

HARDY–HILBERT’S INEQUALITY WITH GENERAL HOMOGENEOUS KERNEL

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Abstract. A general form of recently obtained Hardy–Hilbert’s inequality with perturbed Hilbert’s kernel with the best possible estimation in the case of conjugate exponents is obtained. The multidimensional case is also considered. The case of non-conjugate exponents is briefly given.

1. Introduction

Although classical, Hilbert’s inequality and its generalizations and modifications are still of a great interest. Yongjin Li et al. in [7] considered the Hardy–Hilbert inequality in the case when the kernel is given by $K(x, y) = (x + y + \max\{x, y\})^{-1}$, and they obtained the following result:

THEOREM A. *Suppose $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, $0 < \int_0^\infty g^2(x)dx < \infty$. Then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dx dy < c \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{1/2}, \quad (1)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x + y + \max\{x, y\}} dx \right)^2 dy < c^2 \int_0^\infty f^2(x)dx, \quad (2)$$

where the constant factors $c = \sqrt{2}(\pi - 2 \arctan \sqrt{2})$ and c^2 are the best possible.

In this paper a generalization of Theorem A for a general type of homogeneous kernels is obtained. Recall that for a homogeneous function $K(x, y)$ of degree $-s$, $s > 0$, equality $K(tx, ty) = t^{-s}K(x, y)$ is satisfied for every $t > 0$. Further, we define $k(\alpha) := \int_0^\infty K(1, u)u^{-\alpha} du$ and suppose that $k(\alpha) < \infty$ for $1 - s < \alpha < 1$.

In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

Our results will be based on the following result of Krnić and Pečarić in [6].

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THEOREM B. *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. If $f(x), g(x) \geq 0, f, g \neq 0$, and $K(x, y)$ is nonnegative homogeneous function of degree $-s, s > 0$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy < L \left[\int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{1-s+q(A_2-A_1)} g^q(y) dy \right]^{\frac{1}{q}} \quad (3)$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)} \left(\int_0^\infty K(x, y) f(x) dx \right)^p dy < L^p \int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x) dx, \quad (4)$$

where $L = k(pA_2)^{\frac{1}{p}} k(2-s-qA_1)^{\frac{1}{q}}, A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$.

In Section 3. analogous results for the multidimensional case are given. Note that the best possible constant is obtained in [4] and [5] for the involved kernels $K(x, y) = (x+y)^{-1}$ and $K(x, y) = (x+y)^{-s}$ respectively. Here we obtain the best possible constant for a general homogeneous kernels $K(x_1, \dots, x_n), n \geq 2$.

Another way of generalizing the inequalities (1) and (2) is their extension to the case of non-conjugate exponents p and q . In 1951, Bonsall investigated this problem and established the following conditions for non-conjugate exponents (see [2]). Let p and q be real numbers such that

$$p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} \geq 1, \quad (5)$$

and let p' and q' respectively be their conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \quad (6)$$

and note that $0 < \lambda \leq 1$ for all p and q satisfying (5). Especially, $\lambda = 1$ holds if and only if $q = p'$, that is, only when p and q are mutually conjugate. Otherwise, we have $0 < \lambda < 1$, and in such case p and q will be referred to as non-conjugate exponents.

Applying results from [3] we shall obtain a generalization of inequalities (1) and (2) with non-conjugate exponents, but without a proof that the obtained estimations are the best possible.

2. The conjugate case

By putting the kernel $K(x, y) = (x + y + \max\{x, y\})^{-s}, s > 0$, in Theorem B, we get the following

COROLLARY 1. Let $s > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. If $f(x), g(x) \geq 0, f, g \neq 0$, then the following inequalities hold and are equivalent

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y+\max\{x,y\})^s} dx dy < L_1 \left[\int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{1-s+q(A_2-A_1)} g^q(y) dy \right]^{\frac{1}{q}} \tag{7}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)} \left(\int_0^\infty \frac{f(x)}{(x+y+\max\{x,y\})^s} dx \right)^p dy < L_1^p \int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x) dx, \tag{8}$$

where $L_1 = k(pA_2)^{\frac{1}{p}} k(2-s-qA_1)^{\frac{1}{q}}$, $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ and

$$k(\alpha) = \frac{2^{-s}}{\alpha+s-1} F\left(s, s+\alpha-1; s+\alpha; -\frac{1}{2}\right) + \frac{2^{-s}}{1-\alpha} F\left(s, 1-\alpha; 2-\alpha; -\frac{1}{2}\right).$$

$F(\alpha, \beta; \gamma; z)$ denotes the hypergeometric function defined by

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-zt)^{-\alpha} dt, \quad \gamma > \beta > 0, \quad z < 1.$$

Proof. The proof follows directly from Theorem B setting the kernel $K(x, y) = (x+y+\max\{x,y\})^{-s}$. Namely, in that case we have

$$\begin{aligned} k(\alpha) &= \int_0^\infty K(1, u) u^{-\alpha} du = \int_0^\infty \frac{u^{-\alpha}}{(1+u+\max\{1, u\})^s} du \\ &= \int_0^1 \frac{u^{-\alpha}}{(u+2)^s} du + \int_1^\infty \frac{u^{-\alpha}}{(2u+1)^s} du \\ &= \frac{2^{-s}}{1-\alpha} F(s, 1-\alpha; 2-\alpha; -1/2) \\ &\quad + \frac{2^{-s}}{\alpha+s-1} F(s, s+\alpha-1; s+\alpha; -1/2). \end{aligned}$$

□

REMARK 1. If the function $K(x, y)$ from Theorem B is symmetrical, then $k(2-s-qA_1) = k(qA_1)$. So, if $\max\{\frac{1}{p}, \frac{1}{q}\} < s$, then we can set $A_1 = A_2 = \frac{1}{pq}$ in Corollary 1 and obtain that the constant factor is equal to $L_1 = k\left(\frac{1}{p}\right)^{\frac{1}{q}} k\left(\frac{1}{q}\right)^{\frac{1}{p}}$. Now, setting $s = 1$ and $p = q = 2$, we obtain Theorem A from Introduction.

Here we shall obtain that the constants L and L^p involved in the right-hand side of the inequalities (3) and (4) from Theorem B are the best possible for some choices of the parameters A_1, A_2 and a weak condition on $K(x, y)$. To prove this result we need the next lemma.

LEMMA 1. *Let $s > 0, \frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. If $K(x, y)$ is nonnegative function such that $K(1, t)$ is bounded on $(0, 1)$, then*

$$\int_1^\infty x^{-\varepsilon-1} \left(\int_0^{x^{-1}} K(1, t)t^{-pA_2-\frac{\varepsilon}{q}} dt \right) dx = O(1), \quad \varepsilon \rightarrow 0+, \tag{9}$$

where $A_2 < \frac{1}{p}$.

Proof. Using assumptions we have $K(1, t) \leq C$ for some $C > 0$ and every $t \in (0, 1)$. Let $\varepsilon > 0$ be such that $\varepsilon < pq \left(\frac{1}{p} - A_2 \right)$. We have

$$\begin{aligned} \int_1^\infty x^{-1-\varepsilon} \left(\int_0^{1/x} K(1, t)t^{-pA_2-\frac{\varepsilon}{q}} dt \right) dx &\leq C \int_1^\infty x^{-1-\varepsilon} \left(\int_0^{1/x} t^{-pA_2-\frac{\varepsilon}{q}} dt \right) dx \\ &= \frac{C}{1-pA_2-\frac{\varepsilon}{q}} \int_1^\infty x^{pA_2+\frac{\varepsilon}{q}-\varepsilon-2} dx = \frac{C}{\left(1-pA_2-\frac{\varepsilon}{q}\right)\left(1-pA_2+\frac{\varepsilon}{p}\right)}, \end{aligned}$$

wherefrom (9) follows. □

THEOREM 1. *Let s, A_1, A_2 and $K(x, y)$ be defined as in Theorem B. Let $K(x, y)$ be such that $K(1, t)$ is bounded on $(0, 1)$. If the parameters A_1 and A_2 satisfy condition $pA_2 + qA_1 = 2 - s$, then the constants $L = k(pA_2)$ and L^p from Theorem B are the best possible.*

Proof. For this purpose, with $0 < \varepsilon < pq \left(\frac{1}{p} - A_2 \right)$, set $f(x) = x^{-qA_1-\frac{\varepsilon}{p}} \chi_{[1, \infty)}(x)$ and $g(y) = y^{-pA_2-\frac{\varepsilon}{q}} \chi_{[1, \infty)}(y)$, where χ_A is the characteristic function of a set A . Now, let us suppose that there exists a smaller constant $0 < M < L$ such that the the inequality (3) from Theorem B is valid. Let I denotes the right-hand side of (3). Then

$$I = M \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\varepsilon-1} dy \right)^{\frac{1}{q}} = \frac{M}{\varepsilon}. \tag{10}$$

Applying respectively Fubini's theorem, substitution $t = \frac{y}{x}$ and Lemma 1, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty K(x, y)f(x)g(y)dx dy \\ &= \int_1^\infty x^{-qA_1-\frac{\varepsilon}{p}} \left(\int_1^\infty K(x, y)y^{-pA_2-\frac{\varepsilon}{q}} dy \right) dx \\ &= \int_1^\infty x^{-\varepsilon-1} \left(\int_0^\infty K(1, t)t^{-pA_2-\frac{\varepsilon}{q}} dt - \int_0^{x^{-1}} K(1, t)t^{-pA_2-\frac{\varepsilon}{q}} dt \right) dx \\ &= \frac{1}{\varepsilon} \left(k \left(pA_2 + \frac{\varepsilon}{q} \right) + o(1) \right). \end{aligned} \tag{11}$$

From (3), (10) and (11) we get

$$k \left(pA_2 + \frac{\varepsilon}{q} \right) + o(1) < M. \tag{12}$$

Now, by letting $\varepsilon \rightarrow 0+$, we obtain from (12) a contradiction with assumption $M < L = k(pA_2)$.

Finally, equivalence of the inequalities (3) and (4) means that the constant $L^p = [k(pA_2)]^p$ is the best possible in the inequality (4). This completes the proof. \square

REMARK 2. If we put $s = 1$, $A_1 = A_2 = \frac{1}{pq}$ in Theorem B, then the inequalities (3) and (4) become

$$\int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy < L \|f\|_p \|g\|_q \tag{13}$$

and

$$\int_0^\infty \left(\int_0^\infty K(x, y) f(x) dx \right)^p dy < L^p \|f\|_p^p, \tag{14}$$

where $L = k\left(\frac{1}{q}\right)$. The following kernels $K(x, y)$ are homogenous with bounded $K(1, t)$ on $(0, 1)$. For each of these functions we compute the constant $L = k\left(\frac{1}{q}\right)$:

$$K(x, y) = \frac{1}{x + y + \max\{x, y\}},$$

$$L = \frac{1}{2} qF \left(1, \frac{1}{q}; 1 + \frac{1}{q}; -\frac{1}{2} \right) + \frac{1}{2} pF \left(1, \frac{1}{p}; 1 + \frac{1}{p}; -\frac{1}{2} \right), \quad L_2 = \sqrt{2} \left(\pi - 2 \arctan \sqrt{2} \right).$$

$$K(x, y) = \frac{1}{x + y + \min\{x, y\}},$$

$$L = qF \left(1, \frac{1}{q}; 1 + \frac{1}{q}; -2 \right) + pF \left(1, \frac{1}{p}; 1 + \frac{1}{p}; -2 \right), \quad L_2 = 2\sqrt{2} \arctan \sqrt{2}.$$

$$K(x, y) = \frac{1}{x + y - \min\{x, y\}} = \frac{1}{\max\{x, y\}}, \quad L = pq, \quad L_2 = 4.$$

$$K(x, y) = \frac{1}{x + y + \frac{2}{\frac{1}{x} + \frac{1}{y}}}, \quad L_2 = \sqrt{\frac{2}{3}} \pi.$$

$$K(x, y) = \frac{1}{x + y - \frac{2}{\frac{1}{x} + \frac{1}{y}}}, \quad L = \frac{\pi}{2} \left(\frac{1}{\cos \frac{\pi}{2p}} + \frac{1}{\cos \frac{\pi}{2q}} \right), \quad L_2 = \pi \sqrt{2}.$$

$$K(x, y) = \frac{1}{x + y + \sqrt{xy}}, \quad L_2 = \frac{4\pi}{3\sqrt{3}}.$$

$$K(x, y) = \frac{1}{x + y - \sqrt{xy}}, \quad L_2 = \frac{8\pi}{3\sqrt{3}}.$$

$$K(x, y) = \frac{1}{|x - y| + \max\{x, y\}},$$

$$L = \frac{1}{2}qF\left(1, \frac{1}{q}; 1 + \frac{1}{q}; \frac{1}{2}\right) + \frac{1}{2}pF\left(1, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right), \quad L_2 = 2 \operatorname{artanh} \frac{1}{\sqrt{2}}.$$

$$K(x, y) = \frac{x^{\lambda-1} + y^{\lambda-1}}{x^\lambda + y^\lambda}, \quad L = \frac{\pi}{\lambda} \left(\frac{1}{\sin \frac{\pi}{\lambda p}} + \frac{1}{\sin \frac{\pi}{\lambda q}} \right), \quad \lambda \geq 1.$$

$$K(x, y) = \frac{x^{\lambda-1} - y^{\lambda-1}}{x^\lambda - y^\lambda}, \quad L = \frac{\pi}{\lambda} \left(\cot \frac{\pi}{\lambda p} + \cot \frac{\pi}{\lambda q} \right), \quad \lambda > 1.$$

Note that the parameters $s = 1$, $A_1 = A_2 = \frac{1}{pq}$ satisfy condition $pA_2 + qA_1 = 2 - s$. Therefore, Theorem 1 implies that the constants L and L^p involved in the inequalities (13) and (14) are the best possible for above choices of the kernels $K(x, y)$. This means that the constants L are the norms of the Hilbert type operator $K : \mathcal{L}^p(0, \infty) \rightarrow \mathcal{L}^p(0, \infty)$ defined by $Kf(x) = \int_0^\infty K(x, y)f(y)dy$ for $f \in \mathcal{L}^p(0, \infty)$, where $\mathcal{L}^p(0, \infty)$ denotes the space of Lebesgue measurable function with $\int_0^\infty |f|^p(x)dx < \infty$.

3. The conjugate multidimensional case

Results in this section will be based on the following general form of Hardy-Hilbert's inequality proven in [1]. All the measures are assumed to be σ -finite on some measure space Ω .

THEOREM 2. *Let $n \geq 2$ be an integer and let p_1, \dots, p_n be conjugate parameters such that $p_i > 1$, $i = 1, \dots, n$. Let $K : \Omega^n \rightarrow \mathbf{R}$ and $\phi_{i,j} : \Omega \rightarrow \mathbf{R}$, $i, j = 1, \dots, n$, be nonnegative measurable functions such that $\prod_{i,j=1}^n \phi_{i,j}(x_j) = 1$. Then the following inequalities hold and are equivalent*

$$\int_{\Omega^n} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \cdots d\mu_n(x_n)$$

$$\leq \prod_{i=1}^n \left(\int_{\Omega} F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{1}{p_i}} \quad (15)$$

and

$$\int_{\Omega} h(x_n) \left(\int_{\Omega^{n-1}} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \cdots d\mu_{n-1}(x_{n-1}) \right)^q d\mu_n(x_n)$$

$$\leq \prod_{i=1}^{n-1} \left(\int_{\Omega} F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right)^{\frac{q}{p_i}}, \quad (16)$$

where

$$F_i(x_i) = \int_{\Omega^{n-1}} K(x_1, \dots, x_n) \prod_{j=1, j \neq i}^n \phi_{ij}^{p_i}(x_j) d\mu_1(x_1) \cdots d\mu_{i-1}(x_{i-1}) d\mu_{i+1}(x_{i+1}) \cdots d\mu_n(x_n),$$

$i = 1, \dots, n,$

$$h(x_n) = \phi_{nn}^{-q}(x_n)F_n^{1-q}(x_n) \quad \text{and} \quad \frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}.$$

In the same paper the authors discussed the case of equality in inequalities (15) and (16). They proved that the equality holds in (15) (and analogously in (16)) if and only if

$$f_i(x_i) = C_i \phi_{ii}(x_i)^{\frac{p_i}{1-p_i}}, \quad C_i \geq 0, \quad i = 1, \dots, n. \tag{17}$$

In the following theorem we give the case where $\Omega = (0, \infty)$, the measures $\mu_i, i = 1, \dots, n,$ are Lebesgue measures, K is a nonnegative homogeneous function of degree $-s, s > 0,$ and the functions ϕ_{ij} are of the form $\phi_{ij}(x_j) = x_j^{A_{ij}}$ where $A_{ij} \in \mathbf{R}, i, j = 1, \dots, n.$ We define multidimensional version of the function $k(\alpha),$ but in this section it will be more convenient to slightly change the definition:

$$k(\alpha_1, \dots, \alpha_{n-1}) := \int_{(0, \infty)^{n-1}} K(1, t_1, \dots, t_{n-1}) t_1^{\alpha_1} \cdots t_{n-1}^{\alpha_{n-1}} dt_1 \cdots dt_{n-1},$$

where we suppose that $k(\alpha_1, \dots, \alpha_{n-1}) < \infty$ for $\alpha_1, \dots, \alpha_{n-1} > -1$ and $\alpha_1 + \dots + \alpha_{n-1} + n < s + 1.$

THEOREM 3. *Let $n \geq 2$ be an integer and let p_1, \dots, p_n be conjugate parameters such that $p_i > 1, i = 1, \dots, n.$ Let $K : (0, \infty)^n \rightarrow \mathbf{R}$ be nonnegative measurable homogeneous function of degree $-s, s > 0,$ and let $A_{ij}, i, j = 1, \dots, n,$ be real numbers such that $\sum_{i=1}^n A_{ij} = 0$ for $j = 1, \dots, n.$ If $f_i : (0, \infty) \rightarrow \mathbf{R}, f_i \neq 0, i = 1, \dots, n$ are nonnegative measurable functions, then the following inequalities hold and are equivalent*

$$\int_{(0, \infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^n \left(\int_0^\infty x_i^{n-s-1+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}} \tag{18}$$

and

$$\int_0^\infty x_n^{(1-q)(n-1-s)-q\alpha_n} \left(\int_{(0, \infty)^{n-1}} K(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right)^q dx_n < L^q \prod_{i=1}^{n-1} \left(\int_0^\infty x_i^{n-1-s+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{q}{p_i}}, \tag{19}$$

where

$$L = k(p_1 A_{12}, \dots, p_1 A_{1n})^{\frac{1}{p_1}} \cdot k(s - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \dots, p_2 A_{2n})^{\frac{1}{p_2}} \cdots k(p_n A_{n2}, \dots, p_n A_{n,n-1}, s - n - p_n(\alpha_n - A_{nn}))^{\frac{1}{p_n}}, \tag{20}$$

$$\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}, \quad \alpha_i = \sum_{j=1}^n A_{ij} \quad \text{and} \quad p_i A_{ij} > -1, \quad i \neq j, \quad p_i(A_{ii} - \alpha_i) > n - s - 1.$$

Proof. Set $\phi_{i,j}(x_j) = x_j^{A_{ij}}$ in Theorem 2, where $\sum_{i=1}^n A_{ij} = 0$ for every $j = 1, \dots, n$. It is enough to calculate the functions $F_i(x_i)$, $i = 1, \dots, n$. Using homogeneity of the function K and obvious change of variables we have

$$\begin{aligned} F_1(x_1) &= \int_{(0,\infty)^{n-1}} K(x_1, x_2, \dots, x_n) \prod_{j=2}^n x_j^{p_1 A_{1j}} dx_2 \cdots dx_n \\ &= \int_{(0,\infty)^{n-1}} x_1^{-s} K(1, x_2/x_1, \dots, x_n/x_1) \prod_{j=2}^n x_j^{p_1 A_{1j}} dx_2 \cdots dx_n \\ &= x_1^{n-1-s+p_1(\alpha_1-A_{11})} k(p_1 A_{12}, \dots, p_1 A_{1n}). \end{aligned}$$

Using homogeneity of the function K and the change of variables

$$x_1 = x_2 \cdot \frac{1}{t_2}, \quad x_i = x_2 \cdot \frac{t_i}{t_2}, \quad i = 3, \dots, n, \quad \text{so} \quad \frac{\partial(x_1, x_3, \dots, x_n)}{\partial(t_2, t_3, \dots, t_n)} = x_2^{n-1} t_2^{-n},$$

where $\frac{\partial(x_1, x_3, \dots, x_n)}{\partial(t_2, t_3, \dots, t_n)}$ denotes the Jacobian of the transformation, we have

$$\begin{aligned} F_2(x_2) &= \int_{(0,\infty)^{n-1}} K(x_1, x_2, \dots, x_n) \prod_{j=1, j \neq 2}^n x_j^{p_2 A_{2j}} dx_1 dx_3 \cdots dx_n \\ &= \int_{(0,\infty)^{n-1}} x_1^{-s} K(1, x_2/x_1, \dots, x_n/x_1) \prod_{j=1, j \neq 2}^n x_j^{p_2 A_{2j}} dx_1 dx_3 \cdots dx_n \\ &= x_2^{n-1-s+p_2(\alpha_2-A_{22})} k(s-n-p_2(\alpha_2-A_{22}), p_2 A_{23}, \dots, p_2 A_{2n}). \end{aligned}$$

In a similar manner we obtain

$$F_i(x_i) = x_i^{n-s-1+p_i(\alpha_i-A_{ii})} k(p_i A_{i2}, \dots, p_i A_{i,i-1}, s-n-p_i(\alpha_i-A_{ii}), p_i A_{i,i+1}, \dots, p_i A_{in}),$$

for $i = 3, \dots, n$. This gives inequalities (18) and (19) with inequality sign \leq . Condition (17) immediately gives that nontrivial case of equality in (18) and (19) leads to the divergent integrals. This completes the proof. \square

To obtain a case of the best possible inequality in (18) and (19) it is natural to impose the following conditions on the parameters A_{ij} :

$$\begin{aligned} p_j A_{ji} &= s-n-p_i(\alpha_i-A_{ii}), \quad i=2, \dots, n, \quad j \neq i, \\ p_i A_{ik} &= p_j A_{jk}, \quad k \neq i, j, \quad k \neq 1. \end{aligned} \tag{21}$$

The missing cases $i = 1$ and $k = 1$ can be deduced from (21) as follows:

$$p_1(\alpha_1 - A_{11}) = p_1 A_{1j} + p_1 \sum_{i \neq 1, j} A_{1i} = s-n-p_j(\alpha_j - A_{jj}) + p_j \sum_{i \neq j, 1} A_{ji} = s-n-p_j A_{j1},$$

where $j \neq 1$. Thus, the complete set of conditions is

$$\begin{aligned} p_j A_{ji} &= s-n-p_i(\alpha_i - A_{ii}), \quad i, j = 1, 2, \dots, n, \quad i \neq j, \\ p_i A_{ik} &= p_j A_{jk}, \quad k \neq i, j. \end{aligned} \tag{22}$$

THEOREM 4. *Suppose that the real parameters A_{ij} , $i, j = 1, \dots, n$ satisfy conditions in Theorem 3 and conditions given in (22). If the kernel $K = K(x_1, \dots, x_n)$ is as in Theorem 3 and for every $i = 2, \dots, n$*

$$K(1, t_2, \dots, t_i, \dots, t_n) \leq CK(1, t_2, \dots, 0, \dots, t_n), \quad 0 \leq t_i \leq 1, \quad t_j \geq 0, \quad j \neq i,$$

for some $C > 0$, then the constants L and L^q are the best possible in inequalities (18) and (19), where $L = k(p_1A_{12}, p_1A_{13}, \dots, p_1A_{1n})$.

Proof. It is easy to see that $n - s + p_i\alpha_i = -p_i\tilde{A}_i$ where $\tilde{A}_i = p_1A_{1i}$ for $i \neq 1$ and $\tilde{A}_1 = p_nA_{n1}$. Inequality (18) can now be written as follows:

$$\int_{(0, \infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^n \left(\int_0^\infty x_i^{-1-p_i\tilde{A}_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}}, \tag{23}$$

where $L = k(\tilde{A}_2, \dots, \tilde{A}_n)$.

Now let's suppose that the constant factor L is not the best possible. Then, there exists a positive constant M , smaller than L such that the inequality (23) is still valid if we replace L by M . For this purpose, set

$$\tilde{f}_i(x_i) = \begin{cases} 0 & x \in (0, 1) \\ x_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} & x \in [1, \infty) \end{cases}, \quad i = 1, \dots, n,$$

where $0 < \varepsilon < \min_{1 \leq i \leq n} \{p_i + p_i\tilde{A}_i\}$. If we put these functions in the inequality (23), then the right-hand side of the inequality becomes $\frac{M}{\varepsilon}$, since

$$\prod_{i=1}^n \left[\int_0^\infty x_i^{-1-p_i\tilde{A}_i} \tilde{f}_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} = \frac{1}{\varepsilon}. \tag{24}$$

Further, let J denotes the left-hand side of the inequality (23), for above choice of the functions \tilde{f}_i . By using substitution $u_i = \frac{x_i}{x_1}$, $i = 2, \dots, n$ in J , we find that

$$J = \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{\frac{1}{x_1}}^\infty \cdots \int_{\frac{1}{x_1}}^\infty K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \right] dx_1. \tag{25}$$

It is easy to see that the following inequality holds

$$\begin{aligned} J &\geq \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{(0, \infty)^{n-1}} K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n \right] dx_1 \\ &\quad - \int_1^\infty x_1^{-1-\varepsilon} \sum_{j=2}^n I_j(x_1) dx_1 \\ &= \frac{1}{\varepsilon} k\left(\tilde{A}_2 - \frac{\varepsilon}{p_2}, \dots, \tilde{A}_n - \frac{\varepsilon}{p_n}\right) - \int_1^\infty x_1^{-1-\varepsilon} \sum_{j=2}^n I_j(x_1) dx_1, \end{aligned} \tag{26}$$

where for $j = 2, \dots, n$, $I_j(x_1)$ is defined by

$$I_j(x_1) = \int_{D_j} K(1, u_2, \dots, u_n) \prod_{i=2}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \dots du_n,$$

where $D_j = \{(u_2, u_3, \dots, u_n); 0 < u_j \leq \frac{1}{x_1}, 0 < u_k < \infty, k \neq j\}$. Without losing generality, we only estimate the integral $I_2(x_1)$. The case $n = 2$ is proven in Theorem 1. For $n \geq 3$ we have

$$\begin{aligned} I_2(x_1) &\leq C \left[\int_{(0, \infty)^{n-2}} K(1, 0, u_3, \dots, u_n) \prod_{i=3}^n u_i^{\tilde{A}_i - \frac{\varepsilon}{p_i}} du_3 \dots du_n \right] \cdot \int_0^{\frac{1}{x_1}} u_2^{\tilde{A}_2 - \frac{\varepsilon}{p_2}} du_2 \\ &= C \left(1 - \frac{\varepsilon}{p_2} + \tilde{A}_2 \right)^{-1} x_1^{\frac{\varepsilon}{p_2} - \tilde{A}_2 - 1} k \left(\tilde{A}_3 - \frac{\varepsilon}{p_3}, \dots, \tilde{A}_n - \frac{\varepsilon}{p_n} \right), \end{aligned}$$

where $k \left(\tilde{A}_3 - \frac{\varepsilon}{p_3}, \dots, \tilde{A}_n - \frac{\varepsilon}{p_n} \right)$ is well defined since obviously $\tilde{A}_3 + \dots + \tilde{A}_n < s - n + 2$. Hence, we have $I_j(x_1) = x_1^{\frac{\varepsilon}{p_j} - \tilde{A}_j - 1} O_j(1)$ for $\varepsilon \rightarrow 0+$, $j = 2, \dots, n$, and consequently

$$\int_1^\infty x_1^{-1-\varepsilon} \sum_{j=2}^n I_j(x_1) dx_1 = O(1). \tag{27}$$

We conclude, by using (24), (26) and (27), that $L \leq M$ when $\varepsilon \rightarrow 0+$, which is an obvious contradiction. It follows that the constant L in (23) is the best possible.

Further, since the equivalence keeps the best possible constant, the proof is completed.

□

REMARK 3. If we define the parameters A_{ij} with $A_{ii} = (n-s) \frac{p_i-1}{p_i^2}$ and $A_{ij} = \frac{s-n}{p_i p_j}$, $i \neq j$, $i, j = 1, \dots, n$, then we have

$$\sum_{i=1}^n A_{ij} = (s-n) \sum_{i \neq j} \frac{1}{p_i p_j} + (n-s) \frac{p_j-1}{p_j^2} = \frac{s-n}{p_j} \left(\sum_{i=1}^n \frac{1}{p_i} - 1 \right) = 0,$$

for $j = 1, 2, \dots, n$. Since the parameters A_{ij} are symmetric one obtains $\alpha_i = \sum_{j=1}^n A_{ij} = 0$, for $i = 1, \dots, n$. Also, it is obvious that the parameters A_{ij} , $i, j = 1, \dots, n$ satisfy conditions (22). In this case Theorem 4 gives

$$\int_{(0, \infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n < L \prod_{i=1}^n \left(\int_0^\infty x_i^{n-s-1} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}},$$

where $L = k \left(\frac{s-n}{p_2}, \dots, \frac{s-n}{p_n} \right)$ is the best possible constant. For $s = n - 1$ we obtain non-weighted case with the best possible constant $L = k \left(-\frac{1}{p_2}, \dots, -\frac{1}{p_n} \right)$ (compare Remark 2).

4. The non-conjugate case

To obtain an analogous result for the case of non-conjugate exponents, but without a proof that the obtained estimations are the best possible, we need the following result from [3]:

THEOREM 5. *Let $s > 0$, p, p', q, q' and λ be as in (5) and (6). If $f(x), g(x) \geq 0$, and $K(x, y)$ is homogeneous function of degree $-s, s > 0$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty K^\lambda(x, y) f(x) g(y) dx dy \leq L \left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} g^q(y) dy \right]^{\frac{1}{q}} \tag{28}$$

and

$$\left(\int_0^\infty y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left(\int_0^\infty K^\lambda(x, y) f(x) dx \right)^{q'} dy \right)^{\frac{1}{q'}} \leq L \left(\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x) dx \right)^{\frac{1}{p}}, \tag{29}$$

where $L = k(q'A_2)^{\frac{1}{q'}} k(2 - s - p'A_1)^{\frac{1}{p'}}$, $A_1 \in (\frac{1-s}{p'}, \frac{1}{p'})$ and $A_2 \in (\frac{1-s}{q'}, \frac{1}{q'})$.

By putting the kernel $K(x, y) = (x + y + \max\{x, y\})^{-s}, s > 0$, in Theorem 5, we get the following

COROLLARY 2. *Let $s > 0$, p, p', q, q' and λ be as in (5) and (6). If $f(x), g(y)$ are nonnegative functions, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x + y + \max\{x, y\})^{\lambda s}} dx dy \leq L_2 \left(\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} g^q(y) dy \right)^{\frac{1}{q}} \tag{30}$$

and

$$\left(\int_0^\infty y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left(\int_0^\infty \frac{f(x)}{(x + y + \max\{x, y\})^{\lambda s}} dx \right)^{q'} dy \right)^{\frac{1}{q'}} \leq L_2 \left(\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x) dx \right)^{\frac{1}{p}}, \tag{31}$$

where $L_2 = k(q'A_2)^{\frac{1}{q'}} k(2 - s - p'A_1)^{\frac{1}{p'}}$, $A_1 \in (\frac{1-s}{p'}, \frac{1}{p'})$, $A_2 \in (\frac{1-s}{q'}, \frac{1}{q'})$ and $k(\alpha)$ be as in Corollary 1.

It remains an open problem if the constant $L = k(q'A_2)^\lambda$ is the best possible one under condition $p'A_1 + q'A_2 = 2 - s$, even if the kernel is $K(x, y) = (x + y)^{-s}$? The problem is open even in the case $s = 1$ and $A_1 = A_2$ (see [2]).

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