

CONVOLUTIONS AND HÖLDER–TYPE INEQUALITIES FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract. Applying the coefficient inequalities for functions $f(z)$ belonging to a certain class $\mathcal{M}\mathcal{D}(\alpha, \beta)$ of normalized analytic functions in the open unit disk \mathbb{U} , two subclasses $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ are defined. The object of the present paper is to derive some properties for functions $f(z)$ in the classes $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ involving their generalized convolution by utilizing methods based upon the Hölder-type inequalities.

1. Introduction, definitions and preliminaries

Let \mathcal{A} be the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Nishiwaki and Owa ([3], [5]) considered the subclass $\mathcal{M}\mathcal{D}(\alpha, \beta)$ of \mathcal{A} consisting of functions $f(z)$ which satisfy the following inequality (see also [2] and the references cited therein):

$$\Re \left(\frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U}; \alpha \leq 0; \beta > 1).$$

With a view to discussing some properties of functions $f(z)$ belonging to the class $\mathcal{M}\mathcal{D}(\alpha, \beta)$, we note that, if $f(z) \in \mathcal{M}\mathcal{D}(\alpha, \beta)$, then the function

$$\frac{zf'(z)}{f(z)} = u + iv$$

maps the open unit disk \mathbb{U} onto the *elliptic domain* such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 < \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2} \quad (\alpha < -1),$$

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onto the *parabolic domain* such that

$$u < -\frac{1}{2(\beta-1)}v^2 + \frac{\beta+1}{2} \quad (\alpha = -1),$$

and onto the *hyperbolic domain* such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 - \frac{\alpha^2}{1 - \alpha^2}v^2 > \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2} \quad (-1 < \alpha \leq 0).$$

In fact, Nishiwaki and Owa [3] gave the following coefficient inequality for functions $f(z)$ belonging to the class $\mathcal{M}\mathcal{D}(\alpha, \beta)$.

LEMMA. (Nishiwaki and Owa [3]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta| \quad (1.2)$$

$$(\alpha \leq 0; \beta > 1),$$

then $f(z) \in \mathcal{M}\mathcal{D}(\alpha, \beta)$.

It is easily observed from the above Lemma that

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(n - \beta + 1) + |n - \beta - 1| - 2\alpha(n - 1)}{2(\beta - 1)} |a_n| \\ \leq \sum_{n=2}^{\infty} \frac{(n - \beta + 1) + (n + \beta - 3) - 2\alpha(n - 1)}{2(\beta - 1)} |a_n| \\ \leq 1 \quad (1 < \beta \leq 2) \end{aligned} \quad (1.3)$$

and

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{2} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \\ \leq \sum_{n=2}^{\infty} \frac{1}{2} \{(n + \beta - 3) + (n + \beta - 3) - 2\alpha(n - 1)\} |a_n| \\ \leq 1 \quad (\beta \geq 2). \end{aligned} \quad (1.4)$$

In view of the inequalities (1.3) and (1.4), we define the subclass $\mathcal{M}_1(\alpha, \beta)$ of the class $\mathcal{M}\mathcal{D}(\alpha, \beta)$ consisting of functions $f(z)$ which satisfy the following condition:

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta-1} |a_n| \leq 1 \quad (\alpha \leq 0; 1 < \beta \leq 2) \quad (1.5)$$

and also the subclass $\mathcal{M}_2(\alpha, \beta)$ of the class $\mathcal{M}\mathcal{D}(\alpha, \beta)$ consisting of functions $f(z)$ which satisfy the following condition:

$$\sum_{n=2}^{\infty} \{n(1 - \alpha) - 3 + \alpha + \beta\} |a_n| \leq 1 \quad (\alpha \leq 0; \beta \geq 2). \tag{1.6}$$

In our present investigation, we aim at presenting some interesting properties of functions $f(z)$ in the above-defined function classes $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ involving a certain generalized convolution. We also briefly consider a number of corollaries and illustrative examples associated with some of our main results.

2. Convolution properties for functions in the class $\mathcal{M}_1(\alpha, \beta)$

In this section, some convolution properties of functions $f(z)$ belonging to the classes $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ are discussed. First of all, for functions $f_j(z) \in \mathcal{A}$ ($j = 1, \dots, m$) given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{nj} z^n \quad (j = 1, \dots, m), \tag{2.1}$$

we define the following generalization of the Hadamard product (or convolution):

$$H_m(z) := z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^m a_{nj}^{p_j} \right) z^n \quad (p_j > 0; j = 1, \dots, m). \tag{2.2}$$

The generalized convolution $H_m(z)$ was considered by Choi *et al.* [1] and (*more recently*) by Owa and Srivastava [6] (see also [4]).

For functions $f_j(z) \in \mathcal{A}$ ($j = 1, \dots, m$) given by (2.1), the familiar Hölder inequality assumes the following form:

$$\sum_{n=2}^{\infty} \left(\prod_{j=1}^m |a_{nj}| \right) \leq \prod_{j=1}^m \left(\sum_{n=2}^{\infty} |a_{nj}|^{p_j} \right)^{\frac{1}{p_j}} \tag{2.3}$$

$$\left(p_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{p_j} \geq 1 \right).$$

Our first result for the generalized convolution $H_m(z)$ defined by (2.2) is contained in Theorem 1 below.

THEOREM 1. *If*

$$f_j(z) \in \mathcal{M}_1(\alpha, \beta_j) \quad (\alpha \leq 0; 1 < \beta_j \leq 2; j = 1, \dots, m),$$

then $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$ with

$$\beta^* := 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1 - \alpha)^{s-1}} \quad (2.4)$$

$$\left(s = \sum_{j=1}^m p_j \geq 1; p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

Proof. Let $f_j(z) \in \mathcal{M}_1(\alpha, \beta_j)$ ($j = 1, \dots, m$). Then the inequality (1.5) readily yields

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta_j - 1} |a_{n,j}| \leq 1 \quad (j = 1, \dots, m),$$

which, in turn, implies that

$$\left(\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta_j - 1} |a_{n,j}| \right)^{\frac{1}{q_j}} \leq 1$$

$$\left(q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} = 1 \right).$$

Applying the Hölder inequality (2.3), we arrive at the following inequality:

$$\sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}} \leq 1.$$

Thus we have to determine the largest β^* such that

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j} \right) \leq 1,$$

that is, that

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j} \right) \leq \sum_{n=2}^{\infty} \left[\prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}} \right].$$

Therefore, we need to find the largest β^* such that

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j} \right) \leq \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}},$$

which is equivalent to the following inequality:

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j - \frac{1}{q_j}} \right) \leq \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} \quad (n \geq 2).$$

Since

$$\prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j - \frac{1}{q_j}} |a_{n,j}|^{p_j - \frac{1}{q_j}} \leq 1 \quad \left(p_j - \frac{1}{q_j} \geq 0 \quad (j = 1, \dots, m) \right),$$

we see that

$$\prod_{j=1}^m |a_{n,j}|^{p_j - \frac{1}{q_j}} \leq \frac{1}{\prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j - \frac{1}{q_j}}}. \tag{2.5}$$

This last inequality (2.5) implies that

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \leq \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j} \quad (n \geq 2),$$

so that we find for β^* that

$$\beta^* \geq 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1} (n-1)^{s-1}} \quad \left(s = \sum_{j=1}^m p_j \right). \tag{2.6}$$

The right-hand side of the inequality (2.6) is a *decreasing* function for integers $n \geq 2$. This means that

$$\begin{aligned} \beta^* &= \max_{n \geq 2} \left\{ 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1} (n-1)^{s-1}} \right\} \\ &= 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1}}, \end{aligned}$$

which is precisely the assertion (2.4) of Theorem 1.

EXAMPLE 1. Let us choose the functions $f_j(z)$ ($j = 1, \dots, m$) as given below:

$$f_j(z) = z + \sum_{n=2}^{\infty} \frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} z^n \quad (|\varepsilon_j| = 1 \quad (j = 1, \dots, m)). \tag{2.7}$$

It is easy to see that $f_j(z) \in \mathcal{M}_1(\alpha, \beta_j)$ ($j = 1, \dots, m$). Furthermore, in this case, the definition (2.2) yields

$$H_m(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^m \left(\frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} \right)^{p_j} \right) z^n. \quad (2.8)$$

For the function $H_m(z)$ given by (2.8), we easily find that

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha)}{\beta^* - 1} \right) \left| \prod_{j=1}^m \left(\frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} \right)^{p_j} \right| \\ &= \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)^{2s-1}} \\ &\leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1, \end{aligned}$$

which shows that the generalized convolution $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$, where β^* is given by (2.3).

Upon setting $\beta_j = \beta$ ($j = 1, \dots, m$) in Theorem 1, we obtain Corollary 1 below.

COROLLARY 1. *If*

$$f_j(z) \in \mathcal{M}_1(\alpha, \beta) \quad (j = 1, \dots, m; \alpha \leq 0; 1 < \beta \leq 2),$$

then $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$ with

$$\beta^* := 1 + \frac{(\beta - 1)^s}{(1 - \alpha)^{s-1}} \quad (2.9)$$

$$\left(s = \sum_{j=1}^m p_j \geq 1; p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

By using $\mathcal{M}_1(\alpha_j, \beta)$ instead of $\mathcal{M}_1(\alpha, \beta_j)$ in Theorem 1, we can also derive our next result.

THEOREM 2. *If*

$$f_j(z) \in \mathcal{M}_1(\alpha_j, \beta) \quad (j = 1, \dots, m; \alpha_j \leq 0 \quad (j = 1, \dots, m); 1 < \beta \leq 2),$$

then $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$ with

$$\alpha^* := 1 - \frac{\prod_{j=1}^m (1 - \alpha_j)^{p_j}}{(\beta - 1)^{s-1}} \quad (2.10)$$

$$\left(s = \sum_{j=1}^m p_j \geq 1; p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

Proof. Using the same method as in the proof of Theorem 1, we have

$$\frac{(n-1)(1-\alpha^*)}{\beta-1} \leq \frac{(n-1)^s \prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^s},$$

which implies that

$$\alpha^* \geq 1 - \frac{(n-1)^{s-1} \prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}} \quad (n \geq 2). \tag{2.11}$$

The right-hand side of this last inequality (2.11) is *decreasing* for integers $n \geq 2$. This means that

$$\begin{aligned} \alpha^* &= \max_{n \geq 2} \left\{ 1 - \frac{(n-1)^{s-1} \prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}} \right\} \\ &= 1 - \frac{\prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}}, \end{aligned}$$

which evidently proves Theorem 2.

EXAMPLE 2. Let us consider the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$f_j(z) = z + \sum_{n=2}^{\infty} \frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)} z^n \quad (|\varepsilon_j| = 1 \quad (j = 1, \dots, m)). \tag{2.12}$$

Then we see that $f_j(z) \in \mathcal{M}_1(\alpha_j, \beta)$ ($j = 1, \dots, m$) and that

$$H_m(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^m \left(\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)} \right)^{p_j} \right) z^n.$$

It follows for this generalized convolution $H_m(z)$ that

$$\begin{aligned} &\sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha^*)}{\beta-1} \right) \left| \prod_{j=1}^m \left(\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)} \right)^{p_j} \right| \\ &= \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)^{2s-1}} \\ &\leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1, \end{aligned}$$

which implies that $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$.

Letting $\alpha_j = \alpha$ ($j = 1, 2, \dots, m$) in Theorem 2, we deduce Corollary 2 below.

COROLLARY 2. *If*

$$f_j(z) \in \mathcal{M}_1(\alpha, \beta) \quad (j = 1, \dots, m; \alpha \leq 0; 1 < \beta \leq 2),$$

then $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$ with

$$\alpha^* := 1 + \frac{(1 - \alpha)^s}{(\beta - 1)^{s-1}}, \quad (2.13)$$

$$\left(s = \sum_{j=1}^m p_j \geq 1; p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

3. Convolution Properties for Functions in the Class $\mathcal{M}_2(\alpha, \beta)$

In this section, for the generalized Hadamard product (or convolution) $H_m(z)$ defined by (2.2) for functions in the class $\mathcal{M}_2(\alpha, \beta)$, we first derive Theorem 3 below.

THEOREM 3. *If*

$$f_j(z) \in \mathcal{M}_2(\alpha, \beta_j) \quad (j = 1, \dots, m; \alpha \leq 0; \beta_j \geq 2),$$

then $H_m(z) \in \mathcal{M}_2(\alpha, \beta^*)$ with

$$\beta^* := 1 + \alpha + \prod_{j=1}^m (\beta_j - 1 - \alpha)^{p_j} \quad (3.1)$$

$$\left(p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

Proof. In the same manner as in the proof of Theorem 1, we obtain

$$\beta^* + (n - 1)(1 - \alpha) - 2 \leq \prod_{j=1}^m [n(1 - \alpha) - 3 + \alpha + \beta_j]^{p_j}. \quad (3.2)$$

The right-hand side of the inequality (3.2) is an *increasing* function for integers $n \geq 2$. Thus we get

$$\beta^* + (n - 1)(1 - \alpha) - 2 \leq \prod_{j=1}^m (\beta_j - 1 - \alpha)^{p_j}.$$

Moreover, the left-hand side of the inequality (3.2) is also an *increasing* function for $n \geq 2$, so that we have

$$\beta^* \leq 1 + \alpha + \prod_{j=1}^m (\beta_j - 1 - \alpha)^{p_j},$$

which completes the proof of Theorem 3.

If we take $\beta_j = \beta$ ($j = 1, \dots, m$) in Theorem 3, then we obtain Corollary 3 below.

COROLLARY 3. *If*

$$f_j(z) \in \mathcal{M}_2(\alpha, \beta) \quad (j = 1, \dots, m; \alpha \leq 0; \beta \geq 2),$$

then $H_m(z) \in \mathcal{M}_2(\alpha, \beta^*)$ with

$$\beta^* := 1 + \alpha + (\beta - 1 - \alpha)^s \tag{3.3}$$

$$\left(s = \sum_{j=1}^m p_j \geq 1; p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

Using the function class $\mathcal{M}_2(\alpha_j, \beta)$ instead of $\mathcal{M}_1(\alpha, \beta_j)$ in Theorem 3, we also derive our next result.

THEOREM 4. *If*

$$f_j(z) \in \mathcal{M}_2(\alpha_j, \beta) \quad (j = 1, \dots, m; \alpha_j \leq 0; \beta \geq 2),$$

then $H_m(z) \in \mathcal{M}_2(\alpha^*, \beta)$ with

$$\alpha^* := \max_{n \geq 2} \left\{ 1 - \frac{(\beta - 2) + \prod_{j=1}^m [n(1 - \alpha_j) - 3 + \alpha_j + \beta]^{p_j}}{n - 1} \right\} \tag{3.4}$$

$$\left(p_j \geq \frac{1}{q_j} \quad (j = 1, \dots, m); q_j > 1 \quad (j = 1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

Proof. By using the same method as in the proof of Theorem 1, we see that

$$n - 3 - \alpha^*(n - 1) + \beta \leq \prod_{j=1}^m [n(1 - \alpha_j) - 3 + \alpha_j + \beta]^{p_j},$$

which implies that

$$\alpha^* \geq 1 - \frac{(\beta - 2) + \prod_{j=1}^m (n(1 - \alpha_j) - 3 + \alpha_j + \beta)^{p_j}}{n - 1}.$$

We thus have completed the proof of Theorem 4.

Finally, by setting $\alpha_j = \alpha$ ($j = 1, \dots, m$) in Theorem 4, we deduce Corollary 4 below.

COROLLARY 4. *If*

$$f_j(z) \in \mathcal{M}_2(\alpha, \beta) \quad (j = 1, \dots, m; \alpha \leq 0; \beta \geq 2),$$

then $H_m(z) \in \mathcal{M}_2(\alpha^*, \beta)$ with

$$\alpha^* := 3 - \beta - (\beta - 1 - \alpha)^s \quad (3.5)$$

$$\left(s = \sum_{j=1}^m p_j \geq 1 + \frac{2(\beta-2)}{1-\alpha}; p_j \geq \frac{1}{q_j} \quad (j=1, \dots, m); q_j > 1 \quad (j=1, \dots, m); \sum_{j=1}^m \frac{1}{q_j} \geq 1 \right).$$

Proof. In view of Theorem 4, we have

$$\alpha^* \geq 1 - \frac{(\beta - 2) + [n(1 - \alpha) - 3 + \alpha + \beta]^s}{n - 1}. \quad (3.6)$$

Let $F(n)$ denote the right-hand side of the inequality (3.6). We also denote by $G(n)$ the numerator of $F'(n)$, so that

$$\begin{aligned} G(n) &= -[n(1-\alpha) - 3 + \alpha + \beta]^{s-1} [n(1-\alpha)(s-1) - s(1-\alpha) + 3 - \alpha - \beta] + (\beta-2) \\ &\leq -(\beta - 1 - \alpha)^{s-1} [2(1 - \alpha)(s - 1) - s(1 - \alpha) + 3 - \alpha - \beta] + (\beta - 2) \\ &\leq 2\beta - 3 - \alpha - s(1 - \alpha) \\ &\leq 0 \quad \left(s \geq 1 + \frac{2(\beta - 2)}{1 - \alpha} \right), \end{aligned}$$

which implies that

$$\begin{aligned} \alpha^* &= \max_{n \geq 2} \left\{ 1 - \frac{(\beta - 2) + \prod_{j=1}^m [n(1 - \alpha_j) - 3 + \alpha_j + \beta]^{p_j}}{n - 1} \right\} \\ &= 3 - \beta - (\beta - 1 - \alpha)^s. \end{aligned}$$

This evidently completes the proof of Corollary 4.

We conclude our investigation by remarking that several other special cases and consequences can be deduced from each of our main results presented here.

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