

NEW REFINEMENTS OF SOME CLASSICAL INEQUALITIES

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Abstract. In this paper a refinement of a particular case of an inequality of Radon, referred in the literature as Bergström's inequality, is presented. Later on, this result is used to obtain refinements of some known inequalities among them the classical inequality of Cauchy. Finally, two elementary numerical inequalities are also given.

1. Introduction

In 1913, Radon (see [1], p. 1351 and [3], p. 61) published the following

THEOREM 1. *If $p > 0$, $x_k \geq 0$, $a_k > 0$, ($1 \leq k \leq n$), then*

$$\sum_{k=1}^n \frac{x_k^{p+1}}{a_k^p} \geq \left(\sum_{k=1}^n x_k \right)^{p+1} / \left(\sum_{k=1}^n a_k \right)^p$$

A generalization of this result has been used to obtain inequalities involving manifolds (see [2], p. 692), but surely the most well known is the particular case when $p = 1$, called in the literature Bergström's inequality (see [4], [5], [6], [7], [8]). Namely,

THEOREM 2. *If $x_k \in \mathbb{R}$, $a_k > 0$, ($1 \leq k \leq n$), then*

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n} \quad (1)$$

Notice that in both theorems equality holds when $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_n}{a_n}$. A complex version of (1) in the case when $n = 2$ can be easily obtained from the identity

$$\frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} - \frac{|z_1 + z_2|^2}{u+v} = \frac{|vz_1 - uz_2|^2}{uv(u+v)}$$

valid for all $z_1, z_2 \in \mathbb{C}$ and $u, v \in \mathbb{R} - \{0\}$ such that $u + v \neq 0$ (see [8], p. 315). Furthermore, the preceding result is connected with Cauchy-Buniakowsky-Schwarz (CBS) inequality, as can be easily checked (see [9]). Our goal in this paper is to refine (1) and applying this result to obtain a new refinement of CBS inequality. Finally, with the aid of Lagrange's identity, elementary numerical inequalities will also be obtained.

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2. Main results

Our main results are generalizations of Bergström's and CBS inequalities. We begin with

THEOREM 3. *If $x_k \in \mathbb{R}$, $a_k > 0$, ($1 \leq k \leq n$), then*

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n} + \max_{1 \leq i < j \leq n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)}$$

Proof. Let $\{d_n\}_{n \geq 1}$ be the sequence of nonnegative numbers defined by

$$d_n = \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} - \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n} \geq 0.$$

We claim that $\{d_n\}_{n \geq 1}$ is increasing. In fact,

$$\begin{aligned} d_{n+1} - d_n &= \sum_{k=1}^{n+1} \frac{x_k^2}{a_k} - \left(\sum_{k=1}^{n+1} x_k \right)^2 / \left(\sum_{k=1}^{n+1} a_k \right) \\ &\quad - \sum_{k=1}^n \frac{x_k^2}{a_k} - \left(\sum_{k=1}^n x_k \right)^2 / \left(\sum_{k=1}^n a_k \right) \end{aligned}$$

Setting $\alpha = \sum_{k=1}^n x_k$, $\beta = x_{n+1}$, $a = \sum_{k=1}^n a_k$ and $b = a_{n+1}$ into Bergström's inequality with $n = 2$, namely,

$$\frac{\alpha^2}{a} + \frac{\beta^2}{b} \geq \frac{(\alpha + \beta)^2}{a + b},$$

yields

$$d_{n+1} - d_n = \left(\sum_{k=1}^n x_k \right)^2 / \left(\sum_{k=1}^n a_k \right) + \frac{x_{n+1}^2}{a_{n+1}} - \left(\sum_{k=1}^{n+1} x_k \right)^2 / \left(\sum_{k=1}^{n+1} a_k \right) \geq 0.$$

Therefore, $d_n \geq d_{n-1} \geq \dots \geq d_2 \geq d_1 = 0$. Since

$$d_2 = \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} - \frac{(x_1 + x_2)^2}{a_1 + a_2} = \frac{(a_1 x_2 - a_2 x_1)^2}{a_1 a_2 (a_1 + a_2)},$$

then for all $n \geq 2$, we have $d_n \geq \frac{(a_1 x_2 - a_2 x_1)^2}{a_1 a_2 (a_1 + a_2)}$. Finally, due to the symmetry of d_n respect to the variables a_i and x_j , ($1 \leq i, j \leq n$), we have for all $n \geq 2$,

$$d_n \geq \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)}.$$

Notice that equality holds when $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_n}{a_n}$ and this completes the proof.

□

Setting $x_k = 1$ and changing a_k by x_k into the preceding result immediately follows [10]:

COROLLARY 1. *If $n \geq 2$ and x_1, x_2, \dots, x_n are positive real numbers, then holds*

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - \frac{n^2}{x_1 + x_2 + \dots + x_n} \geq \max_{1 \leq i < j \leq n} \frac{(x_i - x_j)^2}{x_i x_j (x_i + x_j)}.$$

Adding up the expressions $d_n \geq \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)}$, ($1 \leq i, j \leq n$) we obtain

$$\binom{n}{2} d_n \geq \sum_{i,j=1}^n \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)}$$

from which we get

COROLLARY 2. *If $x_k \in \mathbb{R}, a_k > 0, (1 \leq k \leq n)$, then for all $n \geq 2$, holds:*

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{2}{n(n-1)} \sum_{i,j=1}^n \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)}$$

Combining the preceding results the following double refinement of Bergström’s inequality is obtained.

COROLLARY 3. *Under the assumptions of Theorem 3, we have*

$$d_n \geq \max_{1 \leq i < j \leq k} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)} \geq \frac{2}{n(n-1)} \sum_{i,j=1}^n \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)} \geq 0.$$

Another interesting refinement of a generalization of Nesbitt’s inequality, namely,

$$\sum_{k=1}^n \frac{a_k}{s - a_k} \geq \frac{n}{n-1}$$

valid for all $a_1, a_2, \dots, a_n > 0, (n \geq 2)$ where $s = a_1 + a_2 + \dots + a_n$ is stated and proved in the next

THEOREM 4. *Let a_1, a_2, \dots, a_n be strictly positive numbers of sum s . Then, for all $n \geq 2$, holds:*

$$\sum_{k=1}^n \frac{a_k}{s - a_k} \geq \frac{n}{n-1} + \max_{1 \leq i < j \leq n} \frac{a_i a_j (a_i - a_j)^2}{(s - a_i)(s - a_j)[(a_i + a_j)s - (a_i^2 + a_j^2)]}.$$

Proof. Substituting x_k by a_k and a_k by $sa_k - a_k^2$ into Theorem 3, yields

$$\sum_{k=1}^n \frac{a_k}{s - a_k} = \sum_{k=1}^n \frac{a_k^2}{a_k(s - a_k)} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{s(a_1 + \dots + a_n) - (a_1^2 + \dots + a_n^2)}$$

$$+ \max_{1 \leq i < j \leq n} \frac{a_i a_j (a_i - a_j)^2}{(s - a_i)(s - a_j)[(a_i + a_j)s - (a_i^2 + a_j^2)]}$$

Now, taking into account that

$$\begin{aligned} \frac{(a_1 + a_2 + \dots + a_n)^2}{s(a_1 + \dots + a_n) - (a_1^2 + \dots + a_n^2)} &= \frac{s^2}{s^2 - (a_1^2 + \dots + a_n^2)} \\ &\geq \frac{s^2}{s^2 - s^2/n} = \frac{n}{n-1} \end{aligned}$$

the statement follows. Equality holds when $a_1 = a_2 = \dots = a_n$ and the proof is complete. \square

In the sequel a refinement of CBS inequality is presented.

THEOREM 5. *If $x_k, y_k \in \mathbb{R}$, ($1 \leq k \leq n$), then for all $n \geq 2$, holds:*

$$\left(\sum_{k=1}^n x_k^2 \right) \left(\sum_{k=1}^n y_k^2 \right) \geq \left(\sum_{k=1}^n x_k y_k \right)^2 + \left(\sum_{k=1}^n y_k^2 \right) \max_{1 \leq i < j \leq n} \frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2}.$$

Proof. Changing a_k by y_k^2 and x_k by $x_k y_k$, ($1 \leq k \leq n$) into Theorem 3, we obtain

$$\sum_{k=1}^n \frac{(x_k y_k)^2}{y_k^2} \geq \left(\sum_{k=1}^n x_k y_k \right)^2 / \left(\sum_{k=1}^n y_k^2 \right) + \max_{1 \leq i < j \leq n} \frac{(y_i^2 x_j y_j - y_j^2 x_i y_i)^2}{y_i^2 y_j^2 (y_i^2 + y_j^2)}.$$

From the preceding, after simplification, the statement follows. Equality holds when $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}$ and the proof is complete. \square

By the symmetry of variables x_k and y_k , ($1 \leq k \leq n$) into CBS inequality we also have that for all $n \geq 2$,

$$\left(\sum_{k=1}^n x_k^2 \right) \left(\sum_{k=1}^n y_k^2 \right) \geq \left(\sum_{k=1}^n x_k y_k \right)^2 + \left(\sum_{k=1}^n x_k^2 \right) \max_{1 \leq i < j \leq n} \frac{(x_i y_j - x_j y_i)^2}{x_i^2 + x_j^2}.$$

Taking into account Corollary 3, we get

COROLLARY 4. *If $x_k, y_k \in \mathbb{R}$, ($1 \leq k \leq n$) then for all $n \geq 2$, we have*

$$\left(\sum_{k=1}^n x_k^2 \right) \left(\sum_{k=1}^n y_k^2 \right) - \left(\sum_{k=1}^n x_k y_k \right)^2 \geq \frac{2}{n(n-1)} \left(\sum_{k=1}^n x_k^2 \right) \sum_{i,j=1}^n \frac{(x_i y_j - x_j y_i)^2}{x_i^2 + x_j^2}$$

and

$$\left(\sum_{k=1}^n x_k^2 \right) \left(\sum_{k=1}^n y_k^2 \right) - \left(\sum_{k=1}^n x_k y_k \right)^2 \geq \frac{2}{n(n-1)} \left(\sum_{k=1}^n y_k^2 \right) \sum_{i,j=1}^n \frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2}.$$

Finally, with the aid of Lagrange’s identity from Theorem 5, some elementary numerical inequalities are derived. We begin with

THEOREM 6. *Let $x_k, y_k, (1 \leq k \leq n)$ be real numbers, then for all $n \geq 2$, we have*

$$\sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2 \geq \frac{1}{2} \left[\left(\sum_{k=1}^n x_k^2 \right) \max_{1 \leq i < j \leq n} \frac{(x_i y_j - x_j y_i)^2}{x_i^2 + x_j^2} + \left(\sum_{k=1}^n y_k^2 \right) \max_{1 \leq i < j \leq n} \frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2} \right].$$

Proof. Taking into account the well known identity of Lagrange, namely

$$\left(\sum_{k=1}^n x_k^2 \right) \left(\sum_{k=1}^n y_k^2 \right) - \left(\sum_{k=1}^n x_k y_k \right)^2 = \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2,$$

we have

$$\sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2 \geq \left(\sum_{k=1}^n x_k^2 \right) \max_{1 \leq i < j \leq n} \frac{(x_i y_j - x_j y_i)^2}{x_i^2 + x_j^2}$$

and

$$\sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2 \geq \left(\sum_{k=1}^n y_k^2 \right) \max_{1 \leq i < j \leq n} \frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2}.$$

Adding up the preceding expressions we get the inequality claimed. Note that equality holds when $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}$. This completes the proof. □

Applying the same procedure and Corollary 4, we get

THEOREM 7. *Let $x_k, y_k, (1 \leq k \leq n)$ be real numbers, then for all $n \geq 2$, we have*

$$\sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2 \geq \frac{1}{n(n-1)} \left[\left(\sum_{k=1}^n x_k^2 \right) \sum_{i,j=1}^n \frac{(x_i y_j - x_j y_i)^2}{x_i^2 + x_j^2} + \left(\sum_{k=1}^n y_k^2 \right) \sum_{i,j=1}^n \frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2} \right].$$

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