

UNIFORM BOUNDS FOR THE COMPLEMENTARY INCOMPLETE GAMMA FUNCTION

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(communicated by N. Elezović)

Abstract. We prove upper and lower bounds for the complementary incomplete gamma function $\Gamma(a, z)$ with complex parameters a and z . Our bounds are refined within the circular hyperboloid of one sheet $\{(a, z) : |z| > c|a - 1|\}$ with a real and z complex. Our results show that within the hyperboloid, $|\Gamma(a, z)|$ is of order $|z|^{a-1}e^{-\operatorname{Re}(z)}$, and extends an upper estimate of Natalini and Palumbo to complex values of z .

1. Introduction

Euler's gamma function

$$\Gamma(a) := \int_0^\infty e^{-x} x^{a-1} dx, \quad (1.1)$$

defined for complex a , with simple poles at $-1, -2, \dots$, is an important object in many areas of mathematics and has been widely studied. The incomplete gamma function $\gamma(a, z)$ and its complement $\Gamma(a, z)$, defined for $z > 0$ by

$$\gamma(a, z) := \int_0^z e^{-x} x^{a-1} dx \quad (1.2)$$

and

$$\Gamma(a, z) := \int_z^\infty e^{-x} x^{a-1} dx, \quad (1.3)$$

also appear in many different contexts and applications. For example, $\gamma(a, z)$ appears in the asymptotic expansions of the Bessel functions [10, pp. 204–205], and $\Gamma(a, z)$ is closely related to the generalized complementary error function

$$\operatorname{erfc}_p(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^p} dt = \frac{2}{p\sqrt{\pi}} \Gamma\left(\frac{1}{p}, x^p\right). \quad (1.4)$$

One can find an extended and highly readable overview on $\gamma(a, z)$ and $\Gamma(a, z)$ in [3].

Our present investigation was motivated by the need for explicit upper bounds on $|\Gamma(a, z)|$ in large regions of the complex z -plane in order to give effective asymptotic

Mathematics subject classification (2000): 33B20.

Keywords and phrases: Incomplete Gamma Function, Inequalities, Uniform Bounds.

¹ Research supported by NSERC and the Canada Research Chair program.

formulas for Laguerre Polynomials [2]. While there are many asymptotic formulas [5], [7], [8], [9], and inequalities [1], [4], [6] for $\Gamma(a, z)$ in the literature, these results were not applicable in the context of [2] either because the errors to the asymptotics were non-effective, or because the inequalities were only proved for restricted domains.

For example, Alzer [1, eq. 2.6] derived inequalities for the generalized complementary error function, and thus for $\Gamma(1/p, x^p)$ by (1.4), but only for real positive x and p . Of particular note are the inequalities of Natalini and Palumbo [4], which include the following: for $a > 1$, $B > 1$ and $x > \frac{B}{B-1}(a-1)$, we have

$$x^{a-1}e^{-x} < |\Gamma(a, x)| < Bx^{a-1}e^{-x}. \quad (1.5)$$

The inequalities of both Alzer and Natalini-Palumbo were proved for *real* a and z . In this note, we prove upper and lower bounds for $|\Gamma(a, z)|$ that are uniform in the region $|z| > c|a-1|$, where c is a constant and both a and z are complex. We also prove that our upper bound is asymptotically tight.

2. Bounds on $\Gamma(a, z)$

We begin by making a change of variable in (1.3) to obtain the following alternative definition for $\Gamma(a, z)$.

$$\Gamma(a, z) = z^a e^{-z} \int_0^\infty e^{-zs} (1+s)^{a-1} ds. \quad (2.1)$$

It is clear that (2.1) can be analytically continued for complex values of both a and z , and converges for all a when $\operatorname{Re}(z) > 0$, and at least for $\operatorname{Re}(a) < 0$ if $\operatorname{Re}(z) = 0$. With $\Gamma(a, z)$ expressed as in (2.1), estimating $|\Gamma(a, z)|$ reduces to estimating the integral

$$I(a, z) := \int_0^\infty e^{-zs} (1+s)^{a-1} ds. \quad (2.2)$$

For convenience, we write $z = x + iy$, $a = u + iv$, where $u, v, x, y \in \mathbb{R}$. We begin with two trivial estimates.

THEOREM 2.1. *When $x > 0$, we have*

$$|I(a, z)| \leq \begin{cases} \frac{1}{x - (u-1)}, & \text{if } u \geq 1, \\ \frac{1}{x}, & \text{if } u \leq 1. \end{cases} \quad (2.3)$$

Proof. When $u \geq 1$, we have $0 < (1+s)^{u-1} \leq (e^s)^{u-1}$ for $s \in [0, \infty)$. Therefore,

$$\begin{aligned} |I(a, z)| &\leq \int_0^\infty |e^{-zs} (1+s)^{a-1}| ds \\ &\leq \int_0^\infty e^{-xs} e^{(u-1)s} ds = \frac{1}{x - (u-1)}. \end{aligned}$$

When $u \leq 1$, we use the estimate $0 < (1 + s)^{u-1} \leq 1$. □

THEOREM 2.2. For $x \geq 0$ and $u < 0$, we have

$$|I(a, z)| \leq -\frac{1}{u}. \tag{2.4}$$

Proof. We apply the same argument as that of Theorem 2.1 except we bound $|e^{-zs}| \leq 1$ instead of $(1 + s)^{u-1}$. This yields

$$\begin{aligned} |I(a, z)| &\leq \int_0^\infty |e^{-zs}(1 + s)^{a-1}| ds \\ &\leq \int_0^\infty (1 + s)^{u-1} ds = -\frac{1}{u}. \end{aligned}$$

□

We may now prove our key upper bound.

THEOREM 2.3. Let a and z be complex numbers with $\operatorname{Re}(z) > 0$, and set $\theta_n := \arg(a - n)$, where n is any positive integer. Then, for $u = \operatorname{Re}(a) \notin \mathbb{Z}$, we have,

$$|I(a, z)| \leq \begin{cases} \frac{1}{|z|} (1 + |\sec \theta_1|), & \text{if } u < 1, \\ \frac{1}{|z|} \left(\sum_{k=0}^{N-2} \left| \frac{a-1}{z} \right|^k + (1 + |\sec \theta_N|) \left| \frac{a-1}{z} \right|^{N-1} \right), & \text{if } u > 1, \end{cases} \tag{2.5}$$

where $N = \lceil u \rceil$ is the smallest integer greater than or equal to u .

Proof. Since we are integrating with respect to the real variable s , we integrate by parts to find that

$$\begin{aligned} I(a, z) &= \int_0^\infty e^{-zs}(1 + s)^{a-1} ds \\ &= -\frac{e^{-zs}}{z}(1 + s)^{a-1} \Big|_{s=0}^\infty + \int_0^\infty \frac{a-1}{z} e^{-zs}(1 + s)^{a-2} ds \\ &= \frac{1}{z} + \frac{a-1}{z} I(a-1, z). \end{aligned} \tag{2.6}$$

For $u < 1$, we have, by (2.6) and Theorem 2.2,

$$\begin{aligned} |I(a, z)| &\leq \frac{1}{|z|} + \left| \frac{1-a}{z} \right| \cdot \frac{1}{1-u} \\ &= \frac{1}{|z|} \left(1 + \sqrt{1 + \tan^2 \theta_1} \right) = \frac{1}{|z|} (1 + |\sec \theta_1|). \end{aligned} \tag{2.7}$$

Now, suppose $u > 1$. By iterating (2.6), we find that for any fixed integer $N > 0$, we have

$$I(a, z) = \sum_{k=0}^{N-1} \frac{(a-1)(a-2) \cdots (a-k)}{z^{k+1}} + \frac{(a-1) \cdots (a-N)}{z^N} I(a-N, z). \tag{2.8}$$

Set $N = \lceil u \rceil$. Then for $1 \leq k < N$ we have,

$$|(a - 1) \cdots (a - k)| \leq |a - 1|^k,$$

since $u > 1$. Clearly, we also have

$$|(a - 1) \cdots (a - N)| \leq |a - 1|^{N-1} |a - N|. \tag{2.9}$$

Thus, using (2.8), (2.9), and Theorem 2.2, we find that

$$\begin{aligned} |I(a, z)| &\leq \sum_{k=0}^{N-1} \left| \frac{(a - 1)(a - 2) \cdots (a - k)}{z^{k+1}} \right| + \left| \frac{(a - 1) \cdots (a - N)}{z^N} I(a - N, z) \right| \\ &\leq \frac{1}{|z|} \sum_{k=0}^{N-1} \frac{|a - 1|^k}{|z|^k} + \frac{|a - 1|^{N-1} |a - N|}{|z|^N} |I(a - N, z)| \\ &\leq \frac{1}{|z|} \left(\sum_{k=0}^{N-1} \left| \frac{a - 1}{z} \right|^k + \left| \frac{a - 1}{z} \right|^{N-1} |\sec \theta_N| \right). \end{aligned} \tag{2.10}$$

□

Note that θ_N tends to zero as $\text{Im}(a)$ approaches zero, provided that $\text{Re}(a) > 1$ and is not an integer. Hence, for real a , we may refine the estimate in (2.5) by improving the bound in (2.9).

THEOREM 2.4. *Let z be complex with $\text{Re}(z) > 0$ and a be real. Let $N = \lceil a \rceil$ as in Theorem 2.3. Then we have*

$$|I(a, z)| \leq \begin{cases} \frac{2}{|z|}, & \text{if } a < 1, \\ \frac{1}{|z|} \sum_{k=0}^{N-1} \left| \frac{a - 1}{z} \right|^k, & \text{if } a \geq 1, a \in \mathbb{Z}, \\ \frac{1}{|z|} \left(\sum_{k=0}^{N-1} \left| \frac{a - 1}{z} \right|^k + \left| \frac{a - 1}{z} \right|^{N-1} \frac{(N - 1)!}{(N - 1)^{N-1}} \right), & \text{if } a > 1, a \notin \mathbb{Z}. \end{cases} \tag{2.11}$$

Proof. When $a < 1$, the inequality follows from Theorem 2.3 since $\theta_1 = \pi$. When $a > 1$, we follow the proof of Theorem 2.3 except that instead of the estimate (2.9), we use

$$|(a - 1) \cdots (a - N + 1)| \leq |a - 1|^{N-1} \prod_{k=0}^{N-1} \left| \frac{a - 1 - k}{a - 1} \right| \leq |a - 1|^{N-1} \frac{(N - 1)!}{(N - 1)^{N-1}}, \tag{2.12}$$

since for any $0 < m \leq n$ and $\varepsilon > 0$ we have

$$\frac{m}{n} \leq \frac{m + \varepsilon}{n + \varepsilon}.$$

Applying (2.12) and Theorem 2.2 to the N th iterate of (2.6) gives the desired result, since when a is an integer, the product $(a - 1) \cdots (a - N) = 0$. \square

Given the form of the bound in Theorem 2.4, it is natural to consider what happens within the circular hyperboloid of one sheet given by $\{(a, z) : |z| > c|a - 1|\}$. In fact, in this region, we obtain a very clean upper bound. The original requirement in [2] was met with $c = 2$.

COROLLARY 2.5. *Let $c > 1$. For complex z and real a with $\operatorname{Re}(z) > 0$, $a \geq 1$, and $|z| \geq c(a - 1)$, we have*

$$|I(a, z)| \leq \frac{1}{|z|} \cdot \frac{c}{c - 1}. \tag{2.13}$$

This is valid for $a \geq 1$ if $1 < c \leq 440$ and valid for $a > 2$ for all $c > 1$.

Proof. Since $|z| \geq c(a - 1)$ we may bound (2.11) by

$$\begin{aligned} |I(a, z)| &\leq \frac{1}{|z|} \left(\sum_{k=0}^{N-1} c^{-k} + \frac{(N - 1)!}{(N - 1)^{N-1} c^{N-1}} \right) \\ &= \frac{1}{|z|} \left(\frac{c - c^{-N}}{c - 1} + \frac{(N - 1)!}{(N - 1)^{N-1} c^{N-1}} \right). \end{aligned}$$

By Stirling’s Formula, we know that

$$n! < n^n e^{-n+1/12n} \sqrt{2\pi n}.$$

For convenience, let $n = N - 1$. Thus

$$\begin{aligned} |I(a, z)| &\leq \frac{1}{|z|} \left(\frac{c - c^{-n+1}}{c - 1} + c^{-n} e^{-n+1/12n} \sqrt{2\pi n} \right) \\ &= \frac{1}{|z|} \left(\frac{c}{c - 1} + \frac{c^{-n}(c - 1)e^{-n+1/12n} \sqrt{2\pi n} - c^{-n+1}}{c - 1} \right) \\ &\leq \frac{1}{|z|} \cdot \frac{c}{c - 1} \end{aligned}$$

whenever

$$(c - 1)e^{-n+1/12n} \sqrt{2\pi n} - c \leq 0.$$

Or, equivalently,

$$e^{-n+1/12n} \sqrt{2\pi n} \leq 1 + \frac{1}{c - 1}.$$

It is easy to see that the left-hand side is a decreasing function of n for $n \geq 1$, and is less than one if $n \geq 2$. The inequality is valid for $n = 1$ if

$$1 < c \leq 1 + \frac{1}{\sqrt{2\pi}e^{-11/12} - 1} \approx 440.66 \dots$$

\square

The upper bound in Corollary 2.5 is best possible in the following limiting sense.

THEOREM 2.6.

$$\lim_{a \rightarrow \infty} (a - 1)I(a, c(a - 1)) = \frac{1}{c - 1}. \tag{2.14}$$

Proof. We derive the result from the following well-known [3, Eq. 2.12], [9] asymptotic formula for $\Gamma(a, z)$.

$$\Gamma(a + 1, x) = \frac{e^{-x}x^a}{x - a} \left(1 - \frac{a}{(x - a)^2} + \frac{2a}{(x - a)^3} + O\left(\frac{a^2}{(x - a)^4}\right) \right), \tag{2.15}$$

as $\sqrt{a}/(x - a)$ tends to zero. Thus, by the definition of $I(a, z)$, we find that,

$$\begin{aligned} \lim_{a \rightarrow \infty} aI(a + 1, ca) &= \lim_{a \rightarrow \infty} a\Gamma(a + 1, ca)e^{ca}(ca)^{-a} \\ &= \lim_{a \rightarrow \infty} \frac{a}{ca - a} \left(1 + O\left(\frac{a}{(ca - a)^2}\right) \right) \\ &= \frac{1}{c - 1}, \end{aligned} \tag{2.16}$$

since $\sqrt{a}/(ca - a) \rightarrow 0$ as $a \rightarrow \infty$. Replacing a with $a - 1$ gives the stated claim. \square

3. A lower bound

To determine a lower bound for $|I(a, z)|$, we apply the functional equation (2.6).

THEOREM 3.1. *Let $c \geq 2$. Let z be complex and a be real with $\text{Re}(z) > 0$, $a > 3$, and $|z| \geq c(a - 1)$. Then we have*

$$|I(a, z)| \geq \frac{1}{|z|} \cdot \frac{c - 2}{c - 1}. \tag{3.1}$$

Proof. By (2.6), we have

$$I(a, z) - \frac{a - 1}{z}I(a - 1, z) = \frac{1}{z}.$$

Therefore, by the triangle inequality we find that

$$|I(a, z)| \geq \frac{1}{|z|} - \frac{|a - 1|}{|z|}|I(a - 1, z)|.$$

Since $|z| \geq c(a - 1) > c(a - 1 - 1)$ and $a - 1 > 2$ we may apply the upper bound in Corollary 2.5 to $|I(a - 1, z)|$. The result follows after some simplification. \square

We close with a two-sided corollary.

COROLLARY 3.2. *Let c, z , and a be as in Theorem 3.1, and recall that $x = \text{Re}(z)$. Then*

$$\frac{c - 2}{c - 1}|z|^{a-1}e^{-x} \leq |\Gamma(a, z)| \leq \frac{c}{c - 1}|z|^{a-1}e^{-x}. \tag{3.2}$$

Let us compare (3.2) with the Natalini-Palumbo bound (1.5). If we let $c = B/(B - 1)$, we readily find that $B = c/(c - 1)$. Thus, we see that the upper bound in (3.2) extends (1.5) to complex z . However, even though the lower bound in (3.2) is of the same order as that of (1.5), it is much weaker when c is near 2.

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(Received February 16, 2007)

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