha)}$ is decreasing on $\overline{0,k_\alpha}$ and increasing on $\overline{k_\alpha+1,\infty}$.

The observed condition that $r^{(0)}$ is increasing on $\overline{0,\infty}$ implies $\Delta r^{(0)} > 0$ and hence, by (1), $\rho > r^{(0)}$ on $\overline{1,\infty}$. On the other hand, $r^{(\alpha)} = r^{(0)} + \alpha/g$. Therefore, for each $k \in \overline{1,\infty}$, the number $\alpha_k := (\rho_k - r_k^{(0)})\, g_k$ is positive and satisfies the equation $\rho_k = r_k^{(\alpha_k)}$, so that, by (1), $(\Delta r^{(\alpha_k)})_k = 0$; that is, $r_k^{(\alpha_k)} = r_{k-1}^{(\alpha_k)}$. Using again part (ii) of Remark 2, one sees that $r^{(\alpha_k)}$ is decreasing on $\overline{0,k-1}$, constant on $\{k-1,k\}$, and increasing on $\overline{k,\infty}$. It follows, in particular, that one cannot replace $\overline{k+1,b}$ in part (ii) of Remark 2 by $\overline{k,b}$ (keeping the rest unchanged).

In view of (1), observe also that $r_k^{(\alpha)} \geqslant r_{k-1}^{(\alpha)} \iff \rho_k \geqslant r_k^{(\alpha)} \iff \alpha \leqslant \alpha_k$, for all $\alpha \geqslant 0$, and these equivalences hold if all three non-strict inequalities here are replaced by the corresponding strict ones. In particular, one has $\alpha \leqslant \alpha_k \implies r_k^{(\alpha)} \geqslant r_{k-1}^{(\alpha)} \implies r_{k+1}^{(\alpha)} > r_k^{(\alpha)} \implies \alpha < \alpha_{k+1}$; the second implication here follows again by part (ii) of Remark 2. Hence, α_k is increasing in k. Moreover, if $\alpha \in (\alpha_k, \alpha_{k+1})$ then $r_{k-1}^{(\alpha)} > r_k^{(\alpha)} < r_{k+1}^{(\alpha)}$, so that $r_n^{(\alpha)}$ is decreasing in $n \in \overline{1,k}$ and increasing in $n \in \overline{k,\infty}$; recall that here k was taken to be any natural number.



To illustrate, here are parts of the graphs of the linear interpolations of the sequences $n \mapsto r_n^{(\alpha_k)}/r_k^{(\alpha_k)} - 0.97$ for $k = 0,1,3,5,7$ where $\alpha_0 := 0$ and, as above, $p_j = j!$ and $q_j = (j/e)^j$ with $0^0 := 1$.

To visualize the idea of this example, one can imagine a tank with the solution of a liquid in water. Initially, at time $n = 0$, the amounts of the liquid and water are $f_0^{(\alpha)} = \alpha + p_0$ and $g_0 = q_0$, respectively, so that the initial relative concentration of the liquid (with respect to water) is $r_0^{(\alpha)} = (\alpha + p_0)/q_0$. Suppose that, at each of the time moments $n = 1,2,\dots$, the liquid and water are added to the tank in the amounts of $\Delta f_n = p_n$ and $\Delta g_n = q_n$, respectively, so that the relative concentration of the liquid in the nth addition is $\rho_n = p_n/q_n$ and that in the tank at time n is $r_n^{(\alpha)}$. If α is large enough, then initially the relative concentration ρ of the liquid in what is added is less than the relative concentration $r^{(\alpha)}$ of the liquid in the tank, so that $r^{(\alpha)}$ will be decreasing in time. However, at least in the case when ρ is increasing to ∞, ρ will eventually overtake $r^{(\alpha)}$, and the latter will then be forever increasing. $\diamondsuit$

REFERENCES

[1] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY, AND M. VUORINEN, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192** (2000) 1–37.

[2] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *Inequalities for quasiconformal mappings in space*, Pacific J. Math. **160** (1993) 1–18.

[3] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley, New York, 1997.

[4] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *Topics in special functions*, in Papers on Analysis, Rep. Univ. Jyväskylä Dep. Math. Stat., no. 83 (2001), Jyväskylä, Finland, pp. 5–26.

[5] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *M*onotonicity of some functions in calculus, Amer. Math. Monthly, **113** (2006), 805–816.

[6] R. E. BARLOW, A. W. MARSHALL, AND F. PROSHAN, *Properties of probability distributions with monotone hazard rate*, Ann. Math. Statist. **34** (1963) 375–389.

[7] R. P. BOAS, JR., *Lhospital's Rule Without Mean-Value Theorems*, Amer. Math. Monthly **76** 9 (1969) 1051–1053.

[8] I. CHAVEL, *Riemannian Geometry—A Modern Introduction*, Cambridge University Press, Cambridge, 1993.

[9] J. CHEEGER, M. GROMOV, AND M. TAYLOR, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Diff. Geom. **17** (1982) 15–54.

[10] M. GROMOV, *Isoperimetric inequalities in Riemannian manifolds*, in Asymptotic Theory of Finite Dimensional Spaces, Lecture Notes in Math., no. 1200, Springer-Verlag, Berlin, 1986, pp. 114–129.

[11] R. L. HALL, M. KANTER, AND M. D. PERLMAN, *Inequalities for probability content of a rotated square and related convolutions*, Ann. Statist. **8** (1980) 802–813.

[12] I. PINELIS, *Extremal probabilistic problems and Hotelling's T^2 test under symmetry condition*, (preprint, 1991, `http://arXiv:math/0701806v1`); a shorter version of the preprint appeared in Ann. Statist. **22** (1994) 357–368.

[13] I. PINELIS, *Fractional sums and integrals of r-concave tails and applications to comparison probability inequalities. Advances in stochastic inequalities*, (Atlanta, GA, 1997), 149–168, Contemp. Math. **234** Amer. Math. Soc., Providence, RI, 1999.

[14] I. PINELIS, *L'Hospital type rules for monotonicity, with applications*, J. Inequal. Pure Appl. Math. **3** (2002), article 5, 5 pp. (electronic).

[15] I. PINELIS, *L'Hospital type rules for oscillation, with applications*, J. Inequal. Pure Appl. Math. **2** (2001), article 33, 24 pp. (electronic).

[16] I. PINELIS, *Monotonicity properties of the relative error of a Padé approximation for Mills' ratio*, J. Inequal. Pure Appl. Math. **3** (2002), article 20, 8 pp. (electronic).

[17] I. PINELIS, *L'Hospital type rules for monotonicity: Applications to probability inequalities for sums of bounded random variables*, J. Inequal. Pure Appl. Math. **3** (2002), article 7, 9 pp. (electronic).

[18] I. PINELIS, *L'Hospital rules for monotonicity and the Wilker-Anglesio inequality*, Amer. Math. Monthly **111** (2004) 905–909.

[19] I. PINELIS, *L'Hospital-Type Rules for Monotonicity, and the Lambert and Saccheri Quadrilaterals in Hyperbolic Geometry*, J. Inequal. Pure Appl. Math. **6** 4 (2005), Article 99, 12 pp. (electronic).

[20] I. PINELIS, *On L'Hospital-Type Rules for Monotonicity*, J. Inequal. Pure Appl. Math. **7** 2 (2006), Article 40, 19 pp. (electronic).

[21] I. PINELIS, *Binomial upper bounds on generalized moments and tail probabilities of (super)martingales with differences bounded from above*, IMS Lecture Notes-Monograph Series, High Dimensional Probability, Vol. 51 (2006). Institute of Mathematical Statistics, 2006. DOI: 10.1214/074921706000000743. `http://arxiv.org/abs/math.PR/0512301`.

[22] I. PINELIS, *Normal domination of (super)martingales*, (2006) Electronic J. Probab., Vol. 11, Paper 39, 1049–1070, `http://www.math.washington.edu/ ejpecp/viewarticle.php?id=1648&layout=abstract`.

[23] I. PINELIS, *On inequalities for sums of bounded random variables*, (2006) Preprint, `http://arxiv.org/abs/math.PR/0603030`.

[24] I. PINELIS, *Toward the best constant factor for the Rademacher-Gaussian tail comparison*, (2006) Preprint, `http://arxiv.org/abs/math.PR/0605340`. To appear in ESAIM Probability and Statistics

[25] I. PINELIS, *"Non-strict" l'Hospital-type rules for monotonicity: intervals of constancy*, J. Inequal. Pure Appl. Math. **8** 1 (2007), Article 14, 8 pp. (electronic).

(Received January 12, 2007)

Department of Mathematical Sciences
Michigan Technological University
Houghton
Michigan 49931
USA
e-mail: `ipinelis@mtu.edu`