

SOME NEW INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS IN NORMED SPACES

N. S. BARNETT AND S. S. DRAGOMIR

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Abstract. Some new inequalities for convex functions defined on convex subsets in linear spaces with applications for the p -mean absolute deviation of a sequence of vectors are given, in a normed linear space.

1. Introduction

Jensen's inequality is pivotal in the Theory of Inequalities because it implies at once many other classical inequalities including the Hölder, Minkowski, Beckenbach-Dresher and Young inequalities, the arithmetic mean – geometric mean inequality, the generalised triangle inequality.

Let C be a convex subset of the real linear space X and $f : C \rightarrow \mathbb{R}$ a convex function on C . If $x_i \in C$ and $p_i \in (0, 1)$ with $\sum_{i=1}^n p_i = 1$, then the following well-known form of Jensen's discrete inequality holds:

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i). \quad (1.1)$$

In [2], the authors proved, amongst other results, the following refinement of Jensen's inequality in the general setting of linear spaces:

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geq \max_{1 \leq i < j \leq n} \left\{ p_i f(x_i) + p_j f(x_j) - (p_i + p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right) \right\} \geq 0. \quad (1.2)$$

As a natural and important application of the above result (1.2), the authors of [2] considered the case of normed linear spaces $(X, \|\cdot\|)$ and the convex function $f(x) =$

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$\|x\|^r$, $r \geq 1$, obtaining refinements of the generalised triangle inequality:

$$\begin{aligned} & \left\| \sum_{i=1}^n p_i \|x_i\|^r - \left\| \sum_{i=1}^n p_i x_i \right\|^r \right. \\ & \geq \max_{1 \leq i < j \leq n} \left\{ p_i \|x_i\|^r + p_j \|x_j\|^r - (p_i + p_j)^{1-r} \|p_i x_i + p_j x_j\|^r \right\} \geq 0. \end{aligned} \tag{1.3}$$

More recently, the second author [1] has proved the following result:

$$\begin{aligned} & \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j f(x_j) - f \left(\sum_{j=1}^n q_j x_j \right) \right] \geq \sum_{j=1}^n p_j f(x_j) - f \left(\sum_{j=1}^n p_j x_j \right) \\ & \geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j f(x_j) - f \left(\sum_{j=1}^n q_j x_j \right) \right], \end{aligned} \tag{1.4}$$

provided $f : C \rightarrow \mathbb{R}$ is convex on the convex subset C of the linear space X and $p_i, q_i, i \in \{1, \dots, n\}$ are probability sequences with $q_i > 0$ for each $i \in \{1, \dots, n\}$.

In particular, from (1.4) the following is obtained that compares the weighted and unweighted Jensen differences:

$$\begin{aligned} & n \max_{1 \leq i \leq n} \{p_i\} \left[\frac{1}{n} \sum_{j=1}^n f(x_j) - f \left(\frac{1}{n} \sum_{j=1}^n x_j \right) \right] \geq \sum_{j=1}^n p_j f(x_j) - f \left(\sum_{j=1}^n p_j x_j \right) \\ & \geq n \min_{1 \leq i \leq n} \{p_i\} \left[\frac{1}{n} \sum_{j=1}^n f(x_j) - f \left(\frac{1}{n} \sum_{j=1}^n x_j \right) \right]. \end{aligned} \tag{1.5}$$

The above inequalities (1.4) and (1.5) have some nice applications for the generalised triangle inequality in normed linear spaces:

$$\begin{aligned} & \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\|^r - \left\| \sum_{j=1}^n q_j x_j \right\|^r \right] \geq \sum_{j=1}^n p_j \|x_j\|^r - \left\| \sum_{j=1}^n p_j x_j \right\|^r \\ & \geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\|^r - \left\| \sum_{j=1}^n q_j x_j \right\|^r \right] (\geq 0), \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} & \max_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \right] \geq \sum_{j=1}^n p_j \|x_j\|^p - \left\| \sum_{j=1}^n p_j x_j \right\|^p \\ & \geq \min_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \right] (\geq 0), \end{aligned} \tag{1.7}$$

respectively.

For some classical results on convex functions and their inequalities, see [4], while for a contemporary approach on these subjects, see [3] and the references therein.

In this paper some new inequalities for convex functions defined on linear spaces that complement the results from (1.2) and (1.4) are given. Applications for the p -mean absolute deviation of a sequence of vectors in a normed linear space with given probabilities are also provided.

2. The Main Results

THEOREM 1. *Let C be a convex subset in the linear space X , $f : C \rightarrow \mathbb{R}$ a convex function on C , $x_j \in C$, $p_j \in (0, 1)$, $j \in \{1, \dots, n\}$, $n \geq 2$ and $\sum_{j=1}^n p_j = 1$. If $\sum_{j=1}^n p_j x_j = 0$, then,*

$$\begin{aligned} \sum_{j=1}^n p_j f(x_j) &\geq \max_{k \in \{1, \dots, n\}} \left[p_k f(x_k) + (1 - p_k) f\left(\frac{p_k}{p_k - 1} \cdot x_k\right) \right] \\ &\geq \min_{k \in \{1, \dots, n\}} \left[p_k f(x_k) + (1 - p_k) f\left(\frac{p_k}{p_k - 1} \cdot x_k\right) \right] \\ &\geq f(0). \end{aligned} \tag{2.1}$$

In particular, if $\sum_{j=1}^n x_j = 0$, then

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n f(x_j) &\geq \frac{1}{n} \max_{k \in \{1, \dots, n\}} \left[f(x_k) + (n - 1) f\left(\frac{n}{n - 1} \cdot x_k\right) \right] \\ &\geq \frac{1}{n} \min_{k \in \{1, \dots, n\}} \left[f(x_k) + (n - 1) f\left(\frac{n}{n - 1} \cdot x_k\right) \right] \\ &\geq f(0). \end{aligned} \tag{2.2}$$

Proof. Since $\sum_{j=1}^n p_j x_j = 0$, we have,

$$p_k x_k = - \sum_{\substack{j=1 \\ j \neq k}}^n p_j x_j = \frac{p_k - 1}{\sum_{\substack{j=1 \\ j \neq k}}^n p_j} \cdot \sum_{\substack{j=1 \\ j \neq k}}^n p_j x_j$$

for each $k \in \{1, \dots, n\}$, which implies,

$$\frac{p_k}{p_k - 1} \cdot x_k = \frac{1}{\sum_{\substack{j=1 \\ j \neq k}}^n p_j} \cdot \sum_{\substack{j=1 \\ j \neq k}}^n p_j x_j \in C \tag{2.3}$$

for each $k \in \{1, \dots, n\}$, since in the right hand side of (2.3) we have a convex combination of elements from C .

Applying the Jensen inequality, we have from (2.3) that,

$$\begin{aligned} f\left(\frac{p_k}{p_k-1} \cdot x_k\right) &= f\left(\frac{1}{\sum_{\substack{j=1 \\ j \neq k}}^n p_j} \cdot \sum_{\substack{j=1 \\ j \neq k}}^n p_j x_j\right) \\ &\leq \frac{\sum_{\substack{j=1 \\ j \neq k}}^n p_j f(x_j)}{\sum_{\substack{j=1 \\ j \neq k}}^n p_j} = \frac{\sum_{j=1}^n p_j f(x_j) - p_k f(x_k)}{1 - p_k}, \end{aligned}$$

from which it is obvious that,

$$p_k f(x_k) + (1 - p_k) f\left(\frac{p_k}{p_k-1} \cdot x_k\right) \leq \sum_{j=1}^n p_j f(x_j) \quad (2.4)$$

for each $k \in \{1, \dots, n\}$.

Taking the maximum in (2.4) over $k \in \{1, \dots, n\}$, we deduce the first part of (2.1).

Now, since C is convex and $x_k, \frac{p_k}{p_k-1} \cdot x_k \in C$ for $k \in \{1, \dots, n\}$, then by the convexity of f ,

$$p_k f(x_k) + (1 - p_k) f\left(\frac{p_k}{p_k-1} \cdot x_k\right) \geq f(0)$$

for each $k \in \{1, \dots, n\}$, which proves the last part of (2.1). \square

The following result can be useful for applications.

COROLLARY 1. *Let $f : C \rightarrow \mathbb{R}$ be a convex function on the convex set C and $q_j \in (0, 1)$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^n q_j = 1$. If $v_i \in X$, $i \in \{1, \dots, n\}$ are such that,*

$$v_k - \sum_{l=1}^n q_l v_l \in C \quad \text{for each } k \in \{1, \dots, n\}, \quad (2.5)$$

then,

$$\begin{aligned} &\sum_{j=1}^n q_j f\left(v_j - \sum_{l=1}^n q_l v_l\right) \\ &\geq \max_{k \in \{1, \dots, n\}} \left\{ q_k f\left(v_k - \sum_{l=1}^n q_l v_l\right) + (1 - q_k) f\left[\frac{q_k}{1 - q_k} \left(\sum_{l=1}^n q_l v_l - v_k\right)\right] \right\} \\ &\geq \min_{k \in \{1, \dots, n\}} \left\{ q_k f\left(v_k - \sum_{l=1}^n q_l v_l\right) + (1 - q_k) f\left[\frac{q_k}{1 - q_k} \left(\sum_{l=1}^n q_l v_l - v_k\right)\right] \right\} \\ &\geq f(0). \end{aligned} \quad (2.6)$$

In particular, if

$$v_k - \frac{1}{n} \sum_{l=1}^n v_l \in C \quad \text{for each } k \in \{1, \dots, n\}, \tag{2.7}$$

then

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^n f \left(v_j - \frac{1}{n} \sum_{l=1}^n v_l \right) \\ & \geq \frac{1}{n} \max_{k \in \{1, \dots, n\}} \left\{ f \left(v_k - \frac{1}{n} \sum_{l=1}^n v_l \right) + (n-1) f \left[\frac{n}{n-1} \left(\frac{1}{n} \sum_{l=1}^n v_l - v_k \right) \right] \right\} \\ & \geq \frac{1}{n} \min_{k \in \{1, \dots, n\}} \left\{ f \left(v_k - \frac{1}{n} \sum_{l=1}^n v_l \right) + (n-1) f \left[\frac{n}{n-1} \left(\frac{1}{n} \sum_{l=1}^n v_l - v_k \right) \right] \right\} \\ & \geq f(0). \end{aligned} \tag{2.8}$$

The proof follows by Theorem 1 on choosing $x_j = v_j - \sum_{l=1}^n q_l v_l$ and $p_j = q_j$, $j \in \{1, \dots, n\}$.

COROLLARY 2. *Let $f : C \rightarrow \mathbb{R}$ be a convex function on the convex set C and $x_i \in X$, $i \in \{1, \dots, n\}$ such that, for $y_1 := x_1 - x_n$, $y_2 := x_2 - x_1, \dots, y_{n-1} := x_{n-1} - x_{n-2}, y_n := x_n - x_{n-1}$, we have $y_k \in C$ for each $k \in \{1, \dots, n\}$. It follows that,*

$$\begin{aligned} & \frac{1}{n} [f(x_1 - x_n) + f(x_2 - x_1) + \dots + f(x_{n-1} - x_{n-2}) + f(x_n - x_{n-1})] \\ & \geq \frac{1}{n} \max \left\{ f(x_1 - x_n) + (n-1) f \left[\frac{n}{n-1} (x_1 - x_n) \right], \dots, \right. \\ & \quad \left. f(x_n - x_{n-1}) + (n-1) f \left[\frac{n}{n-1} (x_n - x_{n-1}) \right] \right\} \\ & \geq \frac{1}{n} \min \left\{ f(x_1 - x_n) + (n-1) f \left[\frac{n}{n-1} (x_1 - x_n) \right], \dots, \right. \\ & \quad \left. f(x_n - x_{n-1}) + (n-1) f \left[\frac{n}{n-1} (x_n - x_{n-1}) \right] \right\} \\ & \geq f(0). \end{aligned}$$

The proof is obvious by the second part of Theorem 1.

A different result is incorporated in the following.

THEOREM 2. *Let C be a convex set in the linear space X and $f : C \rightarrow \mathbb{R}$ be a convex function on C . If $x_j \in C$, $p_i \in (0, 1)$, $j \in \{1, \dots, n\}$ are such that $\sum_{j=1}^n p_j = 1$*

and $-x_k \in C$ for each $k \in \{1, \dots, n\}$, then,

$$\begin{aligned} & \sum_{j=1}^n p_j \left[\frac{f(x_j) + f(-x_j)}{2} \right] \\ & \geq \max_{k \in \{1, \dots, n\}} \left[\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) \right] \\ & \geq \min_{k \in \{1, \dots, n\}} \left[\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) \right] \\ & \geq f(0). \end{aligned} \tag{2.9}$$

In particular,

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \frac{f(x_j) + f(-x_j)}{2} & \geq \frac{1}{n} \max_{k \in \{1, \dots, n\}} \left\{ \frac{1}{2} f(-x_k) + (2n - 1) f\left(\frac{2n x_k}{2n - 1}\right) \right\} \\ & \geq \frac{1}{n} \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{2} f(-x_k) + (2n - 1) f\left(\frac{2n x_k}{2n - 1}\right) \right\} \\ & \geq f(0). \end{aligned} \tag{2.10}$$

Proof. For any $k \in \{1, \dots, n\}$ we have,

$$\sum_{i=1}^n p_i x_i = p_k x_k + \sum_{\substack{j=1 \\ j \neq k}}^n p_j x_j,$$

which gives,

$$\begin{aligned} p_k x_k & = \sum_{i=1}^n p_i x_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j (-x_j) \\ & = \frac{\sum_{i=1}^n p_i x_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j (-x_j)}{\sum_{i=1}^n p_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j} \cdot \left(\sum_{i=1}^n p_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j \right) \\ & = \frac{\sum_{i=1}^n p_i x_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j (-x_j)}{\sum_{i=1}^n p_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j} (1 + 1 - p_k). \end{aligned}$$

This obviously implies,

$$\frac{p_k x_k}{2 - p_k} = \frac{\sum_{i=1}^n p_i x_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j (-x_j)}{\sum_{i=1}^n p_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j} \in C \tag{2.11}$$

for each $k \in \{1, \dots, n\}$, since in the right side of (2.11) we have a convex combination of elements from C .

Applying Jensen’s inequality, we have from (2.11) that,

$$\begin{aligned} f\left(\frac{p_k x_k}{2 - p_k}\right) &= f\left(\frac{\sum_{i=1}^n p_i x_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j (-x_j)}{\sum_{i=1}^n p_i + \sum_{\substack{j=1 \\ j \neq k}}^n p_j}\right) \\ &\leq \frac{\sum_{i=1}^n p_i f(x_i) + \sum_{\substack{j=1 \\ j \neq k}}^n p_j f(-x_j)}{1 + 1 - p_k} \\ &= \frac{\sum_{i=1}^n p_i [f(x_i) + f(-x_i)] - p_k f(x_k)}{2 - p_k}, \end{aligned}$$

which is clearly equivalent to,

$$\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) \leq \sum_{i=1}^n p_i \left[\frac{f(x_i) + f(-x_i)}{2}\right], \tag{2.12}$$

for each $k \in \{1, \dots, n\}$.

Taking the supremum over $k \in \{1, \dots, n\}$ in (2.12) produces the first inequality in (2.9).

By the convexity of f we also have:

$$\begin{aligned} \frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) &\geq f\left[\frac{p_k}{2} (-x_k) + \left(1 - \frac{p_k}{2}\right) \frac{p_k x_k}{2 - p_k}\right] \\ &= f(0), \end{aligned}$$

and the last part of (2.9) is also established. □

3. Applications for Normed Spaces

Let $(X, \|\cdot\|)$ be a normed space over the real or complex number field \mathbb{K} .

For the probability sequence $\mathbf{p} = (p_1, \dots, p_n)$, $i \in \{1, \dots, n\}$, the sequence of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and a real number $p \geq 1$, we define the p -mean absolute deviation of \mathbf{x} with probability \mathbf{p} by:

$$K_p(\mathbf{p}, \mathbf{x}) := \sum_{j=1}^n p_j \left\| x_j - \sum_{l=1}^n p_l x_l \right\|^p. \tag{3.1}$$

For the uniform probability $\mathbf{u} = (\frac{1}{n}, \dots, \frac{1}{n})$ we have $K_p(\mathbf{u}, \mathbf{x}) = K_p(\mathbf{x})$, where,

$$K_p(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \left\| x_j - \frac{1}{n} \sum_{l=1}^n x_l \right\|^p. \tag{3.2}$$

The following result concerning upper and lower bounds for the p -mean absolute deviation can be stated:

PROPOSITION 1. *With the above, we have,*

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^p &\geq K_p(\mathbf{p}, \mathbf{x}) \\ &\geq \max_{k \in \{1, \dots, n\}} \left\{ \left[p_k + p_k^p (1 - p_k)^{1-p} \right] \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^p \right\}, \end{aligned} \tag{3.3}$$

for any $\mathbf{x} \in X^n$, $p \geq 1$ and \mathbf{p} a probability sequence.

In particular,

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^n x_l \right\|^p &\geq K_p(\mathbf{x}) \\ &\geq \frac{1}{n} \left[1 + (n - 1)^{1-p} \right] \max_{k \in \{1, \dots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^n x_l \right\|^p \end{aligned} \tag{3.4}$$

for all $\mathbf{x} \in X^n$ and $p \geq 1$.

Proof. The first inequality in (3.3) is obvious, the second follows by Corollary 1 applied for the convex function $f : X \rightarrow \mathbb{R}$, $f(x) = \|x\|^p$. The details are omitted. □

REMARK 1. The case $p = 1$ produces the inequalities,

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\| &\geq K(\mathbf{p}, \mathbf{x}) \\ &\geq 2 \max_{k \in \{1, \dots, n\}} \left\{ p_k \left\| x_k - \sum_{l=1}^n p_l x_l \right\| \right\} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^n x_l \right\| &\geq K(\mathbf{x}) \\ &\geq \frac{2}{n} \max_{k \in \{1, \dots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^n x_l \right\|, \end{aligned} \tag{3.6}$$

where $K(\mathbf{p}, \mathbf{x}) = K_1(\mathbf{p}, \mathbf{x})$ and $K(\mathbf{x}) = K_1(\mathbf{x})$.

If $\sigma^2(\mathbf{p}, \mathbf{x}) = K_2(\mathbf{p}, \mathbf{x})$, where $\sigma^2(\mathbf{p}, \mathbf{x})$ denotes the variance of \mathbf{x} with the probability \mathbf{p} , then we have,

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^2 &\geq \sigma^2(\mathbf{p}, \mathbf{x}) \\ &\geq \max_{k \in \{1, \dots, n\}} \left\{ \frac{p_k}{p_k - 1} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^2 \right\}, \end{aligned} \tag{3.7}$$

for any $\mathbf{x} \in X^n$ and \mathbf{p} a probability density.

Also, if $\sigma^2(\mathbf{x}) = K_2(\mathbf{x})$,

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^n x_l \right\| &\geq \sigma^2(\mathbf{x}) \\ &\geq \frac{n}{n-1} \max_{k \in \{1, \dots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^n x_l \right\|^2. \end{aligned} \tag{3.8}$$

We notice that if $X = H$, H an inner product space, then $\sigma(\mathbf{p}, \mathbf{x})$ and $\sigma(\mathbf{x})$ can be represented as,

$$\sigma(\mathbf{p}, \mathbf{x}) = \left(\sum_{j=1}^n p_j \|x_j\|^2 - \left\| \sum_{j=1}^n p_j x_j \right\|^2 \right)^{\frac{1}{2}},$$

while

$$\sigma(\mathbf{x}) = \left(\frac{1}{n} \sum_{j=1}^n \|x_j\|^2 - \left\| \frac{1}{n} \sum_{j=1}^n x_j \right\|^2 \right)^{\frac{1}{2}}.$$

Since the lower bound for $K_p(\mathbf{p}, \mathbf{x})$ may be difficult to use in applications, we provide the following coarser but perhaps more useful bound.

COROLLARY 3. *If $p_m := \min_{k \in \{1, \dots, n\}} p_k$, $p_m \in (0, 1)$, then*

$$K_p(\mathbf{p}, \mathbf{x}) \geq \left[p_m + p_m^p (1 - p_m)^{1-p} \right] \max_{k \in \{1, \dots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^p \tag{3.9}$$

for all $\mathbf{x} \in X^n$.

Proof. For $p \geq 1$, consider the function $h_p : [0, 1) \rightarrow \mathbb{R}$, $h_p(t) := t + t^p(1-t)^{1-p}$. The function h_p is differentiable on $[0, 1)$ and

$$h'_p(t) = 1 + pt^{p-1}(1-t)^{1-p} + (p-1)t^p(1-t)^{-p} > 0,$$

for any $t \in [0, 1)$, showing that h_p is strictly increasing on $[0, 1)$. It follows that,

$$\min_{k \in \{1, \dots, n\}} \left[p_k + p_k^p(1-p_k)^{1-p} \right] = p_m + p_m^p(1-p_m)^{1-p},$$

which together with (3.3) provides the desired bound (3.9). \square

REMARK 2. In particular, we have,

$$K(\mathbf{p}, \mathbf{x}) \geq 2p_m \max_{k \in \{1, \dots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\| \quad (3.10)$$

and

$$\sigma^2(\mathbf{p}, \mathbf{x}) \geq \frac{p_m}{1-p_m} \max_{k \in \{1, \dots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^2. \quad (3.11)$$

From a different perspective, we can state the following inequalities as well.

PROPOSITION 2. Let $(X, \|\cdot\|)$ be a normed linear space, $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, $p \geq 1$ and $p_i \in (0, 1)$ with $\sum_{i=1}^n p_i = 1$, then,

$$\sum_{i=1}^n p_i \|x_i\|^p \geq \frac{1}{2} \max_{k \in \{1, \dots, n\}} \left\{ \left[p_k + p_k^p(2-p_k)^{1-p} \right] \|x_k\|^p \right\}, \quad (3.12)$$

for any \mathbf{x}, \mathbf{p} and p as above.

In particular,

$$\frac{1}{n} \sum_{i=1}^n \|x_i\|^p \geq \frac{1}{2n} \left[1 + (2n-1)^{1-p} \right] \max_{k \in \{1, \dots, n\}} \|x_k\|^p \quad (3.13)$$

for any $\mathbf{x} \in X^n$.

The proof is obvious by Theorem 2 applied for the convex function $f : X \rightarrow \mathbb{R}_+$, $f(x) = \|x\|^p$. The details are omitted.

REMARK 3. The case $p = 2$ gives the simple inequalities:

$$\sum_{i=1}^n p_i \|x_i\|^2 \geq \max_{k \in \{1, \dots, n\}} \left\{ \frac{p_k}{2 - p_k} \|x_k\|^2 \right\} \tag{3.14}$$

and

$$\frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \geq \frac{1}{2n - 1} \max_{k \in \{1, \dots, n\}} \|x_k\|^2. \tag{3.15}$$

As pointed out before, for applications, the lower bound for the quantity $\sum_{i=1}^n p_i \|x_i\|^2$ may not be as useful as one where the p_k 's and x_k 's are separate. This can be achieved, however, by the following coarser result:

COROLLARY 4. If $p_m := \min_{k \in \{1, \dots, n\}} p_k$, $p_m \in (0, 1)$, then,

$$\sum_{i=1}^n p_i \|x_i\|^p \geq \frac{1}{2} \left[p_m + p_m^p (2 - p_m)^{1-p} \right] \max_{k \in \{1, \dots, n\}} \|x_k\|^p, \tag{3.16}$$

for any $\mathbf{x} \in X^n$.

Proof. Consider the function $g_p : [0, 1) \rightarrow \mathbb{R}$, $g_p(t) = t + t^p (2 - t)^{1-p}$ which is differentiable on $[0, 1)$ and

$$g'_p(t) = 1 + p t^{p-1} (2 - t)^{1-p} + (p - 1) t^p (2 - t)^{-p} > 0$$

for any $t \in [0, 1)$, showing that g_p is strictly increasing on $[0, 1)$. Therefore,

$$\min_{k \in \{1, \dots, n\}} \left[p_k + p_k^p (2 - p_k)^{1-p} \right] = p_m + p_m^p (2 - p_m)^{1-p},$$

which, together with (3.12), provides the desired result (3.16). □

REMARK 4. In particular,

$$\sum_{i=1}^n p_i \|x_i\|^2 \geq \frac{p_m}{2 - p_m} \max_{k \in \{1, \dots, n\}} \|x_k\|^2, \tag{3.17}$$

for any $\mathbf{x} \in X^n$.

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N. S. Barnett
School of Computer Science and Mathematics
Victoria University
PO Box 14 428
Melbourne City, VIC 8001
Australia
e-mail: neil.barnett@vu.edu.au

Sever S. Dragomir
School of Computer Science & Mathematics
Victoria University
PO Box 14428
Melbourne City
MC 8001
Australia
e-mail: Sever.Dragomir@vu.edu.au
URL: <http://rgmia.vu.edu.au/dragomir/>